Int. J. Engng Sci. Vol. 21, No. 8, pp. 973-987, 1983
Printed in Great Britain.

0020-7225/83/080973-15$03.00/0

ROTARY OSCILLATIONS OF A SPHEROID IN AN INCOMPRESSIBLE MICROPOLAR FLUID

S. K. LAKSHMANA RAO and T. K. V. IYENGAR
Department of Mathematics, Regional Engineering College, Warangal-506004, India

Abstract—The paper discusses the flow generated by rotary oscillations of a spheroid (prolate and oblate) in incompressible micropolar fluid. The velocity and microrotation components are determined explicitly in terms of spheroidal wave functions and are expressed in infinite series form. The couple on the oscillating spheroid is evaluated and numerical studies are undertaken to examine the effects of the geometric parameter and material constant parameters of the fluid.

1. INTRODUCTION

IN THIS paper we examine the rotary oscillations of a spheroid in an incompressible micropolar fluid[1]. The field equations of micropolar fluids are presentable in terms of the velocity vector $\mathbf{q}$ and the microrotation vector $\boldsymbol{\nu}$ and the theory provides for six material constants. The spheroid is in rotary harmonic oscillation about its axis of symmetry and the amplitude of oscillation is assumed small so that the second order and bilinear terms in the equations describing the flow are omitted. Under the above Stokesian approximation of the flow equations, we obtain analytical expressions for the velocity, microrotation, surface and couple stress components and seek the evaluation for the couple on the spheroid and make a numerical study of its variation with respect to the geometric and physical parameters.

2. BASIC EQUATIONS

The motion of incompressible micropolar fluids is governed by the equations[1]

$$\operatorname{div}\mathbf{q} = 0, \tag{2.1}$$

$$\rho\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \rho\mathbf{f} - \operatorname{grad} p + k\operatorname{curl}\boldsymbol{\nu} - (\mu + k)\operatorname{curl}\operatorname{curl}\mathbf{q} + (\lambda_1 + 2\mu + k)\operatorname{grad}(\operatorname{div}\mathbf{q}), \tag{2.2}$$

$$\rho j\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}t} = \rho\mathbf{l} - 2k\boldsymbol{\nu} + k\operatorname{curl}\mathbf{q} - \gamma\operatorname{curl}\operatorname{curl}\boldsymbol{\nu} + (\alpha + \beta + \gamma)\operatorname{grad}(\operatorname{div}\boldsymbol{\nu}). \tag{2.3}$$

In the above, the scalar quantities ρ and j denote respectively the density and gyration parameters of the fluid and are constant. The vectors $\mathbf{q}$, $\boldsymbol{\nu}$, $\mathbf{f}$, $\mathbf{l}$ are the velocity, microrotation, body force per unit mass and body couple per unit mass. The constants $\{\lambda, \mu, k\}$ and $\{\alpha, \beta, \gamma\}$ are the viscosity and gyroviscosity coefficients and these conform to the inequalities

$$\begin{gathered} k \geqslant 0; \quad 2\mu + k \geqslant 0; \quad 3\lambda_1 + 2\mu + k \geqslant 0; \\ \gamma \geqslant 0; \quad |\beta| \leqslant \gamma; \quad 3\alpha + \beta + \gamma \geqslant 0. \end{gathered} \tag{2.4}$$

The stress tensor t_{ij} and the couple stress tensor m_{ij} are given by [1]

$$t_{ij} = (-p + \lambda_1 \operatorname{div}\mathbf{q})\,\delta_{ij} + (2\mu + k)d_{ij} + k\epsilon_{ijm}(\omega_m - \nu_m), \tag{2.5}$$

$$m_{ij} = \alpha(\operatorname{div}\boldsymbol{\nu})\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i} \tag{2.6}$$

in which the symbols p, δ_{ij}, d_{ij}, ω_m, ν_m and $\nu_{i,j}$ have their usual meanings.

Let (ξ, η, ϕ) be an axially symmetric system of coordinates with base vectors $(\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\phi)$ and scale factors (h_1, h_2, h_3). The spheroid oscillates harmonically about its axis of symmetry with angular speed $\Omega\, e^{i\omega t}$. The fluid flow generated by this oscillation is axially symmetric and the flow field vectors can be chosen in the form

$$\mathbf{q} = \mathbf{Q}\, e^{iwt} = V(\xi, \eta)\, e^{iwt}\, \mathbf{e}_\phi \tag{2.7}$$

and

$$\boldsymbol{\nu} = [A(\xi, \eta)\, \mathbf{e}_\xi + B(\xi, \eta)\mathbf{e}_\eta]\, e^{iwt}. \tag{2.8}$$

Under the assumptions of Stokesian flow, the field eqns (2.2), (2.3) simplify to

$$\rho \frac{\partial \mathbf{q}}{\partial t} = -\text{grad}\, p + k\, \text{curl}\, \boldsymbol{\nu} - (\mu + k)\, \text{curl curl}\, \mathbf{q}, \tag{2.9}$$

$$\rho j \frac{\partial \boldsymbol{\nu}}{\partial t} = -2k\boldsymbol{\nu} + k\, \text{curl}\, \mathbf{q} - \gamma\, \text{curl curl}\, \boldsymbol{\nu} + (\alpha + \beta + \gamma)\, \text{grad}\, (\text{div}\, \boldsymbol{\nu}). \tag{2.10}$$

Defining the functions $f(\xi, \eta)$ and $g(\xi, \eta)$ and the operator E^2 in the form

$$\text{div}\, \boldsymbol{\nu} = f(\xi, \eta)\, e^{iwt}, \tag{2.11}$$

$$\text{curl}\, \boldsymbol{\nu} = g(\xi, \eta)\, e^{iwt}\, \mathbf{e}_\phi \tag{2.12}$$

$$E^2 = \frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial \xi}\left(\frac{h_2}{h_1 h_3}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial \eta}\right)\right] \tag{2.13}$$

we see from eqns (2.7)–(2.13) that

$$i\rho w V(\xi, \eta) = \text{k}g + ((\mu + k)/h_3)E^2(h_3 V), \tag{2.14}$$

$$\begin{aligned}(2k + i\rho j\omega)(A\mathbf{e}_\xi + B\mathbf{e}_\eta) &= (1/h_2 h_3)(\partial/\partial\eta)[h_3(kV - \gamma g)]\mathbf{e}_\xi - (1/h_1 h_3)(\partial/\partial\xi)\, h_3(kV - \gamma g)]\mathbf{e}_\eta \\ &+ (\alpha + \beta + \gamma)\left[\frac{1}{h_1}\frac{\partial f}{\partial \xi}\mathbf{e}_\xi + \frac{1}{h_2}\frac{\partial f}{\partial \eta}\mathbf{e}_\eta\right]\end{aligned} \tag{2.15}$$

$$\frac{\partial p}{\partial \xi} = \frac{\partial p}{\partial \eta} = 0. \tag{2.16}$$

From (2.15) it follows that

$$\left(\nabla^2 - \frac{p^2}{c^2}\right) f = 0 \tag{2.17}$$

where

$$(p^2/c^2) = (2k + i\rho j\omega)/(\alpha + \beta + \gamma) \tag{2.18}$$

and

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \xi}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial \eta}\right)\right]. \tag{2.19}$$

From (2.15) we can also deduce that

$$\left[i\rho j\omega + \frac{k(2\mu + k)}{\mu + k}\right] g = -\frac{i\rho\omega k}{\mu + k} V + \frac{\gamma}{h_3} E^2(h_3 g). \tag{2.20}$$

Eliminating g between (2.14) and (2.20), we obtain the following differential equation for the determination of $V(\xi, \eta)$

$$\gamma(\mu + k)E^4(h_3 V) - [k(2\mu + k) + i\rho\omega(\gamma + j(\mu + k))]E^2(h_3 V) + i\rho\omega(2k + i\rho j\omega)(h_3 V) = 0. \tag{2.21}$$

On finding $V(\xi, \eta)$ from the above equation, we can find $g(\xi, \eta)$ from (2.14). The function $f(\xi, \eta)$ can be determined from (2.17). The functions $A(\xi, \eta)$, $B(\xi, \eta)$ can then be determined from the following two equations

$$(2k + i\rho j\omega)A = \frac{1}{h_2 h_3}\frac{\partial}{\partial \eta}\left[\frac{\gamma(\mu + k)}{k}E^2(h_3 V) + \left(k - \frac{i\rho\omega\gamma}{k}\right)(h_3 V)\right] + \frac{(\alpha + \beta + \gamma)}{h_1}\frac{\partial f}{\partial \xi}, \tag{2.22}$$

$$(2k + i\rho j\omega)B = -\frac{1}{h_1 h_3}\frac{\partial}{\partial \xi}\left[\frac{\gamma(\mu + k)}{k}E^2(h_3 V) + \left(k - \frac{i\rho\omega\gamma}{k}\right)(h_3 V)\right] + \frac{(\alpha + \beta + \gamma)}{h_2}\frac{\partial f}{\partial \eta}. \tag{2.23}$$

