## Flexure of a Beam of Curvilinear Polygonal Section

### Introduction

Recently L. M. MILNE-THOMSON [1] has developed complex variable techniques to obtain the solution of flexure problem of beams of different cross-sections which can be mapped conformally onto the unit circle in the $\zeta$-plane. Expression for the required single flexure function $\Phi(z)$ can be obtained by using CAUCHY's theorem. The torsion function, the shearing stress and the centre of flexure can be determined very easily. Solution of the flexure problem of beams of the following cross-sections are obtained by L. M. MILNE-THOMSON: (a) Circle (b) Cardioid (c) One loop of BERNOULLIS Leminscate.

In this paper complex variable methods developed by L. M. MILNE-THOMSON have been employed to obtain the solution of the flexure problem of a beam of curvilinear polygonal cross-section.

Let us suppose that the beam is of homogeneous material bounded by planes normal to the generating lines. One end of the beam is fixed and at the other end say $R = L$ there is a transverse force applied. The lateral surface of the beam is free of applied force and the body forces are neglected. The transverse force $W$ applied at the end of the beam $R = L$ is supposed to be applied by a distribution of shear stresses with components $\widehat{xz}_L$, $\widehat{yz}_L$ parallel to $o\,x$, $o\,y$. This shear system is equivalent to forces $W_x$ and $W_y$ through $(0,0,L)$ and a moment $M$ about $o\,R$, where

$$\left.\begin{array}{l} W_x = \int \widehat{xz}_L\,ds\,, \quad W_y = \int \widehat{yz}_L\,ds\,, \\ M = \int (x\,\widehat{yz}_L - y\,\widehat{xz})_L\,ds \end{array}\right\} \quad . . (1).$$

The integrals are to be evaluated over the area of the cross section of the beam $R = L$. Suppose the load is applied in such a manner that the stress components $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ vanish and then the non-vanishing stress component $\widehat{zz}$ is given by the formula

$$\widehat{zz} = \frac{1}{2} E\,(R - L)\,(\beta\,\bar z + \bar\beta\,z + 2\,r_1) \qquad (2),$$

and the shears $\widehat{xz}$ and $\widehat{yz}$ can be determined from the formula

$$\widehat{xz} - i\,\widehat{yz} = \mu\,[\Phi(z) - p\,\bar z - q\,\bar z^2 - r\,z\,\bar z] \quad (3),$$

where

$$z = x + i\,y\,,\ \bar z = x - i\,y\,,\ E = 2\,\mu\,(1 + \eta) \quad (4),$$

(YOUNG's modulus)

$$P = W_x + i\,W_y \quad . . . . . . (5),$$

$$Z_G = x_G + i\,y_G \quad . . . . . . . (6).$$

$x_G$, $y_G$ are the coordinates of the centre of gravity of the cross section.

$$S = \int ds\,,\ z_G\,S = \int z\,ds \quad . . . . . . (7),$$

$$p = i\,\tau - \frac{1}{2}(1 + \eta)\,(\beta\,\bar z_G + \bar\beta\,z_G) \quad . . (8),$$

$$q = \frac{1}{4}(1 + 2\,\eta)\,\beta \quad . . . . . . . . . (9),$$

$$2\,r = \bar\beta \quad . . . . . . . . . . . . . . (10,)$$

$$2\,r_1 = -(\beta\,\bar z_G + \bar\beta\,z_G) \quad . . . . . . . (11),$$

$$\beta = (A + B)\,P - (B - A + 2\,i\,H)\,\bar P / 2\,E\,(A\,B - H^2) \quad (12),$$

$$A + B = -\frac{i}{4}\int (\bar z - \bar z_G)^2\,(z - z_G)\,dz \quad (13),$$

$$B - A - 2\,i\,H = \frac{i}{2}\int (z - z_G)^2\,(\bar z - \bar z_G)\,dz \qquad (14).$$

