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Drag on an elliptic cylinder in a fluid particle suspension

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Introduction

Several researchers over years have studied the oscillatory flow problems concerning various classes of fluids. Kanwal [1, 2] has studied analytically the oscillations of symmetric bodies like circular cylinder, sphere, spheroid and elliptic cylinder in classical viscous liquids. Frater [3, 4] has examined the oscillations of a sphere and a circular cylinder in a viscoelastic fluid of Oldroyd's–B type and studied numerically the effects of the viscoelastic and frequency parameters on the drag experienced by the bodies. S. K. Lakshman Rao et al. [5, 6, 7, 8, 9] have examined the harmonic oscillations of a circular cylinder, sphere, spheroid as well as an elliptic cylinder in micropolar and couple stress fluids. Kumar [10] has discussed the oscillations of a circular cylinder and a sphere in a fluid-particle suspension and studied the effects on the drag experienced by the bodies with reference to the geometric and fluid characteristic parameters.

The present paper aims at obtaining an expression for the drag experienced by an elliptic cylinder performing rectilinear oscillations along major/minor axis of the cross-sectional ellipse in an infinite expanse of a fluid-particle suspension [11]. The equations of motion are linearized under the assumption of smallness of the amplitude U of oscillation. The stream function governing the fluid flow is determined and is expressed as a series of Mathieu functions. The drag on a part of the cylinder of length L units is obtained and is expressed in terms of parameters K and K'. The variation of K and K' is studied numerically with reference to the eccentricity parameter, as well as the physical parameters of the fluid.

Basic equations and statement of the problem

The equations of motion of an incompressible fluid particle suspension were derived by Saffman [11] under the assumptions:

(i) the dust particles present in the fluid are uniform in size and shape;
(ii) the net effect of the dust on the fluid is equivalent to an extra force proportional to the "dust particle-fluid particle" relative velocity and
(iii) the number of dust particles per unit volume is constant. The equations of motion are [11]

$$\operatorname{div} \bar{q} = 0 \tag{1}$$

$$\varrho \left\{ \frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla) \bar{q} \right\} = -\nabla p - \mu \underline{\nabla x (\nabla x \bar{q})} + K_1 N (\bar{q}_s - \bar{q}) \tag{2}$$

$$m \left\{ \frac{\partial \bar{q}_s}{\partial t} + (\bar{q}_s \cdot \nabla) \bar{q}_s \right\} = -K_1 (\bar{q}_s - \bar{q}) \tag{3}$$

$$\frac{\partial N}{\partial t} + \operatorname{div} (N \bar{q}_s) = 0 \tag{4}$$

where $\bar{q}$ is the fluid velocity vector, $\bar{q}_s$ is the particle velocity vector, ϱ is the density of the fluid, m is the mass of each suspended particle, p is the pressure at any point, μ is the fluid viscosity and N is the number of particles per unit volume. The parameter $K_1 = m/\tau$ where τ is the relaxation time parameter: τ is the time taken by a suspended particle to adjust itself to the motion when there is a disturbance.

An elliptic cylinder is assumed to perform rectilinear oscillations along the major or minor axis of the cross-sectional ellipse with velocity $U \exp(i\sigma t)$. Assuming $U/(c\sigma)$ to be small where c is the semifocal distance of the cross-sectional ellipse, under the Stokesian approximation, neglecting the nonlinear terms, the linearized version of the Eqs. (1) to (4) is given by

$$\operatorname{div} \bar{q} = 0 \tag{5}$$

$$\varrho \frac{\partial \bar{q}}{\partial t} = -\nabla p - \mu \underline{\nabla x (\nabla x \bar{q})} + K_1 N (\bar{q}_s - \bar{q}) \tag{6}$$

$$m \frac{\partial \bar{q}_s}{\partial t} = -K_1 (\bar{q}_s - \bar{q}) \tag{7}$$

$$\frac{\partial N}{\partial t} + \nabla \cdot (N \bar{q}_s) = 0 \tag{8}$$

The number density N is assumed to be constant and in view of this Eq. (8) takes the form

$$\operatorname{div} \bar{q}_s = 0. \tag{9}$$

Let (α, β, z) be a curvilinear system with z axis along the axis of the cylinder and (α, β) a plane elliptic coordinate system.