The eqn (2.21) can be expressed in the form

$$(E^2 - \alpha^2)(E^2 - \beta^2)(h_3 V) = 0 \tag{2.24}$$

where the constants α^2, β^2 are obtained from the two equations

$$\alpha^2 + \beta^2 = \frac{k(2\mu + k) + i\rho\omega(\gamma + j\mu + jk)}{\gamma(\mu + k)}, \tag{2.25}$$

$$\alpha^2\beta^2 = i\rho\omega(2k + i\rho j\omega)/(\gamma(\mu + k)).$$

When $\alpha^2 \neq \beta^2$, the solution V can be obtained by the superposition† of the solutions V_1, V_2 where

$$(E^2 - \alpha^2)(h_3 V_1) = 0, \tag{2.26}$$

$$(E^2 - \beta^2)(h_3 V_2) = 0. \tag{2.27}$$

The rotary oscillation of the body and the flow arising from it are presumed to have negligible effect at a great distance from the body and we may therefore presume that each of the quantities V, A and B tends to zero at infinity. On the boundary of the spheroid we assume the hyperstick condition of adherence [2]. This means that, *on the boundary*

$$\mathbf{q} = (\Omega h_3 e^{i\omega t})\mathbf{e}_\phi, \tag{2.28}$$

$$\boldsymbol{\nu} = \frac{1}{2}\operatorname{curl}(\Omega h_3 e^{i\omega t}\mathbf{e}_\phi). \tag{2.29}$$

3. PROLATE SPHEROID

Let a prolate spheroid (focal distance $= 2c$) perform rotary oscillations about its axis of symmetry with angular speed $\Omega e^{i\omega t}$. If (ξ, η, ϕ) are prolate spheroidal coordinates, the scale factors h_1, h_2, h_3 and the operators E^2 and ∇^2 are given by

$$h_1 = h_2 = c\sqrt{(s^2 - t^2)},\ h_3 = c\sqrt{((s^2 - 1)(1 - t^2))}, \tag{3.1}$$

†We thank the referee for drawing our attention to the possibility of "resonance ($\alpha^2 = \beta^2$)" in which case the solutions of (2.26) and (2.27) would no longer be independent. As pointed out by the referee, this case arises when $(\gamma/j) = (2\mu + k)(\mu + k)/(2\mu + 3k)$ and $\rho\omega = (2\mu + k).(2\mu + 3k)/(2(\mu + k)j)$ and this situation can occur in rotary as well as rectilinear oscillation problems. We then require solutions of $(E^2 - \alpha^2)(h_3 V) = 0$ and $(E^2 - \alpha^2)^2(h_3 V) = 0$ to build the solution V in this case of resonance. The case of resonance covering both the rectilinear and rotary oscillations will be treated in a separate paper.

$$E^2 = \frac{1}{c^2(s^2 - t^2)}\left[(s^2-1)\frac{\partial^2}{\partial s^2} + (1-t^2)\frac{\partial^2}{\partial t^2}\right] \tag{3.2}$$

$$\nabla^2 = \frac{1}{c^2(s^2 - t^2)}\left[(s^2-1)\frac{\partial^2}{\partial s^2} + (1-t^2)\frac{\partial^2}{\partial t^2} + 2s\frac{\partial}{\partial s} - 2t\frac{\partial}{\partial t}\right] \tag{3.3}$$

where $s = \cosh \xi$ and $t = \cos \eta$. The prolate spheroid in oscillation is given by $\xi = \xi_0$ (i.e. $s = s_0$).

The solution $f(\cosh \xi, \cos \eta) = f(s, t)$ satisfying (2.17) can be expressed in terms of the prolate spheroidal wave functions[3]. To ensure the regularity of $f(s, t)$ at infinity and everywhere on the axis of symmetry in the region $s > s_0$, we select the wave functions $R_{0n}^{(3)}(ip, s)$, $S_{0n}^{(1)}(ip, t)$[3] and choose $f(s, t)$ in the form

$$f(s,t) = \sum A_n R_{0n}^{(3)}(ip, s)\, S_{0n}^{(1)}(ip, t) \tag{3.4}$$

where $\{A_n\}$ is an infinite set of unknown constants. The regularity requirement at infinity is satisfied if we choose p from (2.18) such that its real part is positive. The prolate spheroidal functions $R_{0n}^{(3)}(ip, s)$ and $S_{0n}^{(1)}(ip, t)$ have the expansions

$$R_{0n}^{(3)}(ip, s) = \left[i^{n+2} \sum_{r=0,1}^{\infty\prime} d_r^{0n}(ip)\right]^{-1} \sum_{r=0,1}^{\infty\prime} d_r^{0n}(ip)\, K_{r+(1/2)}(ps). \tag{3.5}$$

$$S_{0n}^{(1)}(ip, t) = \sum_{r=0,1}^{\infty\prime} d_r^{0n}(ip)\, P_r(t) \tag{3.6}$$

where $K_{r+(1/2)}(ps)$, $P_r(t)$ denote the modified Bessel function of the second kind and Legrendre polynomial respectively.

The solution V_1 of eqn (2.26) can be expressed in terms of the spheroidal wave functions $R_{1n}^{(3)}(i\alpha c, s)$ and $S_{1n}^{(1)}(i\alpha c, t)$[3]. Likewise the solution V_2 of (2.27) is expressible in terms of the functions $R_{1n}^{(3)}(i\beta c, s)$, $S_{1n}^{(1)}(i\beta c, t)$. In the above functions, the values of α, β are to be such that the regularity of V_1 and V_2 at infinity is ensured and this is attained by selecting the roots α, β from (2.25) such that each of them has a positive real part. The restriction of the radial function to the type $R_{1n}^{(3)}$ and the angular function to the type $S_{1n}^{(1)}$ is also motivated by the requirements of regularity in the region $s > s_0$. The solutions V_1 and V_2 take the form

$$V_1 = \sum B_n R_{1n}^{(3)}(i\alpha c, s)\, S_{1n}^{(1)}(i\alpha c, t) \tag{3.7}$$

$$V_2 = \sum C_n R_{1n}^{(3)}(i\beta c, s)\, S_{1n}^{(1)}(i\beta c, t) \tag{3.8}$$

where $R_{1n}^{(3)}(i\alpha c, s)$, $S_{1n}^{(1)}(i\alpha c, t)$ are given by

$$R_{1n}^{(3)}(i\alpha c, s) = \left[i^{n+2} \sum_{r=0,1}^{\infty\prime} (r+1)(r+2)\, d_r^{1n}(i\alpha c)\right]^{-1} \left(\frac{2(s^2-1)}{\pi \alpha c s^3}\right)^{1/2} \times \sum_{r=0,1}^{\infty\prime} (r+1)(r+2)\, d_r^{1n}(i\alpha c)\, K_{r+(3/2)}(\alpha c s), \tag{3.9}$$

$$S_{1n}^{(1)}(i\alpha c, t) = \sum_{r=0,1}^{\infty\prime} d_r^{1n}(i\alpha c)\, P_{(r+1)}^{(1)}(t) \tag{3.10}$$

in which $K_{r+(3/2)}(\alpha c s)$ denotes the modified Bessel function of the second kind and $P_{r+1}^{(1)}(t)$ denotes the associated Legendre function of the first kind. The functions $R_{1n}^{(3)}(i\beta c, s)$ and $S_{1n}^{(1)}(i\beta c, t)$ are obtained from (3.9) and (3.10) by changing α to β. The velocity component

$V(s,t)$ is given by

$$V(s,t)=\sum B_nR_{1n}^{(3)}(i\alpha c,s)\,S_{1n}^{(1)}(i\alpha c,t)+\sum C_nR_{1n}^{(3)}(i\beta c,s)\,S_{1n}^{(1)}(i\beta c,t). \tag{3.11}$$

From eqns (2.22), (2.23), (3.4), (3.11) we can obtain the expression for the microrotation components $A(s,t)$ and $B(s,t)$ in explicit form. These are given by

$$\begin{aligned}
&k(2k+ipj\omega)c\surd((s^2-t^2))\,A(s,t)\\
&\quad=k(\alpha+\beta+\gamma)\surd((s^2-1))\sum A_n\left(\frac{\mathrm{d}}{\mathrm{d}s}(R_{0n}^{(3)}(ip,s))\right)S_{0n}^{(1)}(ip,t)\\
&\quad\quad-\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum B_nR_{1n}^{(3)}(i\alpha c,s)\frac{\mathrm{d}}{\mathrm{d}t}[\surd((1-t^2))S_{1n}^{(1)}(i\alpha c,t)]\\
&\quad\quad-\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum C_nR_{1n}^{(3)}(i\beta c,s)\frac{\mathrm{d}}{\mathrm{d}t}[\surd((1-t^2))S_{1n}^{(1)}(i\beta c,t)],
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&k(2k+ipj\omega)c\surd((s^2-t^2))\,B(s,t)\\
&\quad=-k(\alpha+\beta+\gamma)\sum A_nR_{0n}^{(3)}(ip,s)\surd((1-t^2))\frac{\mathrm{d}}{\mathrm{d}t}S_{0n}^{(1)}(ip,t)\\
&\quad\quad-\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum B_n\frac{\mathrm{d}}{\mathrm{d}s}[\surd((s^2-1))R_{1n}^{(3)}(i\alpha c,s)]\,S_{1n}^{(1)}(i\alpha c,t)\\
&\quad\quad-\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum C_n\frac{\mathrm{d}}{\mathrm{d}s}[\surd((s^2-1))R_{1n}^{(3)}(i\beta c,s)]\,S_{1n}^{(1)}(i\beta c,t).
\end{aligned} \tag{3.13}$$

The functions $V(s,t)$, $A(s,t)$ and $B(s,t)$ in the eqns (3.11)–(3.13) involve three infinite sets of unknown constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ and these are to be determined by using the boundary conditions on $s=s_0$.