$\Phi(z)$ is the required complex flexure function which can be determined from the integral equation

$$\psi(\zeta)\,w'(\zeta) = \frac{1}{2\,\pi\,i}\int_\gamma \frac{H(\sigma)}{\sigma - \zeta}\,d\sigma \quad . . . (15),$$

where $\zeta$ is a point inside the unit circle $\gamma$ in the $\zeta$-plane and

$$\Phi(z) = \Phi(w(\zeta)) = \psi(\zeta) \quad . . . . (16),$$

$$z = w(\zeta) \quad . . . . . . . . . (17),$$

is the mapping formula which maps conformally the cross-section of the beam under consideration onto the unit circle $|\zeta| \leq 1$ and

$$H(\sigma) = \overline{w(\sigma)}\,w'(\sigma)\,[p + q\,\overline{w(\sigma)} + r\,w(\sigma)] + \sigma^{-2}\,w(\sigma)\,\overline{w'(\sigma)}\,[\bar p + \bar q\,w(\sigma) + \bar r\,\overline{w(\sigma)}] \quad (18).$$

### Determination of Displacements

The displacement components $u$, $v$, $w$ can be determined from the formulae

$$u - i\,v = -\frac{1}{6}\,\bar\beta\,(R - L)^3 - (R - L)\left[\eta\left(\frac{1}{2}\,\beta\,\bar z^2 + r_1\,\bar z\right) + i\,\tau\,\bar z\right] \quad (19),$$

$$w = \mathrm{Re}\int \Phi(z)\,dz - \frac{1}{4}\,r\,\bar z\,z^2 - \frac{1}{4}\,\bar r\,z\,\bar z^2 - \frac{1}{2}\,r_1\,z\,\bar z + \frac{1}{4}(R - L)^2\,(\bar\beta\,z + \beta\,\bar z + 2\,r_1) \quad . . . . (20),$$

where Re denotes the real part.

### The Centre of Flexure

The rotation about $oR$ is given by the formula

$$\Omega = \frac{1}{2}(R - L)\,[2\,\tau - i\,\eta\,(\beta\,\bar z - \bar\beta\,z)] \quad (21),$$

and the local twist is given by the formula

$$\frac{\partial\Omega}{\partial R} = \tau - \frac{1}{2}\,i\,\eta\,(\beta\,\bar z - \bar\beta\,z) \quad . . . (22).$$

For the case of torsion $\beta = 0$ and $\tau$ is the twist per unit length. In the case of general flexure $\beta$ is not equal to zero, there is usually torsion accompanying flexure.

### Expression for the Moment

If $M$ is the moment of the applied load with respect to $0_L$ then

$$M\,\bar\mu' = \frac{1}{2}\,\mathrm{Re}\,K + \tau\,I + \frac{1}{48}(1 - 2\,\eta)\,(\beta\,J + \bar\beta\,\bar J) \quad (23),$$

where

$$I = -\frac{i}{4}\int_\gamma w(\sigma)\,\overline{w(\sigma)}^2\,w'(\sigma)\,d\sigma \quad . . (24),$$

$$J = \int_\gamma w(\sigma)\,\overline{w(\sigma)}^3\,w'(\sigma)\,d\sigma \quad . . . . . (25),$$

$$K = \int_\gamma w(\sigma)\,\overline{w(\sigma)}\,H_1(\sigma)\,d\sigma \quad . . . . . (26).$$

If $\beta = 0$, then $q = r = 0$ and we easily obtain

$$K = \tau\int_\gamma w(\sigma)\,\overline{w(\sigma)}\,f_0'(\sigma)\,d\sigma \quad . . . . . (27),$$

where

$$f_0(\zeta) = \frac{1}{2\,\pi}\int_\gamma \frac{w(\sigma)\,\overline{w(\sigma)}}{\sigma - \zeta}\,d\sigma \quad . . . (28).$$

### Conformal Transformation

The mapping funktion

$$z = w(\zeta) = w(\varrho\, e^{i\theta}) = c\,\zeta \,/\, (1 + m\,\zeta^n) \qquad (29),$$

