S. K. LAKSHMANA RAO

Stagnation Point — Line Vortex Flow of Non-Linear Viscous Liquids

Introduction

In his classical work on vortex motion H. HELMHOLTZ dealt with the irrotational motion of inviscid fluids and established three basic theorems. Lord KELVIN added another basic theorem concerning the constancy of circulation around a vortex core. However, to be able to explain the decay of vortices the viscosity of the fluid must be taken into account. The solution of C. W. OSEEN viz.,

$$u = 0\,, \qquad v = \frac{\Omega_0}{r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right], \qquad w = 0,$$

where u, v, w denote the velocity components of a viscous liquid of kinematic viscosity coefficient ν along the directions r, θ, z of a cylindrical system of coordinates, was almost the first to indicate the diffusion of vorticity explicitly. Subsequently there have been several other exact solutions which also indicate clearly the decay of vorticity in a viscous liquid. In his paper on the dying vortex, ALBERT DE NEUFVILLE [1] has given solutions of the NAVIER-STOKES equations which include the above solution as a special case and serve to explain the diffusion of vorticity and the decay of the motion. The solutions given by N. ROTT [2] are in effect the result of combining a stagnation point flow and the flow due to a vortex core.

In the present paper we seek to extend these results to non-linear viscous liquids of the REINER-RIVLIN type characterized by the constitution relation

$$t_{ij} = -p\,\delta_{ij} + 2\mu\, d_{ij} + 2\mu_c\, d_{ik}\, d_{kj}\,,$$

assuming the coefficients of viscosity μ and cross viscosity μ_c to be constant.

§ 1. Line vortex in a non-Newtonian liquid.

If the velocity components in the r, θ, z directions are

$$(1.1)\qquad u = 0\,, \qquad v = V(r, t)\,, \qquad w = 0\,,$$

the equations of motion of REINER-RIVLIN liquids are

$$(1.2)\qquad \begin{cases} -\varrho\dfrac{V^2}{r} = -\dfrac{\partial p}{\partial r} + \dfrac{1}{2}\mu_c\dfrac{\partial}{\partial r}\left[\left(\dfrac{\partial V}{\partial r} - \dfrac{V}{r}\right)^2\right], \\ \varrho\dfrac{\partial V}{\partial t} = \mu\left(\dfrac{\partial}{\partial r} + \dfrac{2}{r}\right)\left(\dfrac{\partial V}{\partial r} - \dfrac{V}{r}\right), \\ 0 = -\dfrac{\partial p}{\partial z}\,. \end{cases}$$

On putting $V = \Omega/r$ the middle equation becomes

$$(1.3)\qquad \frac{\partial\Omega}{\partial t} = \nu\left(\frac{\partial^2\Omega}{\partial r^2} - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right).$$

Apart from the obvious solution $\Omega = \alpha + \beta r^2$, this equation has the solution

$$(1.4)\qquad \Omega = \Omega_0\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right],$$

which is the same as mentioned earlier. It is also possible to obtain a class of solutions for the vorticity equation by introducing the new independent variables

$$(1.5)\qquad x = \frac{r^2}{4\nu t}\,, \qquad t = t\,.$$

The equation for vorticity

$$\zeta = \frac{1}{r}\frac{\partial\Omega}{\partial r}$$

viz.,

$$(1.6)\qquad \frac{\partial\zeta}{\partial t} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta}{\partial r}\right)$$

then takes on the form

$$(1.7)\qquad t\frac{\partial\zeta}{\partial t} = x\frac{\partial^2\zeta}{\partial x^2} + (x+1)\frac{\partial\zeta}{\partial x}$$

and has the solutions

$$(1.8)\qquad \zeta = (\text{constant})\, t^{-(n+1)}\, e^{-x}\, L_n^0(x) \qquad (n = 0, 1, 2, \ldots),$$

where

$$(1.9)\qquad L_n^\alpha(x) = (n!)^{-1}\, x^{-\alpha}\, e^x \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right)$$

denotes the LAGUERRE polynomial of degree n. The velocity component is found to be