Boundary conditions

The spheroid $s=s_0$ is in rotary oscillation with angular speed $\Omega\,e^{iwt}$ and the hyperstick condition [2] implies that on $s=s_0$

$$V(s,t)=\Omega c\surd((s^2-1)(1-t^2)), \tag{3.14}$$

$$A(s,t)=\frac{\Omega}{h_2}\frac{\partial h_3}{\partial\eta}=\frac{\Omega\surd(s^2-1)t}{\surd((s^2-t^2))}, \tag{3.15}$$

$$B(s,t)=-\frac{\Omega}{h_1}\frac{\partial h_3}{\partial\xi}=-\frac{\Omega s\surd((1-t^2))}{\surd((s^2-t^2))}. \tag{3.16}$$

The above three equations are valid on the interval $-1\leqslant t\leqslant 1$ and will in general enable us to determine the three sets of constants $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$. The details are shown in Appendix A1.

Calculation of the couple on the body

The calculation of the couple on the body has to reckon with the contributions to it from both the force stress and couple stress tensors. We can see easily that $t_{\xi\phi}$ is the only component of the force stress tensor and $m_{\xi\xi}$, $m_{\xi\eta}$ are the only two components of the couple stress tensor that contribute to the couple on the body. The expressions for these three quantities are obtainable from eqns (2.5) and (2.6) and after the evaluations of the relevant components of the tensors d_{ij}, $\nu_{i,j}$ we see that

$$(t_{\xi\phi}, m_{\xi\xi}, m_{\xi\eta}) = (T_{\xi\phi}, M_{\xi\xi}, M_{\xi\eta})\, e^{i\omega t} \tag{3.17}$$

with

$$\begin{aligned} T_{\xi\phi} &= \frac{1}{2c\surd((s^2-t^2))}\Big\{(2\mu+k)(s^2-1)\frac{\partial}{\partial s}\left(\frac{V}{\surd((s^2-1))}\right) \\ &\quad + \frac{k}{2k+ipj\omega}\left[ipj\omega\frac{\partial}{\partial s}(\surd((s^2-1))V)+2\gamma\frac{\partial}{\partial s}(\surd((s^2-1))g)-2(\alpha+\beta+\gamma)\surd((1-t^2))\frac{\partial f}{\partial t}\right]\Big\} \\ &= \frac{(2\mu+k)(s^2-1)}{2c\surd((s^2-t^2))}\frac{\partial}{\partial s}\left(\frac{V}{\surd((s^2-1))}\right)+k\left\{\frac{1}{2c\surd((s^2-t^2))}\frac{\partial}{\partial s}(\surd((s^2-1))V)+B\right\} \end{aligned} \tag{3.18}$$

$$M_{\xi\xi} = \alpha f + \frac{\beta+\gamma}{c\surd((s^2-t^2))}\left\{\surd(s^2-1))\frac{\partial A}{\partial s}+\frac{t\surd((1-t^2))}{(s^2-t^2)}B\right\} \tag{3.19}$$

$$M_{\xi\eta} = -\beta\left\{\frac{\surd((1-t^2))}{c\surd((s^2-t^2))}\frac{\partial A}{\partial t}+\frac{s\surd((s^2-1))}{c((s^2-t^2))^{3/2}}B\right\}+\gamma\left\{\frac{\surd((s^2-1))}{c\surd((s^2-t^2))}\frac{\partial B}{\partial s}-\frac{t\surd((1-t^2))}{c((s^2-t^2))^{3/2}}A\right\}. \tag{3.20}$$

The contribution of the force stress tensor to the couple on the body is given by $\mathbf{r}\times\mathbf{t}\cdot\mathbf{e}_z$ where $\mathbf{r}$ is the position vector of the point on the body from the centre of the spheroid, $\mathbf{t}$ is the stress vector and $\mathbf{e}_z$ is the unit axial vector. This simplifies to

$$c\surd((s^2-1)(1-t^2))t_{\xi\phi} \tag{3.21}$$

and the couple on the spheroid due to the surface traction equals

$$\begin{aligned} C_I &= 2\pi c^3(s_0^2-1)\int_{-1}^{1}\surd(((s_0^2-1))(1-t^2)))[t_{\xi\phi}]_{s_0}\,dt \\ &= \frac{8\pi}{3}c^2(\mu+k)(s_0^2-1)\Big[\surd((s_0^2-1))\Big\{\sum B_n d_0^{1n}(i\alpha c)\left(\frac{d}{ds}R_{1n}^{(3)}(i\alpha c,s)\right)_{s_0} \\ &\quad + \sum C_n d_0^{1n}(i\beta c)\left(\frac{d}{ds}R_{1n}^{(3)}(i\beta c,s)\right)_{s_0}\Big\}-\Omega c s_0\Big]e^{i\omega t}. \end{aligned} \tag{3.22}$$

The contribution of the couple stress tensor to the couple on the body is $\mathbf{m}\cdot\mathbf{e}_z$ where $\mathbf{m}$ is the couple stress vector and it is seen that the contribution to the couple on the body is

$$\begin{aligned} C_{II} &= 2\pi c^2\surd((s_0^2-1))\int_{-1}^{1}[t\surd((s^2-1))m_{\xi\xi}-s\surd((1-t^2))m_{\xi\eta}]_{s_0}\,dt \\ &= \left[2\pi c^2\surd((s_0^2-1))\int_{-1}^{1}\{(\alpha+\beta+\gamma)\surd(s^2-1)tf(s,t)-\gamma s\surd((1-t^2))g(s,t)\}_{s_0}\,dt\right]e^{i\omega t}. \end{aligned} \tag{3.23}$$

The integrand can be further simplified by replacing the functions $f(s,t)$ and $g(s,t)$ by their actual expressions. The expression for $f(s,t)$ is seen earlier in eqn (3.4) while that for g can be obtained by means of eqns (2.14) and (3.11). We have, in fact

$$g(s,t) = (ip\omega/k)V - \frac{\mu+k}{k}(\alpha^2 V_1+\beta^2 V_2). \tag{3.24}$$

We can see that

$$\begin{aligned} C_{II} &= \frac{4}{3}\pi c^2\surd((s_0^2-1))\Big\{(\alpha+\beta+\gamma)\surd((s_0^2-1))\sum A_n d_1^{0n}(ip)R_{0n}^{(3)}(ip,s_0) \\ &\quad -\frac{2(\mu+k)}{k}(\alpha^2-\beta^2)\gamma s_0\sum C_n d_0^{1n}(i\beta c)R_{1n}^{(3)}(i\beta c,s_0)-\frac{2\Omega c s_0\surd((s_0^2-1))}{k}\gamma(ip\omega-(\mu+k)\alpha^2)\Big\}e^{i\omega t}. \end{aligned} \tag{3.25}$$

The series involving the constants $\{A_n\}$ in the above expression for C_{II} can be eliminated by the use of the eqns (3.13) and (3.16) and this results in the following expression for C_{II} containing only two sets of constants $\{B_n\}$ and $\{C_n\}$:

$$C_{II} = \frac{4\pi}{3k} c^2\sqrt{((s_0^2-1))}\Big[\Omega c s_0 \sqrt{((s_0^2-1))}\Big\{\gamma(\mu+k)\alpha^2 + k^2 - i\rho\omega\gamma + i\rho j\omega k\Big\} + \gamma(\mu+k)(\beta^2-\alpha^2)s_0X_2 - \{\gamma(\mu+k)\alpha^2 + k^2 - i\rho\omega\gamma\}(s_0^2-1)X_3 - \{\gamma(\mu+k)\beta^2 + k^2 - i\rho\omega\gamma\}(s_0^2-1)X_4\Big] e^{i\omega t} \tag{3.26}$$

where

$$\left.\begin{aligned} X_2 &= \sum C_n d_0^{1n}(i\beta c) R_{1n}^{(3)}(i\beta c, s_0) \\ X_3 &= \sum B_n d_0^{1n}(i\alpha c)\left(\frac{d}{ds}R_{1n}^{(3)}(i\alpha c, s)\right)_{s_0} \\ X_4 &= \sum C_n d_0^{1n}(i\beta c)\left(\frac{d}{ds}R_{1n}^{(3)}(i\beta c, s)\right)_{s_0} \end{aligned}\right\} \tag{3.27}$$

From (3.22), (3.26) and (3.27), we can obtain the total couple C on the spheroid in the form

$$= \frac{4\pi}{3k}c^2\sqrt{((s_0^2-1))}[\Omega c s_0\sqrt{((s_0^2-1))}\{-k(2\mu+k) + i\rho j\omega k + \gamma(\mu+k)\alpha^2 - i\rho\omega\gamma\} + \gamma(\mu+k)(\beta^2-\alpha^2)s_0X_2 + (s_0^2-1)\{k(2\mu+k) - \gamma(\mu+k)\alpha^2 + i\rho\omega\gamma\}X_3 + (s_0^2-1)\{k(2\mu+k) - \gamma(\mu+k)\alpha^2 + i\rho\omega\gamma)\}X_4]\, e^{i\omega t}. \tag{3.28}$$

It has not been found possible to express the couple in a finite number of terms.