$c > 0$, $n \geqq 2$, $-1 \leqq m\,(n-1) \leqq 1$, maps conformally the cross-section of the beam on to the unit circle $|\zeta| \leqq 1$. Using the relation (29) in the formulae (7), (13), (14), (24), (25), (27) and (28) we obtain the following results after simplification

$$S = -\frac{1}{2}\, i \int_\gamma \overline{w(\sigma)}\, w'(\sigma)\, d\sigma$$

$$= \pi\, c^2\, [1 + m^2\,(n-1)]/(1 - m^2)^2 \quad . \; . \quad (30),$$

$$z_G\, S = -\frac{1}{2}\, i \int_\gamma w(\sigma)\, \overline{w(\sigma)}\, w'(\sigma)\, d\sigma = 0 \qquad (31),$$

$$A + B = -\frac{i}{4} \int_\gamma \overline{w(\sigma)}^2\, w(\sigma)\, w'(\sigma)\, d\sigma$$

$$= \frac{\pi\, c^4}{2}\, [1 + 2\,m^2 n + m^4 (n-1)]\, (1 - m^2)^{-4} \quad (32),$$

$$B - A - 2\,i\,H = \frac{1}{2}\, i \int_\gamma [w(\sigma)]^2\, \overline{w(\sigma)}\, w'(\sigma)\, d\sigma = 0 \quad (33).$$

Hence $H = 0$, $A + B = I$ and

$$A = B = \frac{\pi\, c^4}{4}\, [1 + 2\,m^2 n + m^4 (n-1)]\, (1 - m^2)^{-4} \quad (34),$$

$$J = \int_\gamma w(\sigma)\, \overline{w(\sigma)}^3\, w'(\sigma)\, d\sigma = 0 \quad . \; . \quad (35),$$

$$f_0(\zeta) = \frac{1}{2\,\pi} \int_\gamma \frac{w(\sigma)\, \overline{w(\sigma)}\, d\sigma}{\sigma - \zeta}$$

$$= i\, c^2 \,/\, (1 - m^2)\, (1 + m\,\zeta^n) + \text{Const.} \qquad (36)$$

and

$$K = \tau \int_\gamma w(\sigma)\, \overline{w(\sigma)}\, f_0'(\sigma)\, d\sigma = -2\,\pi\,\tau\, c^4\, m^2 n \,/\, (1 - m^2)^4 \qquad (37).$$

From (8), (9), (10), (11) and (12) we find

$$\left.\begin{aligned} &p = i\tau, \quad q = \frac{1}{4}(1 + 2\eta)\beta, \quad 2r = \beta, \quad r_1 = 0, \\ &\beta = 4\, W\, (1 - m^2)^4 \,/\, \pi\, E\, c^4\, [1 + 2\, m^2\, n + m^4\, (n-1)] \end{aligned}\right\} \qquad (38).$$

Let us take

$$P = W_x = W\,, \text{ then } \beta = \bar{\beta} \quad . \; . \; . \quad (39).$$

From (29), (18) and (15) we obtain after integration the expression for the complex flexure function in the form

$$\psi(\zeta)\, w'(\zeta) = \frac{c^2}{\zeta\,(1 + m\,\zeta^n)} \left(\bar{p} + \bar{\gamma}\, \frac{c}{\zeta}\right)$$

$$+ c^3\, \gamma \left\{\frac{1 + m\,(1-n)\,\zeta^n}{(1 + m\,\zeta^n)^3}\right\} - \frac{c^2}{\zeta}\,(p + \bar{p})$$

$$+ \gamma\, c^3\, m^2\,(n-1) + \frac{c^2}{(1 + m\,\zeta^n)^2} \left[p\left\{\frac{1}{\zeta} - m\,(n-1)\,\zeta^{n-1}\right\}\right.$$

$$\left. + q\,c\left\{\frac{1}{\zeta^2} - m\,(n-1)\,\zeta^{n-2}\right\} + \bar{q}\,c\right] - \frac{c^3}{\zeta^2}\,(\bar{\gamma} + q)$$

$$+ p\, c^2 \Big[\sum_{k_1, k_2, = 1, 2, \ldots, \alpha} (-1)^{k_1 + k_2}\, (k_2 + 1)\, m^{k_1 + k_2}$$

$$\times \left\{\zeta^{n\,(k_2 - k_1) - 1} - m\,(n-1)\, \zeta^{n\,(k_2 - k - 1) + 1}\right\}\Big]$$