$$(1.10)\qquad V = \frac{\Omega_0}{r}\, t^{-n}\left[\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x} x^n\right)\right]_{x = \frac{r^2}{4\nu t}} \qquad (n = 1, 2, 3, \ldots),$$

and when $n = 0$, we recover the solution (1.4). The above analysis shows that the velocity distribution (1.1) with $v = V(r, t)$ as defined in (1.4) and (1.10) which is known to satisfy the NAVIER-STOKES's equations also constitutes a solution for the equations of non-Newtonian viscous liquids. The effect of the cross viscosity is absent in the velocity distribution and is shown up only in the pressure distribution. The pressure is given by

$$(1.11)\qquad \begin{aligned} p &= p_0 + \varrho\int\frac{V^2}{r}\,dr + \frac{1}{2}\mu_c\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right)^2 \\ &= p_0 + p_{II} + p_{III} \end{aligned}$$

and is made up of the components p_{II} due to the circulatory motion and p_{III} contributed by the cross viscosity.

The solutions (1.8), (1.10) correspond to vortices in a non-Newtonian liquid, having a core along the axis ($r = 0$) and a number of layers of alternating directions of rotation. The circulation is zero at infinity for all the solutions except when $t \to \infty$. The axis is a singular line and the velocity and vorticity tend to zero for all finite values of r.

§ 2. Stagnation point aligned with line vortex.

If the velocity components are chosen in the form

$$(2.1)\qquad u = -A\,r\,, \qquad v = V(r, t)\,, \qquad w = 2\,(A\,z + C)$$

the equations of motion of non-Newtonian fluids are

$$(2.2)\qquad \begin{cases} \varrho\left(A^2 r - \dfrac{V^2}{r}\right) = -\dfrac{\partial p}{\partial r} + \dfrac{1}{2}\mu_c\dfrac{\partial}{\partial r}\left[\left(\dfrac{\partial V}{\partial r} - \dfrac{V}{r}\right)^2\right], \\ \varrho\left(\dfrac{\partial V}{\partial t} - A\,r\dfrac{\partial V}{\partial r} - A\,V\right) \\ \qquad = (\mu - 2\,A\,\mu_c)\left(\dfrac{\partial}{\partial r} + \dfrac{2}{r}\right)\left(\dfrac{\partial V}{\partial r} - \dfrac{V}{r}\right), \\ \varrho\,(4\,A^2 z) = -\dfrac{\partial p}{\partial z}\,. \end{cases}$$

On putting $v = \Omega/r$ and $k = (\mu - 2\,A\,\mu_c)/\varrho = \nu - 2A\,\nu_c$ we have from the middle equation in (2.2)

$$(2.3)\qquad \frac{\partial\Omega}{\partial t} - A\,r\frac{\partial\Omega}{\partial r} = k\left(\frac{\partial^2\Omega}{\partial r^2} - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right)$$

and this has be the steady state solution

$$(2.4)\qquad \Omega = \Omega_0\left[1 - \exp\left(-\frac{A\,r^2}{2\,k}\right)\right].$$

Comparing this with the unsteady solution (1.5) in the case of a line vortex without the alignment of stagnation point ($A = 0$), we see that the present solution can be regarded as that in (1.5) frozen at time $t = (\nu - 2 A \nu_c)/2 A \nu$. We may reduce the equation (2.3) to the form

$$k\left(\frac{d^2\Omega}{d\sigma^2} - \frac{1}{\sigma}\frac{d\Omega}{d\sigma}\right) + A\lambda\sigma\frac{d\Omega}{d\sigma} = 0 \tag{2.5}$$

by regarding Ω as function of $\sigma = (\lambda + c_1 e^{-2At})^{-1/2}$ where λ and c_1 are arbitrary constants. The equation (2.5) is exactly like the steady state form of (2.3) and so, analogous to (2.4) we have the following unsteady solution

$$\Omega = \Omega_0\left\{1 - \exp\left[-\frac{A r^2}{2k(1 + c_2 e^{-2At})}\right]\right\}. \tag{2.6}$$