Numerical work

Defining the nondimensional couple C_{ND} by means of the equation

$$C = \frac{4\pi}{3}(2\mu+k)\Omega c^3 C_{ND}\, e^{i\omega t} \tag{3.29}$$

we see that

$$\begin{aligned} C_{ND} = {} & s_0(s_0^2-1)\left(\frac{2i\theta^2\Theta^2}{\lambda^2\Lambda^2} + \frac{\alpha^2c^2 - i\theta^2}{\lambda^2} - 1\right) \\ & + \frac{(\beta^2-\alpha^2)}{\lambda^2} s_0\sqrt{((s_0^2-1))}(X_2/(\Omega c)) \\ & + (s_0^2-1)^{3/2}\left[\ 1 - \frac{\alpha^2c^2 - i\theta^2}{2}\right)X_3/(\Omega c) \\ & + \left(1 - \frac{\beta^2c^2 - i\theta^2}{\lambda^2}\right)X_4/(\Omega c)\Big] \end{aligned} \tag{3.30}$$

wherein the parameters θ^2, λ^2, Λ^2, Θ^2 are given by

$$\theta^2 = \frac{\rho\omega c^2}{\mu+k}, \qquad \lambda^2 = \frac{k(2\mu+k)}{\gamma(\mu+k)}c^2$$
$$\Lambda^2 = \frac{2(\mu+k)}{k}, \qquad \Theta^2 = \frac{j(\mu+k)}{\gamma} \tag{3.31}$$

The couple parameters K, K' are introduced by means of the relation

$$C_{ND} = \theta^2 s_0(s_0^2-1)(-K' - iK). \tag{3.32}$$

The quantities K and K' are numerically evaluated for a series of parameter values s_0, θ^2, λ^2, Λ^2, Θ^2 by evaluating the constants $\{C_m\}$ from the system of eqns (A17) and the constants $\{B_m\}$ from the system of eqns (A20) in Appendix A by truncating each system to a 5 by 5 set. This order of truncation is motivated by the fact that the coefficients needed for the evaluation of the constants $d_r^{mn}(i\alpha c)$, $d_r^{mn}(i\beta c)$, $d_r^{mn}(ip)$ are available only to a limited extent in the published literature[3]. The variation of K and K' is graphically presented for the polar as well as the nonpolar case[5] in Fig. 1–12.

4. OBLATE SPHEROID

Let an oblate spheroid (focal distance = $2c$) perform rotary oscillations about its axis of symmetry with angular speed $\Omega\, e^{i\omega t}$. If (ξ, η, ϕ) are oblate spheroidal coordinates, the scale factors (h_1, h_2, h_3) and operators E^2, ∇^2 are given by

$$h_1 = h_2 = c\sqrt{((\tau^2 + t^2))}, \qquad h_3 = c\sqrt{((\tau^2+1)(1-t^2))} \tag{4.1}$$

$$E^2 = \frac{1}{c^2(\tau^2+t^2)}\left[(\tau^2+1)\frac{\partial^2}{\partial\tau^2} + (1-t^2)\frac{\partial^2}{\partial t^2}\right] \tag{4.2}$$

$$\nabla^2 = \frac{1}{c^2(\tau^2+t^2)}\left[(\tau^2+1)\frac{\partial^2}{\partial\tau^2} + (1-t^2)\frac{\partial^2}{\partial t^2} + 2\tau\frac{\partial}{\partial} - 2t\frac{\partial}{\partial t}\right] \tag{4.3}$$

where $\tau = \sinh \xi$ and $t = \cos \eta$. The oblate spheroid in oscillation is given by $\xi = \xi_0$ (i.e. $\tau = \tau_0$).

The solution $f(\xi, \eta) = f(\tau, t)$ satisfying (2.17) and the regularity requirements on the axis of symmetry for $\tau > \tau_0$ as well as at infinity can be expressed in terms of the oblate spheroidal wave functions $\{R_{0n}^{(3)}(ip, \tau),\ S_{0n}^{(1)}(ip, t)\}$[3]. Likewise the functions V_1 and V_2 satisfying the eqns (2.26), (2.27) can be expressed in terms of the oblate spheroidal wave functions $\{R_{1n}^{(3)}(i\alpha c, \tau),\ S_{1n}^{(1)}(i\alpha c, t)\}$ and $\{R_{1n}^{(3)}(i\beta c, \tau),\ S_{1n}^{(1)}(i\beta c, t)\}$. It is known that the oblate spheroidal wave functions can also be expressed in terms of prolate spheroidal wave functions by multiplying the parameters $(ip)/(i\alpha c)/(i\beta c)$ by $-i$ and the variable τ by i[3]. Thus the functions $f(\tau, t)$, $V_1(\tau, t)$, $V_2(\tau, t)$ can be expressed in terms of the *prolate* spheroidal wave functions. Taking the regularity of these functions on the axis of symmetry at points $\tau > \tau_0$ and the regularity at $\tau = \infty$, we can write

$$f(\tau, t) = \sum A_n R_{0n}^{(3)}(p, i\tau) S_{0n}^{(1)}(p, t) \tag{4.4}$$

$$V_1(\tau, t) = \sum B_n R_{1n}^{(3)}(\alpha c, i\tau) S_{1n}^{(1)}(\alpha c, t) \tag{4.5}$$

$$V_2(\tau, t) = \sum C_n R_{1n}^{(3)}(\beta c, i\tau) S_{1n}^{(1)}(\beta c, t) \tag{4.6}$$

where the functions R and S are *prolate* spheroidal wave functions.

The function $V(\tau, t)$ is given by

$$\begin{aligned} V &= V_1 + V_2 \\ &= \sum B_n R_{1n}^{(3)}(\alpha c, i\tau) S_{1n}^{(1)}(\alpha c, t) + \sum C_n R_{1n}^{(3)}(\beta c, i\tau) S_{1n}^{(1)}(\beta c, t). \end{aligned} \tag{4.7}$$

The functions $A(\tau, t)$, $B(\tau, t)$ can be obtained on using (2.22), (2.23), (4.4) and (4.7). We find that

$$\begin{aligned} &k(2k + ip j\omega)c\sqrt{((\tau^2+t^2))}A(\tau, t) \\ &\quad = k(\alpha+\beta+\gamma)\sum A_n \sqrt{((\tau^2+1))}\frac{\mathrm{d}}{\mathrm{d}\tau}R_{0n}^{(3)}(p, i\tau) S_{0n}^{(1)}(p, t) \\ &\qquad - \{\gamma(\mu+k)\alpha^2 + k^2 - ip\omega\gamma\}\sum B_n R_{1n}^{(3)}(\alpha c, i\tau)\frac{\mathrm{d}}{\mathrm{d}t}[\sqrt{((1-t^2))}S_{1n}^{(1)}(\alpha c, t)] \\ &\qquad - \{\gamma(\mu+k)\beta^2 + k^2 - ip\omega\gamma\}\sum C_n R_{1n}^{(3)}(\beta c, i\tau)\frac{\mathrm{d}}{\mathrm{d}t}[\sqrt{((1-t^2))}S_{1n}^{(1)}(\beta c, t)], \end{aligned} \tag{4.8}$$

$$\begin{aligned}
&k(2k+i\rho j\omega)c\sqrt{((\tau^2+t^2))}B(\tau,t)\\
&\quad=-(\alpha+\beta+\gamma)\sum A_nR_{0n}^{(3)}(p,i\tau)\sqrt{((1-t^2))}\frac{\mathrm{d}}{\mathrm{d}t}(S_{0n}^{(1)}(p,t))\\
&\qquad-\{\gamma(\mu+k)\alpha^2+k^2-i\rho\omega\gamma\}\sum B_n\frac{\mathrm{d}}{\mathrm{d}\tau}[\sqrt{((\tau^2+1))}R_{1n}^{(3)}(\alpha c,i\tau)]\,S_{1n}^{(1)}(\alpha c,t)\\
&\qquad-\{\gamma(\mu+k)\beta^2+k^2-i\rho\omega\gamma\}\sum C_n\frac{\mathrm{d}}{\mathrm{d}\tau}[\sqrt{((\tau^2+1))}R_{1n}^{(3)}(\beta c,i\tau)]S_{1n}^{(1)}(\beta c,t).
\end{aligned}\tag{4.9}$$

The above expansions for V, A and B involve the three infinite sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ and these are to be determined by invoking the boundary conditions on $\tau=\tau_0$.

Boundary conditions

The spheroid $\tau=\tau_0$ is in rotary oscillation with angular speed $\Omega\,\mathrm{e}^{i\omega t}$ and the hyperstick condition[2] implies that on $\tau=\tau_0$

$$V(\tau_0,t)=\Omega c\sqrt{((\tau_0^2+1)(1-t^2))}\tag{4.10}$$

$$A(\tau_0,t)=\left(\frac{\Omega}{h_2}\frac{\partial h_3}{\partial\eta}\right)_{\tau_0}=\frac{\Omega\sqrt{((\tau_0^2+1))}}{\sqrt{((\tau_0^2+t^2))}}t\tag{4.11}$$

$$B(\tau_0,t)=-\left(\frac{\Omega}{h_1}\frac{\partial h_3}{\partial\xi}\right)_{\tau_0}=\frac{-\Omega\tau_0\sqrt{((1-t^2))}}{\sqrt{((\tau_0^2+t^2))}}.\tag{4.12}$$

The above three conditions are valid on the interval $-1\leqslant t\leqslant 1$ and will in general enable us to determine the three sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$. The details are shown in Appendix B.