$$+ q\, c^3 \Big[\sum_{k_3, k_4 = 1, 2, \ldots, \alpha} (-1)^{k_3 + k_4}\, (k_3 + 1)\, (k_4 + 1)\, m^{(k_3 + k_4)}$$

$$\times \left\{\zeta^{n\,(k_4 - k_3) - 2} - m\,(n-1)\, \zeta^{n\,(k_4 - k_3 + 1) - 2}\right\}\Big]$$

$$+ \gamma\, c^3 \Big[\sum_{k_5, k_6 = 1, 2, \ldots, \alpha} (-1)^{k_5 + k_6}\, \frac{(k_6 + 1)\, (k_6 + 2)}{2}\, m^{(k_6 + k_5)}$$

$$\times \left\{\zeta^{n\,(k_6 - k_5)} - m\,(n-1)\, \zeta^{n\,(k_6 - k_5 + 1)}\right\}\Big]$$

$$+ \bar{p}\, c^2 \Big[\sum_{t_1, t_2 = 1, 2, \ldots, \alpha} (-1)^{t_1 + t_2}\, m^{(t_1 + t_2)}\, \zeta^{n\,(t_2 - t_1) - 1}$$

$$- m\, n \sum (-1)^{t_1 + t_2}\, (t_1 + 1)\, m^{(t_1 + t_2)}\, \zeta^{n\,(t_2 - t_1 - 1) - 1}\Big]$$

$$+ \bar{q}\, c^3 \Big[\sum_{t_3, t_4 = 1, 2, \ldots, \alpha} (-1)^{t_3 + t_4}\, (t_4 + 1)\, m^{(t_4 + t_3)}\, \zeta^{n\,(t_4 - t_3)}$$

$$- m\, n \sum (-1)^{t_4 + t_3}\, (t_4 + 1)\, (t_3 + 1)\, m^{(t_4 + t_3)}\, \zeta^{n\,(t_4 - t_3) - 1}\Big]$$

$$+ \bar{\gamma}\, c^3 \Big[\sum_{t_5, t_6 = 1, 2, \ldots, \alpha} (-1)^{t_5 + t_6}\, m^{(t_5 + t_6)}\, \zeta^{n\,(t_6 - t_5) - 2}$$

$$- m\, n \sum (-1)^{t_5 + t_6}\, (t_5 + 1)\, (t_5 + 2)\, m^{(t_5 + t_6)}\, \zeta^{n\,(t_6 - t_5 - 1) + 2}\Big] \qquad (40),$$

where

$$\left.\begin{aligned} k_2 - k_1 > 0\,, &\quad n\,(k_2 - k_1 - 1) > 1\,, \\ n\,(k_4 - k_3) > 2\,, &\quad n\,(k_4 - k_3 + 1) > 2\,, \\ k_6 - k_5 > 0\,, &\quad k_6 - k_5 + 1 > 0\,, \\ n\,(t_2 - t_1) > 1\,, &\quad n\,(t_2 - t_1 - 1) > 1\,, \\ t_4 - t_3 > 0\,, &\quad t_4 - t_3 - 1 > 0\,, \\ n\,(t_6 - t_5) - 2 > 0\,, &\quad n\,(t_6 - t_5 - 1) + 2 > 0 \end{aligned}\right\} \quad (41).$$

### Calculation of the Stress Components

Substituting (29) in (2) we find

$$\widehat{zz} = E\,(R - L)\,\beta\,\varrho\, c\, [\cos\theta + m\,\varrho^n \cos\,(n-1)\,\theta]$$

$$\times [1 + 2\, m\,\varrho^n \cos n\,\theta + m^2 \varrho^{2n}]^{-1} \quad . \; . \quad (42).$$

The expression for the shears $\widehat{yz}$ and $\widehat{zx}$ can be determined from (3), (29) and (40).