We can get here again a class of solutions analogous to (1.10) in the case of no stagnation point by integrating the equation for vorticity

$$\frac{\partial\zeta}{\partial t} - A\left(r\frac{\partial\zeta}{\partial r} + 2\zeta\right) = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta}{\partial r}\right) \tag{2.7}$$

after changing to the new independent variables

$$x = \frac{A r^2}{2k(1 + c_2 e^{-2At})}, \quad T = 1 + c_2 e^{-2At}, \tag{2.8}$$

The equation (2.8) then takes the form

$$(1 - T)\frac{\partial}{\partial T}(T\zeta) - \frac{\partial}{\partial x}(x\zeta) = \frac{\partial}{\partial x}\left(x\frac{\partial\zeta}{\partial x}\right) \tag{2.9}$$

and has the solutions

$$\zeta = (\text{constant})\,(T - 1)^n\, T^{-(n+1)}\, e^{-x} L_n^0(x) \quad (n = 0, 1, 2, \ldots), \tag{2.10}$$

The velocity component v is then found to be

$$\frac{\Omega_0}{r}(e^{2At} + c_2)^{-n}\left[\frac{d^{n-1}}{dx^{n-1}}(e^{-x} x^n)\right]_{x = \frac{A r^2}{2kT}} \quad (n = 1, 2, 3, \ldots), \tag{2.11}$$

and when $n = 0$, we recover the solution (2.6).

The radial velocity u is directed towards the axis and the velocity gradient A is positive. The validity of the solutions (2.4) (2.6), (2.11) is conditioned by the fact that the dissipation function

(2.12)

$$\Phi = 12 A^2(\mu + A\mu_c) + (\mu - 3A\mu_c)\frac{1}{r^2}\left(\frac{d\Omega}{dr} - \frac{2}{r}\Omega\right)^2$$

is positive definite and it is possible to choose $\mu > 0$, $A > 0$ such that $\Phi \geqq 0$ and $k = (\mu - 2A\mu_c)/\varrho > 0$.

The unsteady solution (2.6) tends to the steady state solution (2.4) as time increases to infinity while the solutions in (2.11) corresponding to $n = 1, 2, \ldots$ decay in course of time.

The solutions

$$u = -A r,$$

$$v = \frac{\Omega_0}{r}(e^{2At} + c_2)^{-n}\left[\frac{d^{n-1}}{dx^{n-1}}(e^{-x} x^n)\right]_{x = \frac{A r^2}{2kT}}, \tag{2.13}$$

$$w = 2(A z + C)$$

correspond to vortices with a core along the axis ($r = 0$) and a stagnation point on the axis. While the vortices tend to decay, the on-rushing stagnation point flow carries new circulation from infinity towards the axis. Solutions of this type in Newtonian viscous flow have already been noticed [3] and it is significant that both the viscosity and cross viscosity coefficients enter the expression for the velocity field if the flow is due to a line vortex aligned with a stagnation point. The zeros and the relative extrema of the expressions for the velocity about the vortex core occur at fixed distances from the axis of the core and the speed of propagation of such characteristic values is found from

$$dx = d\left[\frac{A r^2}{2k(1 + c_2 e^{-2At})}\right] = 0,$$

showing that the speed is

$$\frac{dr}{dt} = -\frac{A c_2 r e^{-2At}}{1 + c_2 e^{-2At}} = \sqrt{\frac{2Akx}{T}}(1 - T) \tag{2.14}$$

and that it depends on both the coefficients of viscosity and cross viscosity besides the flow parameter A. In the case of no stagnation point the speed is found from

$$dx = d\left(\frac{r^2}{4\nu t}\right) = 0$$

to have the value

$$\frac{dr}{dt} = \sqrt{\frac{\nu x}{t}}, \tag{2.15}$$

which depends on the viscosity coefficient only.

In view of the carriage of circulation towards the axis by the stagnation point flow and the decay of the vortex, in equilibrium there is a viscous radius r^* given by

$$r^* = \sqrt{\frac{2k}{A}} = \sqrt{\frac{2(\mu - 2A\mu_c)}{A}}. \tag{2.16}$$

For $r \gg r^*$ the steady solution (2.4) leads to the potential flow $\Omega = \Omega_0$. The viscous effect is restricted only to a cylinder of radius of the order r^*.