Calculation of the couple on the body

The calculation of the couple on the body has to take note of the contributions to it from the force stress as well as the couple stress tensors. It is easily seen that $t_{\xi\phi}$ is the only component of the force stress tensor and $m_{\xi\xi}$, $m_{\xi\eta}$ are the only components of the couple stress tensor that contribute to the couple on the body and these expressions can be obtained from the eqns (2.5), (2.6). After some preliminary calculations, we find that

$$(t_{\xi\phi},m_{\xi\xi},m_{\xi\eta})=(T_{\xi\phi},M_{\xi\xi},M_{\xi\eta})\,\mathrm{e}^{i\omega t}\tag{4.13}$$

with

$$T_{\xi\phi}=\frac{(2\mu+k)(\tau^2+1)}{2c\sqrt{((\tau^2+t^2))}}\frac{\partial}{\partial\tau}\left(\frac{V}{\sqrt{((\tau^2+1))}}\right)+k\left\{\frac{1}{2c\sqrt{((\tau^2+t^2))}}\frac{\partial}{\partial\tau}(\sqrt{((\tau^2+1))}V)+B\right\}\tag{4.14}$$

$$M_{\xi\xi}=\alpha f+(\beta+\gamma)\left\{\frac{\sqrt{((\tau^2+1))}}{c\sqrt{((\tau^2+t^2))}}\frac{\partial A}{\partial\tau}-\frac{t\sqrt{((1-t^2))}}{c(\tau^2+t^2)^{3/2}}B\right\}\tag{4.15}$$

$$\begin{aligned}
M_{\xi\eta}=&-\beta\left[\frac{\sqrt{((1-t^2))}}{c\sqrt{((\tau^2+t^2))}}\frac{\partial A}{\partial t}+\frac{\tau\sqrt{((\tau^2+1))}}{c(\tau^2+t^2)^{3/2}}B\right]\\
&+\gamma\left[\frac{t\sqrt{((1-t^2))}}{c(\tau^2+t^2)^{3/2}}A+\frac{\sqrt{((\tau^2+1))}}{c\sqrt{((\tau^2+t^2))}}\frac{\partial B}{\partial\tau}\right].
\end{aligned}\tag{4.16}$$

The contribution of the force stress tensor to the couple on the body is given by

$$\begin{aligned}
C_I&=2\pi c^3(\tau_0^2+1)\int_{-1}^{1}\sqrt{((\tau_0^2+t^2)(1-t^2))}(t_{\xi\phi})_{\tau_0}\,\mathrm{d}t\\
&=\frac{8\pi}{3}c^2(\mu+k)(\tau_0^2+1)\Big[\sqrt{((\tau_0^2+1))}\Big\{\sum B_n\Big[\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}R_{1n}^{(3)}(\alpha c,i\tau)\Big]_{\tau_0}\mathrm{d}_0^{1n}(\alpha c)\\
&\quad+\sum C_n\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}R_{1n}^{(3)}(\beta c,i\tau)\Big)_{\tau_0}\mathrm{d}_0^{1n}(\beta c)\Big\}-\Omega c\tau_0\Big]\,\mathrm{e}^{i\omega t}.
\end{aligned}\tag{4.17}$$

The contribution of the couple stress to the couple on the body equals

$$C_{II} = 2\pi c^2\sqrt{((\tau_0^2+1))}\int_{-1}^{1}[\sqrt{((\tau^2+1))}tm_{\xi\xi} - \tau\sqrt{((1-t^2))}m_{\xi\eta}]_{\tau_0}\,dt$$
$$= 2\pi c^2\sqrt{((\tau_0^2+1))}\left[\int_{-1}^{1}\{(\alpha+\beta+\gamma)(\tau^2+1)tf(\tau,t) - \gamma g(\tau,t)\sqrt{((1-t^2))}\}_{\tau_0}\,dt\right]e^{i\omega t} \quad (4.18)$$

and this simplifies to

$$C_{II} = \frac{4\pi}{3k}c^2\sqrt{((\tau_0^2+1))}\,e^{i\omega t}\times[\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma+ipj\omega k\}\Omega c\tau_0\sqrt{((\tau_0^2+1))}$$
$$+\gamma(\mu+k)(\beta^2-\alpha^2)\tau_0 Y_2 - \{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}(\tau_0^2+1)Y_4]$$
$$-\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}(\tau_0^2+1))Y_4] \quad (4.19)$$

where

$$Y_2 = \sum C_n d_0^{1n}(\beta c)R_{1n}^{(3)}(\beta c, i\tau_0)$$

$$Y_3 = \sum B_n d_0^{1n}(\alpha c)\left(\frac{d}{d\tau}R_{1n}^{(3)}(\alpha c, i\tau)\right)_{\tau_0} \quad (4.20)$$

$$Y_4 = \sum C_n d_0^{1n}(\beta c)\left(\frac{d}{d\tau}R_{1n}^{(3)}(\beta c, i\tau)\right)_{\tau_0}.$$

From (4.17) and (4.19) we obtain the total couple on the spheroid in the form

$$C = C_I + C_{II} = \frac{4\pi}{3k}c^2\sqrt{((\tau_0^2+1))}\,e^{i\omega t}\times[\{-k(2\mu+k)+ipj\omega k$$
$$+\gamma(\mu+k)\alpha^2 - ip\omega\gamma\}\Omega c\tau_0\sqrt{((\tau_0^2+1))} + \gamma(\mu+k)(\beta^2-\alpha^2)\tau_0 Y_2$$
$$+\{k(2\mu+k)-\gamma(\mu+k)\alpha^2+ip\omega\gamma\}(\tau_0^2+1)Y_3$$
$$+\{k(2\mu+k)-\gamma(\mu+k)\beta^2+ip\omega\gamma\}(\tau_0^2+1)Y_4]. \quad (4.21)$$

Numerical work

Defining the nondimensional couple C_{ND} by means of the equation

$$C = \frac{4\pi}{3}(2\mu+k)\Omega c^3 C_{ND}\,e^{i\omega t} \quad (4.22)$$

we see that

$$C_{ND} = \tau_0(\tau_0^2+1)\left(\frac{2i\theta^2\Theta^2}{\lambda^2\Lambda^2}+\frac{\alpha^2c^2-i\theta^2}{\lambda^2}-1\right)+\frac{(\beta^2-\alpha^2)c^2}{\lambda^2}\tau_0\sqrt{((\tau_0^2+1))}Y_2/(\Omega c)$$
$$+(\tau_0^2+1)^{3/2}\left[\left(1-\frac{\alpha^2c^2-i\theta^2}{\lambda^2}\right)Y_3/(\Omega c)+\left(1-\frac{\beta^2c^2-i\theta^2}{\lambda^2}\right)Y_4/(\Omega c)\right] \quad (4.23)$$

where the parameters θ^2, λ^2, Λ^2, Θ^2, are given in (3.32).

The couple parameters K and K' are introduced by means of the relation

$$C_{ND} = \theta^2\tau_0(\tau_0^2+1)(-K'-iK). \quad (4.24)$$

The quantities K and K' are numerically evaluated for a series of parameter values τ_0, θ^2, λ^2, Λ^2, Θ^2 by evaluating the constants $\{C_n\}$ from the system of eqns (B11) and the constants $\{B_n\}$ from the system (B14) in Appendix B by truncating each system to a 5×5 set. The variation of K and K' is graphically presented for the polar case and nonpolar case[5] in Figs. 13–22.

REFERENCES

[1] A. C. ERINGEN, *J. Math. Mech.* **16** (1966). pp. 1-18.
[2] S. C. COWIN, *Advances in Applied Mechanics*, Vol. 14. Academic Press, New York (1974).
[3] M. ABRAMOWITZ and I. A. STEGUN, *Handbook of Mathematical Functions with Formulae, Graphs and Mathematics Tables.* Dover, New York (1966).
[4] N. W. McLACHLAN, *Bessel Functions for Engineers.* Oxford University Press (1961).
[5] R. P. KANWAL, *Q. J. Mech. Appl. Math.* **13**, 146 (1955).

(*Received* 12 *June* 1980; *in revised form* 24 *November* 1981)

APPENDIX A

Determination of the constants $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ *that occur in the expressions for* $V(s,t)$, $A(s,t)$ and $B(s,t)$ *in eqns* (3.11)–(3.13)

From eqn (3.14) we have

$$\sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(i\alpha c, s_0) S_{1n}^{(1)}(i\alpha c, t) + \sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\beta c, s_0)\, S_{1n}^{(1)}(i\beta c, t) = \Omega c\sqrt{((s_0^2-1)(1-t^2))}. \tag{A1}$$

The boundary conditions in (3.15) and (3.16) concerning the micro-rotation vector provide the following two equations involving the three sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$:

$$\begin{aligned} &k(\alpha+\beta+\gamma)\sqrt{((s_0^2-1))}\sum_{n=1}^{\infty} A_n\left(\frac{d}{ds}R_{0n}^{(3)}(ip,s)\right)_{s_0} S_{0n}^{(1)}(ip,t)\\ &-\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(i\alpha c,s_0)\frac{d}{dt}[\sqrt{((1-t^2))}\, S_{1n}^{(1)}(i\alpha c,t)]\\ &-\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\beta c,s_0)\frac{d}{dt}[\sqrt{((1-t^2))}S_{1n}^{(1)}(i\beta c,t)] = k(2k+ipj\omega)\Omega c\sqrt{((s_0^2-1))}t, \end{aligned} \tag{A2}$$

$$\begin{aligned} &k(\alpha+\beta+\gamma)\sum_{n=1}^{\infty} A_n R_{0n}^{(3)}(ip,s_0)\sqrt{((1-t^2))}\frac{d}{dt}S_{0n}^{(1)}(ip,t)\\ &+\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty} B_n\left(\frac{d}{ds}(\sqrt{((s^2-1))}R_{1n}^{(3)}(i\alpha c,s))\right)_{s_0}S_{1n}^{(1)}(i\alpha c,t)\\ &+\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty} C_n\left(\frac{d}{ds}(\sqrt{((s^2-1))}R_{1n}^{(3)}(i\beta c,s))\right)_{s_0} S_{1n}^{(1)}(i\beta c,t) = k(2k+ipj\omega)\Omega c s_0\sqrt{((1-t^2))}. \end{aligned} \tag{A3}$$

The constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ may be presumed zero for even values of n in view of the following symmetry/antisymmetry in the field variables:

$$\left.\begin{aligned} V(s,t) &= V(s,-t)\\ A(s,t) &= -A(s,-t).\\ B(s,t) &= B(s,-t) \end{aligned}\right\} \tag{A4}$$

From (A1) we can express each one of the constants B_n in terms of the set $\{C_m\}$ and also *vice versa*. The relations are

$$B_n R_{1n}^{(3)}(i\alpha c,s_0) N_{nn}^{(1)}(i\alpha c) = \frac{4}{3}\Omega c\sqrt{((s_0^2-1))}\, d_0^{1n}(i\alpha c) - \sum_{m=1}^{\infty} M_{nm}(i\alpha c, i\beta c) C_m R_{1m}^{(3)}(i\beta c, s_0); \tag{A5}$$

$$C_n R_{1n}^{(3)}(i\beta c,s_0) N_{nn}^{(1)}(i\beta c) = \frac{4}{3}\Omega c\sqrt{((s_0^2-1))}\, d_0^{1n}(i\beta c) - \sum_{m=1}^{\infty} M_{mn}(i\alpha c, i\beta c) B_m R_{1m}^{(3)}(i\alpha c, s_0) \tag{A6}$$

where

$$N_{nn}^{(1)}(i\alpha c) = \int_{-1}^{1}\{S_{1n}^{(1)}(i\alpha c,t)\}^2\, dt = \sum_{r=0,1}^{\infty\,\prime}\frac{2(r+1)(r+2)}{(2r+3)}\{d_r^{1n}(i\alpha c)\}^2 \tag{A7}$$

with a corresponding expression for $N_{nn}^{(1)}(i\beta c)$ and

$$M_{nm}(i\alpha c, i\beta c) = \int_{-1}^{1} S_{1n}^{(1)}(i\alpha c,t) S_{1m}^{(1)}(i\beta c,t)\, dt = \sum_{r=0,1}^{\infty\,\prime}\frac{2(r+1)(r+2)}{(2r+3)} d_r^{1m}(i\beta c)\, d_r^{1n}(i\alpha c). \tag{A8}$$

We can divide eqn (A3) by $\sqrt{((1-t^2))}$ and integrate the result with respect to t from 0 to t. This gives rise to the equation

$$\begin{aligned} &k(\alpha+\beta+\gamma)\sum_{n=1}^{\infty} A_n R_{0n}^{(3)}(ip,s_0)\{S_{0n}^{(1)}(ip,t) - S_{0n}^{(1)}(ip,0)\}\\ &+\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty} B_n\left(\frac{d}{ds}(\sqrt{((s^2-1))}R_{1n}^{(3)}(i\alpha c,s))\right)_{s_0} T_{1n}^{(1)}(i\alpha c,t)\\ &+\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty} C_n\left(\frac{d}{ds}(\sqrt{((s^2-1))}R_{1n}^{(3)}(i\beta c,s))\right)_{s_0} T_{1n}^{(1)}(i\beta c,t) = k(2k+ipj\omega)\Omega c s_0 t \end{aligned} \tag{A9}$$

where

$$T_{1n}^{(1)}(i\alpha c, t) = \int_0^t \frac{S_{1n}^{(1)}(i\alpha c, t)}{\sqrt{((1-t^2))}}\mathrm{d}t = \sum_{r=0,1}^{\infty'} d_r^{1n}(i\alpha c)[P_{r+1}(t) - P_{r+1}(0)] \tag{A10}$$

and it is easily seen that the terms $S_{0n}^{(1)}(ip, 0)$ and $P_{r+1}(0)$ are zero in view of (A4).

From eqns (A2) and (A9) we obtain

$$k(\alpha+\beta+\gamma)\sqrt{((s_0^2-1))}\left(\frac{\mathrm{d}}{\mathrm{d}s}R_{0n}^{(3)}(ip, s)\right)_{s_0} N_{nn}^{(0)}(ip)A_n + \{\gamma(\mu+k)\alpha^2 + k^2 - ip\omega\gamma\}\sum_{m=1}^{\infty} B_m R_{1m}^{(3)}(i\alpha c, s_0)\beta_{mn}(i\alpha c, ip)$$
$$+ \{\gamma(\mu+k)\beta^2 + k^2 - ip\omega\gamma\}\sum_{m=1}^{\infty} C_m R_{1m}^{(3)}(i\beta c, s_0)\beta_{mn}(i\beta c, ip) = \frac{2}{3}k(2k + ipj\omega)\Omega c\sqrt{((s_0^2-1))}\, d_1^{0n}(ip), \tag{A11}$$

$$k(\alpha+\beta+\gamma)R_{0n}^{(3)}(ip, s_0)N_{nn}^{(0)}(ip)A_n + \{\gamma(\mu+k)\alpha^2 + k^2 - ip\omega\gamma\}\sum_{m=1}^{\infty} B_m\left(\frac{\mathrm{d}}{\mathrm{d}s}(\sqrt{(s^2-1)})R_{1m}^{(3)}(i\alpha c, s)\right)_{s_0}\alpha_{mn}(i\alpha c, ip)$$
$$+ \{\gamma(\mu+k)\beta^2 + k^2 - ip\omega\gamma\}\sum_{m=1}^{\infty} C_m\left(\frac{\mathrm{d}}{\mathrm{d}s}(\sqrt{(s^2-1)})R_{1m}^{(3)}(i\beta c, s)\right)_{s_0}\alpha_{mn}(i\beta c, ip) = \frac{2}{3}k(2k + ipj\omega)\Omega c s_0 d_1^{0n}(ip). \tag{A12}$$

where

$$N_{nn}^{(0)}(ip) = \int_{-1}^{1}[S_{0n}^{(1)}(ip, t)]^2\,\mathrm{d}t = \sum_{r=0,1}^{\infty'}\frac{2}{(2r+1)}[d_r^{0n}(ip)]^2, \tag{A13}$$

$$\alpha_{mn}(i\alpha c, ip) = \int_{-1}^{1} S_{0n}^{(1)}(ip, t)\,T_{1m}^{(1)}(i\alpha c, t)\,\mathrm{d}t = \sum_{r=0,1}^{\infty'}\frac{2}{(2r+3)}d_r^{1m}(i\alpha c)\,d_{r+1}^{0n}(ip), \tag{A14}$$

$$\beta_{mn}(i\alpha c, ip) = -\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{((1-t^2))}S_{1m}^{(1)}(i\alpha c, t))\,S_{0n}^{(1)}(ip, t)\,\mathrm{d}t = \sum_{r=0,1}^{\infty'}\frac{2(r+1)(r+2)}{(2r+3)}d_r^{1m}(i\alpha c)\,d_{r+1}^{0n}(ip) \tag{A15}$$

with similar expressions for $\alpha_{mn}(i\beta c, ip)$ and $\beta_{mn}(i\beta c, ip)$. From (A11) and (A12) we can eliminate A_n and this results in the following relation between the constants $\{B_m\}$ and $\{C_m\}$:

$$\{\gamma(\mu+k)\alpha^2 + k^2 - ip\omega\gamma\}\sum_{m=1}^{\infty}\left[\left\{\beta_{mn}(i\alpha c, ip) - \alpha_{mn}(i\alpha c, ip)\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{0n}^{(3)}(ip, s)}{R_{0n}^{(3)}(ip, s)}\left[(s^2-1)\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{1m}^{(3)}(i\alpha c, s)}{R_{1m}^{(3)}(i\alpha c, s)} + s\right]\right\}_{s_0} B_m R_{1m}^{(3)}(i\alpha c, s_0)\right]$$
$$+ \{\gamma(\mu+k)\beta^2 + k^2 - ip\omega\gamma\}\sum_{m=1}^{\infty}\left[\left[\left\{\beta_{mn}(i\beta c, ip) - \alpha_{mn}(i\beta c, ip)\left[\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{0n}^{(3)}(ip, s)}{R_{0n}^{(3)}(ip, s)}\right]\left[(s^2-1)\left[\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{1m}^{(3)}(i\beta c, s)}{R_{1m}^{(3)}(i\beta c, s)}\right] + s\right]\right\}_{s_0}\right.$$
$$\left.\times\, C_m R_{1m}^{(3)}(i\beta c, s_0)\right] = \frac{2}{3}k(2k + ipj\omega)\Omega c\, d_1^{0n}(ip)\sqrt{((s_0^2-1))}.\left\{1 - s\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{0n}^{(3)}(ip, s)}{R_{0n}^{(3)}(ip, s)}\right\}_{s_0}. \tag{A16}$$

From (A5) and (A16) we can eliminate the constants B_n and obtain a nonhomogeneous linear system for the determination of the constants $\{C_l\}$, viz.

$$\sum_{l=l}^{\infty}\Delta_{nl}c_l R_{1l}^{(3)}(i\beta c, s_0) = \delta_n \tag{A17}$$