### Determination of Displacements

Using (29) in the formula (19) we obtain after simplification the expressions for the displacements in the form

$$u = -\frac{1}{6}\,\beta\,(R - L)^3 - (R - L) \Big[\frac{1}{2}\,\eta\,\beta\,\varrho^2\, c^2 \{\cos 2\,\theta$$

$$+ m^2 \varrho^{2n} \cos 2\,(n-1)\,\theta + 2\, m\,\varrho^n \cos\,(n-2)\,\theta\}\, D^{-2}$$

$$+ \tau\,\varrho\, c\, \{\sin\theta - m\,\varrho^n \sin\,(n-1)\,\theta\}\, D^{-1}\Big] \quad . \; . \quad (43),$$

$$v = (R - L) \Big[\frac{1}{2}\,\eta\,\beta\,\varrho^2\, c^2 \{m^2 \varrho^{2n} \sin\, 2\,(n-1)\,\theta$$

$$+ 2\, m\,\varrho^n \sin\,(n-2)\,\theta - \sin\, 2\,\theta\}\, D^{-2}$$

$$+ \tau\,\varrho\, c\, \{\cos\theta + m\,\varrho^n \cos\,(n-1)\,\theta\}\, D^{-1}\Big] \qquad (44),$$

where

$$D = (1 + 2\, m\,\varrho^n \cos n\,\theta + m^2 \varrho^{2n}) \quad . \; . \quad (45).$$

The displacement component $w$ can be determined from (40), (29), (31), (38) and (20).

### Centre of Flexure

From (29) and (21) and (22) we easily obtain

$$\Omega = (R - L)\, [\tau - \eta\,\beta\,\varrho\, c\, (\sin\theta - m\,\varrho^n \sin\,(n-1)\,\theta)\, D^{-1}] \qquad (46),$$

$$\frac{\partial \Omega}{\partial R} = \text{(local twist)}$$

$$= \tau - \eta\, \beta\, \varrho\, c\, (\sin \theta - m\, \varrho^n \sin(n-1)\, \theta)\, D^{-1} \quad (47).$$

Expression for the Moment

Substituting (37), (32), (35) in the formula (23) we easily obtain

$$M\, \mu^{-1} = \pi\, \tau\, c\, 4\, [1 + m\, 4\, (n-1)] \,/\, 2\, (1 - m^2)^4 \quad (48),$$

which agrees with the result obtained by W. A. BASSALI [2].

When the cross-section of the beam is an inverse of an ellipse write $n = 2$ and $m = 1/4$ (say) in the above formulae and we obtain

$$\widehat{zz} = E\, (R - L)\, \beta_1\, \varrho\, c \cos \theta\, (1 + 0{,}25\, \varrho^2)\, D_1^{-1} \quad (49.1),$$

$$u = -\frac{1}{6} \beta_1 (R - L)^3 - (R - L) \left[\frac{1}{2} \eta\, \beta_1\, c^2 \varrho^2 \times \{\cos 2\theta + 0.0\,625\, \varrho^4 \cos 2\, \theta + 0.5\, \varrho^2\} D_1^{-2} + \tau\, c\, \varrho \sin \theta (1 - 0.25\, \varrho^2)\, D_1^{-1}\right] \quad \ldots\ldots (49.2),$$

$$v = (R - L) \left[\frac{1}{2} \eta\, \beta_1\, \varrho^2\, c^2 \{0.0625\, \varrho^4 \sin 2\, \theta + 0.5\, \varrho^2 - \sin 2\, \theta\} D_1^{-2} + \tau\, c\, \varrho \cos \theta\, (1 + 0.25\, \varrho^2)\, D_1^{-1}\right] \quad (49.3),$$

where

$$D_1 = (1 + 0.5\, \varrho^2 \cos 2\, \theta + 0.0625\, \varrho^4) \quad (49.4),$$

$$\Omega = (R - L)\, [\tau - \eta\, \beta_1\, c\, \varrho \sin \theta\, (1 - 0.25\, \varrho^2)\, D_1^{-1}] \quad (49.5),$$

$$\frac{\partial \Omega}{\partial R} = \tau - \eta\, \beta_1\, c\, \varrho \sin \theta\, (1 - 0.25\, \varrho^2)\, D_1^{-1} \quad (49.6),$$

$$\beta_1 = W \,/\, 0.4057\, E\, \pi\, c^4 \quad \ldots . (49.7),$$

$$M\, \mu^{-1} = 0.6497\, \pi\, \tau\, c^4 \quad \ldots . (49.8).$$

When the cross section of the beam is a hexagon we put $n = 6$ and $m = -1/10$ (say) and we easily obtain