The pressure sustaining the flow (2.13) is found from the equations (2.2) to be

$$p = p_0 - \frac{1}{2}\varrho A^2(r^2 + 4z^2) + \varrho\int\frac{V^2}{r}dr + \frac{1}{2}\mu_c\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right)^2 = p_0 + p_{\mathrm{I}} + p_{\mathrm{II}} + p_{\mathrm{III}} \tag{2.17}$$

and the component p_{I} gives the contribution on account of the stagnation point flow, while p_{II} and p_{III} are due to the circulatory motion and the explicit contribution from cross viscosity. The pressure due to the viscous vortex is

$$p_{\mathrm{II}} = \varrho\int_\infty^r \frac{V^2}{r}dr$$

and for the steady stagnation point-line vortex flow (2.4) this has the value

$$p_{\mathrm{II}} = -\varrho\Omega_0^2\int_r^\infty \frac{1}{r^3}\left[1 - \exp\left(-\frac{A r^2}{2k}\right)\right]^2 dr. \tag{2.18}$$

On putting $A r^2/2k = x$ we have

$$p_{\mathrm{II}} = -\frac{\varrho\Omega_0^2 A}{4k}\int_x^\infty\left(\frac{1 - e^{-x}}{x}\right)^2 dx \tag{2.19}$$

$$= -\frac{\varrho\Omega_0^2 A}{4k}\frac{(1 - e^{-x})^2}{x} - \frac{\varrho\Omega_0^2 A}{2k}\int_x^\infty\frac{e^{-x} - e^{-2x}}{x}dx$$

$$= -\frac{1}{2}\varrho v^2 - \frac{\varrho\Omega_0^2 A}{2k}\int_x^\infty[(e^{-x} - e^{-2x})/x]\,dx.$$

The last integral above measures the change in stagnation pressure p_{II} and depends on the inward flow gradient A and both the coefficients μ and μ_c.

The total pressure difference from infinity to the centre of the core due to the vortex is

$$(2.18') \quad p_{\mathrm{II}}^{s}(x=0) = -\frac{\varrho\,\Omega_0^2 A}{2k}\int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx = \frac{\varrho\,\Omega_0^2 A}{2k}\log 2\,.$$

In the case of the unsteady solution (2.6) the circulation is restricted to a domain outside a cylinder of radius $R(t)$ given by

$$(2.20) \quad (R(t))^2 = \frac{2k}{A}\,(1 + c_2\,e^{-2At})\,.$$

For small values of t the circulation is brought in, in an almost inviscid way until the core is reached and the steady solution (2.4) is approached for large t.

§ 3.

If the inflow gradient is a positive valued function $f(t)$ we can obtain the solution for the line vortex-stagnation point alignment in very much the same way as in § 2. We find that the solution analogous to (2.6) is given by

$$(3.1) \quad \begin{cases} u = -f(t)\,r\,, \\ v = \dfrac{\Omega_0}{r}\,[1 - \exp[-g(t)\,r^2]]\,, \\ w = 2\,f(t)\,z \end{cases}$$

with

(3.2)

$$g(t) = \frac{\varrho\exp(\int 2 f(t)\,dt)}{4\int[\mu - 2\mu_c f(t)]\exp(\int 2f(t)\,dt)\,dt + \text{constant}}\,,$$

and the pressure distribution is

$$(3.3) \quad p = p_0 - \frac{1}{2}\varrho\,[\{(f(t))^2 - f'(t)\}\,r^2 + 4(f(t))^2 z^2] + \varrho\int\frac{V^2}{r}\,dr + \frac{1}{2}\mu_c\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right)^2,$$

and the solution is valid if

$$\mu + \mu_c f(t) \geqq 0\,, \qquad \mu - 3\mu_c f(t) \geqq 0\,,$$
$$\mu - 2\mu_c f(t) \geqq 0\,.$$

References

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Verfasser: S. K. LAKSHMANA RAO, Dept. of Math., Regional Engineering College, Warangal (A. P.), India.