$n = 1,3,5,\ldots$ where

$$\Delta_{nl} = \{\gamma(\mu+k)\beta^2 + k^2 - ip\omega\gamma\}\left[\beta_{1n}(i\beta c, ip) - \alpha_{1n}(i\beta c, ip)\left[\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{0n}^{(3)}(ip, s)}{R_{0n}^{(3)}(ip, s)}\right]_{s_0}\left\{(s^2-1)\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{1n}^{(3)}(i\beta c, s)}{R_{1n}^{(3)}(i\beta c, s)} + s\right\}_{s_0}\right]$$
$$-\{\gamma(\mu+k)\alpha^2 + k^2 - ip\omega\gamma\}\sum_{m=1}^{\infty}\left\{\frac{M_{ml}(i\alpha c, i\beta c)}{N_{mm}^{(1)}(i\alpha c)}\left[\beta_{mn}(i\alpha c, ip) - \alpha_{mn}(i\alpha c, ip)\left[\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{0n}^{(3)}(ip, s)}{R_{0n}^{(3)}(ip, s)}\right]_{s_0}\left\{(s^2-1)\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{1m}^{(3)}(i\alpha c, s)}{R_{1m}^{(3)}(i\alpha c, s)} + s\right\}_{s_0}\right]\right\}, \tag{A18}$$

$$\delta_n = \frac{2}{3}k(2k + ipj\omega)\Omega c d_1^{0n}(ip)\sqrt{((s_0^2-1))}\left[1 - s\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{0n}^{(3)}(ip, s)}{R_{0n}^{(3)}(ip, s)}\right]_{s_0}$$
$$-\frac{4}{3}\{\gamma(\mu+k)\alpha^2 + k^2 - ip\omega\gamma\}\Omega c\sqrt{((s_0^2-1))}\sum_{m=1}^{\infty}\left\{\left[\frac{d_0^{1m}(i\alpha c)}{N_{mm}^{(1)}(i\alpha c)}\cdot\right.\right.$$
$$\beta_{mn}(i\alpha c, ip) - \alpha_{mn}(i\alpha c, ip)\left[\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{0n}^{(3)}(ips)}{R_{0n}^{(3)}(ip, s)}\right]_{s_0}$$
$$\left.\left.\times\left[(s^2-1)\frac{\frac{\mathrm{d}}{\mathrm{d}s}R_{1m}^{(3)}(i\alpha c, s)}{R_{1m}^{(3)}(i\alpha c, s)} + s\right]_{s_0}\right\}\right] \tag{A19}$$

From the above infinite linear system the constants $\{C_n\}$ are to be determined by a numerical procedure. Then the constants B_n can be determined from (A5) and the constants A_n can be determined from either of (A11) or (A12). Thus we have a feasible procedure for the determination of the unknown constants that occur in the expressions for the field functions $V(s,t)$, $A(s,t)$ and $B(s,t)$.

The numerical evaluation of the couple C is of particular interest and the expression for C in (3.28) involves infinite summations on both $\{B_n\}$ and $\{C_n\}$. To check the error in numerical evaluation of C to the extent possible, it is desirable to obtain the constants B_n as well as C_n directly. The system of equations for B_n can be obtained on the same lines as seen above for the constants C_m by eliminating C_n's between (A6) and (A16). The B_n's are directly evaluated numerically from the following system of linear equations:

$$\sum_{l=l}^{\infty}\Gamma_{nl}B_l R_{1l}^{(3)}(i\alpha c, s_0)=\gamma_n \tag{A20}$$

$n=1, 3, 5,\ldots$ where

$$\Gamma_{nl}=\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\left[\beta_{1n}(i\alpha c,ip)-\alpha_{1n}(i\alpha c,ip)\left[\frac{\frac{d}{ds}R_{0n}^{(3)}(ip,s)}{R_{0n}^{(3)}(ip,s)}\right]_{s_0}\left[(s^2-1)\frac{\frac{d}{ds}R_{1l}^{(3)}(i\alpha c,s)}{R_{1l}^{(3)}(i\alpha c,s)}+s\right]\right]_{s_0}$$
$$-\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{m=1}^{\infty}\left\{\frac{M_{1m}(i\alpha c,i\beta c)}{N_{mm}^{(1)}(i\beta c)}\left[\beta_{mn}(i\beta c,ip)-\alpha_{mn}(i\beta c,ip)\left[\frac{\frac{d}{ds}R_{0n}^{(3)}(ip,s)}{R_{0n}^{(3)}(ip,s)}\right]_{s_0}\left[(s^2-1)\frac{\frac{d}{ds}R_{1m}^{(3)}(i\beta c,s)}{R_{1m}^{(3)}(i\beta c,s)}+s\right]_{s^0}\right]\right\} \tag{A21}$$

$$\gamma_n=\frac{2}{3}k(2k+ipj\omega)\Omega c d_1^{0n}(ip)\surd((s_0^2-1))$$
$$\left[1-\frac{s\frac{d}{ds}R_{0n}^{(3)}(ip,s)}{R_{0n}^{(3)}(ip,s)}\right]_{s_0}$$
$$-\frac{4}{3}\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\Omega c\surd((s_0^2-1))$$
$$\sum_{m=1}^{\infty}\left[\frac{d_0^{1m}(i\beta c)}{N_{mm}^{(1)}(i\beta c)}\left\{\beta_{mn}(i\beta c,ip)-\alpha_{mn}(i\beta c,ip).\right.\right.$$
$$\left.\left.\left[\frac{\frac{d}{ds}R_{0n}^{(3)}(ip,s)}{R_{0n}^{(3)}(ip,s)}\right]_{s_0}\left[(s^2-1)\frac{\frac{d}{ds}R_{1m}^{(3)}(i\beta c,s)}{R_{1m}^{(3)}(i\beta c,s)}+s\right]_{s_0}\right\}\right] \tag{A22}$$

APPENDIX B

Determination of the constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ occurring in the expressions for $V(\tau, t)$, $A(\tau, t)$ and $B(\tau, t)$ in eqns (4.7)–(4.9)

The boundary conditions (4.10)–(4.12) yield the following equations

$$\sum B_nR_{1n}^{(3)}(\alpha c,i\tau_0)S_{1n}^{(1)}(\alpha c,t)+\sum C_nR_{1n}^{(3)}(\beta c,i\tau_0)S_{1n}^{(1)}(\beta c,t)=\Omega c\surd((\tau_0^2+1)(1-t^2)), \tag{B1}$$

$$k(\alpha+\beta+\gamma)\sum_{n=1}^{\infty}A_n\surd((\tau_0^2+1))\left(\frac{d}{d\tau}R_{0n}^{(3)}(p,i\tau)\right)_{\tau_0}S_{0n}^{(1)}(p,t)-\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty}B_nR_{1n}^{(3)}(\alpha c,i\tau_0)\frac{d}{dt}(\surd(1-t^2)S_{1n}^{(1)}(\alpha c,t))$$
$$-\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty}C_nR_{1n}^{(3)}(\beta c,i\tau_0)\frac{d}{dt}(\surd((1-t^2))\,S_{1n}^{(1)}(\beta c,t))=k(2k+ipj\omega)\Omega c\surd((\tau_0^2+1))t, \tag{B2}$$

$$k(\alpha+\beta+\gamma)\sum_{n=1}^{\infty}A_nR_{0n}^{(3)}(p,i\tau_0)\surd\left((1-t^2)\frac{d}{dt}(S_{0n}^{(1)}(p,t)\right)$$
$$+\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty}B_n\left(\frac{d}{d\tau}(\surd((\tau^2+1))R_{1n}^{(3)}(\alpha c,i\tau))\right)_{\tau_0}S_{1n}^{(1)}(\alpha c,t)$$
$$+\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty}C_n\left(\frac{d}{d\tau}(\surd((\tau^2+1))\,R_{1n}^{(3)}(\beta c,i\tau))\right)_{\tau_0}S_{1n}^{(1)}(\beta c,t)=k(2k+ipj\omega)\Omega c\tau_0\surd((1-t^2)). \tag{B3}$$

As in the case of the prolate spheroid, we may presume that the constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ are zero for even values of n. From (B1) we have

$$B_nR_{1n}^{(3)}(\alpha c,i\tau_0)N_{nn}^{(1)}(\alpha c)=\frac{4}{3}\Omega c d_0^{1n}(\alpha c)\surd((\tau_0^2+1))-\sum_{m=1}^{\infty}C_mR_{1m}^{(3)}(\beta c,i\tau_0)M_{nm}(\alpha c,\beta c), \tag{B4}$$

$$C_nR_{1n}^{(3)}(\beta c,i\tau_0)N_{nn}^{(1)}(\beta c)=\frac{4}{3}\Omega c d_0^{1n}(\beta c)\surd((\tau_0^2+1)),-\sum_{m=1}^{\infty}B_mR_{1m}^{(3)}(\alpha c,i\tau_0)M_{mn}(\alpha c,\beta c), \tag{B5}$$

where

$$N_{nn}^{(1)}(\alpha c)=\int_{-1}^{1}[S_{1n}^{(1)}(\alpha c,t)]^2\,dt=\sum_{r=0,1}^{\infty\,'}\frac{2(r+1)(r+2)}{2r+3}[d_r^{1n}(\alpha c)]^2 \tag{B6}$$

$$M_{mn}(\alpha c,\beta c)=\int_{-1}^{1} S_{1m}^{(1)}(\alpha c,t)\,S_{1n}^{(1)}(\beta c,t)\,dt=\sum_{r=0,1}^{\infty'}\frac{2(r+1)(r+2)}{2r+3}d_r^{1m}(\alpha c)\,d_r^{1n}(\beta c). \tag{B7}$$