$$\widehat{zz} = E\, (R - L)\, \beta_2\, c\, \varrho\, (\cos \theta - 0.1\, \varrho^6 \cos 5\, \theta)\, D_2^{-1} \quad (50.1),$$

$$u = -\frac{1}{6} \beta_2\, (R - L)^3 - (R - L) \left[\tau\, c\, \varrho\, (\sin \theta + 0.1\, \varrho^6 \sin 5\, \theta)\, D_2^{-1} - \frac{1}{2} \eta\, \beta_2\, c^2\, \varrho^2\, \{\cos 2\, \theta + 0.01\, \varrho^{12} \cos 10\, \theta - 0.2\, \varrho^6 \cos 4\, \theta\}\, D_2^{-2}\right] \quad (50.2),$$

$$v = (R - L) \left[\frac{1}{2} \eta\, \beta_2\, c^2\, \varrho^2 \times \{0.01\, \varrho^{12} \sin 10\, \theta - 0.2\, \varrho^6 \sin 4\, \theta - \sin 2\, \theta\}\, D_2^{-2} + \tau\, c\, \varrho\, \{\cos \theta - 0.1\, \varrho^6 \cos 5\, \theta\}\, D_2^{-1}\right] \quad . . (50.3),$$

$$D_2 = (1 - 0.2\, \varrho^6 \cos 6\, \theta + 0.01\, \varrho^{12}) \quad (50.4),$$

$$\Omega = (R - L)\, [\tau - \eta\, \beta_2\, c\, \varrho\, (\sin \theta + 0.1\, \varrho^6 \sin 5\, \theta)\, D_2^{-1}] \quad (50.5),$$

$$\frac{\partial \Omega}{\partial R} = \tau - \eta\, \beta_2\, c\, \varrho\, (\sin \theta + 0.1\, \varrho^6 \sin 5\, \theta)\, D_2^{-1} \quad (50.6),$$

$$\beta_2 = W \,/\, 0.2915\, E\, \pi\, c^4 \quad \ldots . (50.7),$$

and

$$M\, \mu^{-1} = 0.5207\, \pi\, \tau\, c^4 \quad \ldots (50.8).$$

If the cross section of the beam is a circle we put $m = 0$, and we easily find the following results.

$$\psi(\zeta)\, w'(\zeta) = \frac{c^3\, \beta_0}{4} (3 + 2\, \eta) \quad . . (51.1),$$

$$\widehat{zz} = 4\, W\, (R - L)\, x \,/\, \pi\, c^4 \quad \ldots (51.2),$$

$$u = -\frac{1}{6} \beta_0\, (R - L)^3 - (R - L) \left[\frac{1}{2} \eta\, \beta_0\, (x^2 - y^2)\right] \quad (51.3),$$

$$v = -(R - L)\, \beta_0\, x\, y \quad \ldots . (51.4),$$

$$\Omega = -\eta\, (R - L)\, \beta_0\, y \quad \ldots . (51.5),$$

$$\frac{\partial \Omega}{\partial R} = -\eta\, \beta_0\, y \quad \ldots\ldots (51.6),$$

$$M\, \mu^{-1} = 0.5\, \pi\, \tau\, c^4 \quad \ldots\ldots (51.7),$$

$$\beta_0 = 2\, W \,/\, \pi\, \mu\, (1 + \eta)\, c^4\,, \quad (\tau = 0\,,\ P = W) \quad (51.8),$$

$$\widehat{xz}\, \mu^{-1} = \frac{\beta_0}{4} [c^2 (3 + 2\, \eta) - 3\, x^2 - y^2 - 2\, \eta\, (x^2 - y^2)] \quad (51.9),$$

$$\widehat{yz}\, \mu^{-1} = -\frac{(1 + 2\, \eta)}{2} \beta_0\, x\, y \quad \ldots (60).$$

These results agree with the results given by L. M. MILNE-THOMSON.

References

[1] L. M. MILNE-THOMSON, Trans. Amer. Math. Soc. 90 (1959), pp. 143—160.
[2] W. A. BASSALI, Torsion of Elastic Cylinders with regular curvilinear cross-section, J. Math. Phys. xxxv (1960), p. 111.

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