From (B3) we can deduce that

$$k(\alpha+\beta+\gamma)\sum_{n=1}^{\infty}A_nR_{0n}^{(3)}(p,i\tau_0)\{S_{0n}^{(1)}(p,t)-S_{0n}^{(1)}(p,0)\}+\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty}B_n\left(\frac{d}{d\tau}(\surd(\tau^2+1)R_{1n}^{(3)}(\alpha c,i\tau))\right)_{\tau_0}T_{1n}^{(1)}(\alpha c,t)$$
$$+\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{n=1}^{\infty}c_n\frac{d}{d\tau}(\surd((\tau^2+1))R_{1n}^{(3)}(\beta c,i\tau))_{\tau_0}T_{1n}^{(1)}(i\beta c,t)=k(2k+ipj\omega)\Omega c\tau_0 t, \tag{B8}$$

where

$$T_{1n}^{(1)}(\alpha c,t)=\int_0^t\frac{S_{1n}^{(1)}(\alpha c,t)}{\surd((1-t^2))}dt=\sum_{r=0,1}^{\infty'}d_r^{1n}(\alpha c)[P_{r+1}(t)-P_{r+1}(0)] \tag{B9}$$

and it is easily seen that the terms $S_{0n}^{(1)}(p,0)$ and $P_{r+1}(0)$ are zero as only odd values of n are involved. From each of (B2) and (B3) we can express each A_n in terms of $\{B_n\}$ and $\{C_n\}$. Elimination of A_n between these expressions yields

$$\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\sum_{m=1}^{\infty}\left[\left\{\beta_{mn}(\alpha c,p)-\alpha_{mn}(\alpha c,p)\left[\frac{\frac{d}{d\tau}R_{0n}^{(3)}(p,i\tau)}{R_{0n}^{(3)}(p,i\tau)}\right]_{\tau_0}\left[(\tau_0^2+1)\left[\frac{\frac{d}{d\tau}R_{1m}^{(3)}(\alpha c,i\tau)}{R_{1m}^{(3)}(\alpha c,i\tau)}\right]_{\tau_0}+\tau_0\right]B_mR_{1m}^{(3)}(\alpha c,i\tau_0)\right]$$
$$+\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{m=1}^{\infty}\left\{\beta_{mn}(\beta c,p)-\alpha_{mn}(\beta c,p)\left[\left[\frac{\frac{d}{d\tau}R_{0n}^{(3)}(p,i\tau)}{R_{0n}^{(3)}(p,i\tau)}\right]_{\tau_0}\left[(\tau_0^2+1)\left[\frac{\frac{d}{d\tau}R_{1m}^{(3)}(\beta c,i\tau)}{R_{1m}^{(3)}(\beta c,i\tau)}\right]_{\tau_0}+\tau_0\right]\right]\right\}C_mR_{1m}^{(3)}(\beta c,i\tau_0)$$
$$=\frac{2}{3}k(2k+ipj\omega)\Omega c d_1^{0n}(p)\surd((\tau_0^2+1))\left[1-\frac{\frac{d}{d\tau}R_{0n}^{(3)}(p,i\tau)}{R_{0n}^{(3)}(p,i\tau)}\right]_{\tau_0}. \tag{B10}$$

The quantities $\alpha_{mn}(\alpha c,p)$, $\beta_{mn}(\alpha c,p)$ are given by espressions analogous to those in (A14), (A15).

From (B4) and (B10) we can eliminate the constants B_n and obtain a nonhomogeneous linear system for the determination of the constants $\{C_l\}$, viz.

$$\sum_{l=1}^{\infty}\Delta_{nl}C_lR_{1l}^{(3)}(\beta c,i\tau_0)=\delta_n \tag{B11}$$

where

$$\Delta_{nl}=\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\left[\beta_{ln}(\beta c,p)-\alpha_{ln}(\beta c,p)\left[\frac{\frac{d}{d\tau}R_{0n}^{(3)}(p,i\tau)}{R_{0n}^{(3)}(p,i\tau)}\right]_{\tau_0}\left\{(\tau_0^2+1)\left[\frac{\frac{d}{d\tau}R_{1l}^{(3)}(\beta c,i\tau)}{R_{1l}^{(3)}(\beta c,i\tau)}\right]_{\tau_0}+\tau_0\right\}\right]$$
$$-\{\gamma(\mu+k)\beta^2+k^2-ip\omega\gamma\}\sum_{m=1}^{\infty}\frac{M_{ml}(\alpha c,\beta c)}{N_{mm}^{(1)}(\alpha c)}\left[\beta_{mn}(\alpha c,p)-\alpha_{mn}(\alpha c,p)\left[\frac{\frac{d}{d\tau}R_{0n}^{(3)}(p,i\tau)}{R_{0n}^{(3)}(p,i\tau)}\right]_{\tau_0}\right.$$
$$\left.\times\left\{(\tau_0^2+1)\left[\frac{\frac{d}{d\tau}R_{1m}^{(3)}(\alpha c,i\tau)}{R_{1m}^{(3)}(\alpha c,i\tau)}\right]_{\tau_0}+\tau_0\right\}\right] \tag{B12}$$

$$\delta_n=\frac{2}{3}k(2k+ipj\omega)\Omega c d_1^{0n}(p)\surd((\tau_0^2+1))\left(1-\frac{\frac{d}{d\tau}R_{0n}^{(3)}(p,i\tau)}{R_{0n}^{(3)}(p,i\tau)}\right)_{\tau_0}$$
$$=\frac{4}{3}\{\gamma(\mu+k)\alpha^2+k^2-ip\omega\gamma\}\Omega c\surd((\tau_0^2+1)).$$
$$\times\sum_{m=1}^{\infty}\frac{d_0^{1m}(\alpha c)}{N_{mm}^{(1)}(\alpha c)}\left\{\beta_{mn}(\alpha c,p)-\alpha_{mn}(\alpha c,p)\left[\frac{\frac{d}{d\tau}R_{0n}^{(3)}(p,i\tau)}{R_{0n}^{(3)}(p,i\tau)}\right]\tau_0.\right.$$
$$\left.\left\{(\tau_0^2+1)\left[\frac{\frac{d}{d\tau}R_{1m}^{(3)}(\alpha c,i\tau)}{R_{1m}^{(3)}(\alpha c,i\tau)}\right]_{\tau_0}+\tau_0\right\}\right\}. \tag{B13}$$

From the above infinite linear system the constants $\{C_l\}$ are to be determined by a numerical procedure. The constants $\{B_n\}$ can then be determined from (B4) and the constants $\{A_n\}$ can be found thereafter on integrating (B2) or (B3). We have thus a feasible procedure for the determination of the unknown constants that occur in the expressions for the field functions $V(\tau, t)$, $A(\tau, t)$ and $B(\tau, t)$.

The numerical evaluation of the couple C is of particular interest and the expression for C in (4.21) involves infinite summations on both $\{B_n\}$ as well as $\{C_n\}$. It is desirable to obtain the constants $\{B_n\}$ as well as $\{C_n\}$ directly. The constants $\{B_n\}$ are evaluated from the system

$$\sum_{l=1}^{\infty}\Gamma_{nl}B_lR_{1l}^{(3)}(\alpha c,i\tau_0)=\gamma_n. \tag{B14}$$

$n=1,3,5,\ldots,$

where

$$\Gamma_{n1} = \{\gamma(\mu+k)\alpha^2 + k^2 - i\rho\omega\gamma\}\left[\beta_{1n}(\alpha c, p) - \alpha_{1n}(\alpha c, p)\left(\frac{\frac{\mathrm{d}}{\mathrm{d}\tau} R_{0n}^{(3)}(p, i\tau)}{R_{0n}^{(3)}(p, i\tau)}\right)_{\tau_0}\left(\frac{(\tau^2+1)\frac{\mathrm{d}}{\mathrm{d}\tau} R_{1l}^{(3)}(\alpha c, i\tau)}{R_{1l}^{(3)}(\alpha c, i\tau)} + \tau\right)_{\tau_0}\right]$$

$$- \{\gamma(\mu+k)\beta^2 + k^2 - i\rho\omega\gamma\}\sum_{m=1}^{\infty}\left\{\frac{M_{1m}(\alpha c, \beta c)}{N_{mm}^{(1)}(\beta c)}\ \beta_{mn}(\beta c, p)\right.$$

$$\left. - \alpha_{mn}(\beta c, p)\left(\frac{\frac{\mathrm{d}}{\mathrm{d}\tau} R_{0n}^{(3)}(p, i\tau)}{R_{0n}^{(3)}(p, i\tau)}\right)_{\tau_0}\left\{(\tau_0^2+1)\left(\frac{\frac{\mathrm{d}}{\mathrm{d}\tau} R_{1m}^{(3)}(\beta c, i\tau)}{R_{1m}^{(3)}(\beta c, i\tau)}\right)_{\tau_0} + \tau_0\right\}\right\} \tag{B15}$$

$$\gamma_n = \frac{2}{3} k(2k + i\rho j\omega)\Omega c d_1^{0n}(p)\sqrt{((\tau_0^2+1))}\left(1 - \frac{\frac{\mathrm{d}}{\mathrm{d}\tau} R_{0n}^{(3)}(p, i\tau)}{R_{0n}^{(3)}(p, i\tau)}\right)_{\tau_0} - \frac{4}{3}\{\gamma(\mu+k)\beta^2 + k^2 - i\rho\omega\gamma\}\Omega c\sqrt{((\tau_0^2 + +1))}$$

$$\times \sum_{m=1}^{\infty} \frac{d_0^{1m}(\beta c)}{N_{mm}^{(1)}(\beta c)}\left\{\beta_{mn}(\beta c, p) - \alpha_{mn}(\beta c, p)\left(\frac{\frac{\mathrm{d}}{\mathrm{d}\tau} R_{0n}^{(3)}(p, i\tau)}{R_{0n}^{(3)}(p, i\tau)}\right)_{\tau_0}\left(\tau^2+1)\frac{\frac{\mathrm{d}}{\mathrm{d}\tau} R_{1m}^{(3)}(\beta c, i\tau)}{R_{1m}^{(3)}(\beta c, i\tau)} + \tau\right)_{\tau_0}\right\} \tag{B16}$$