Let h_α, h_β, $h_z(=1)$ be the scale factors and $\bar{e}_\alpha$, $\bar{e}_\beta$, $\bar{e}_z$ the unit vectors along α, β, z directions. The velocity vector appropriate to the problem is

$$\bar{q} = u(\alpha, \beta, t)\,\bar{e}_\alpha + v(\alpha, \beta, t)\,\bar{e}_\beta \tag{10}$$

and we introduce the stream function $\psi(\alpha, \beta, t)$ through

$$h_\beta u = -\frac{\partial \psi}{\partial \beta},\ h_\alpha v = \frac{\partial \psi}{\partial \alpha} \tag{11}$$

The particle phase velocity vector can be assumed to be

$$\bar{q}_s = u_s\,\bar{e}_\alpha + v_s\,\bar{e}_\beta\,. \tag{12}$$

Using (10), (11) and (12) in the Eq. (6), the linearized versions of the equations of motion are

$$\varrho h \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial \alpha} - \mu \frac{\partial}{\partial \beta}(\nabla_1^2 \psi) + K_1 N (u_s - u) \tag{13}$$

$$\varrho h \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial \beta} + \mu \frac{\partial}{\partial \alpha}(\nabla_1^2 \psi) + K_1 N (v_s - v) \tag{14}$$

in which

$$h = h_\alpha = h_\beta\,;\ \nabla_1^2 = \frac{1}{h^2}\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right). \tag{15}$$

Since the oscillation of the cylinder is harmonic, we may take

$$\{\psi(\alpha, \beta, t), p(\alpha, \beta, t), u_s(\alpha, \beta, t), v_s(\alpha, \beta, t)\} = \{F(\alpha, \beta), P(\alpha, \beta), U_s(\alpha, \beta), V_s(\alpha, \beta)\} \exp(i\sigma t)\,. \tag{16}$$

Using Eq. (16) in (13) and (14)

$$\frac{\partial P}{\partial \alpha} = i\varrho\sigma \frac{\partial F}{\partial \beta} - \mu \frac{\partial}{\partial \beta}(\nabla_1^2 F) + K_1 N (U_S - U) \tag{17}$$

$$\frac{\partial P}{\partial \beta} = -i\varrho\sigma \frac{\partial F}{\partial \alpha} + \mu \frac{\partial}{\partial \alpha}(\nabla_1^2 F) + K_1 N (V_S - V)\,. \tag{18}$$

Eliminating U_S and V_S from (17), (18) using (7), we get

$$\nabla_1^2(\nabla_1^2 - a^2) F(\alpha, \beta) = 0 \tag{19}$$

where

$$a^2 = \frac{(1+f)\,i\sigma - \tau\sigma^2}{\nu + i\nu\sigma\tau},\quad f = \frac{Nm}{\varrho}. \tag{20}$$

It is to be noted that the parameter a^2 in (20) is complex and for no values of the material parameters can be purely imaginary.

The operators ∇_1^2 and $(\nabla_1^2 - a^2)$ that appear in Eq. (19) commute and Eq. (19) is linear and homogeneous. It can be checked directly that if $F_0(\alpha, \beta)$ and $F_1(\alpha, \beta)$ are solutions of

$$\nabla_1^2 F_0 = 0 \tag{21}$$

$$(\nabla_1^2 - a^2) F_1 = 0 \tag{22}$$

respectively, then

$$F = F_0 + F_1 \tag{23}$$

is a solution of (19).

Conversely, taking

$$(\nabla_1^2 - a^2) F = G \tag{24}$$

(19) becomes

$$\nabla_1^2 G = 0 . \tag{25}$$

If $G = F_0$ is the general solution of (25) and F_1 is the general solution of $(\nabla_1^2 - a^2) F_1 = 0$, the solution of (24) is seen to be

$$F = F_1 - \frac{F_0}{a^2} \tag{26}$$

and there is no loss of generality of F is written as $F_0 + F_1$ (as in (23)).

Thus the solution of (19) can be written in the form

$$F = F_0 + F_1 \tag{27}$$

where F_0 and F_1 are solutions of

$$\nabla_1^2 F_0 = 0; \; (\nabla_1^2 - a^2) F_1 = 0 . \tag{28}$$

The arbitrary constants that appear in $F(\alpha, \beta)$ are to be determined using the no slip condition on the boundary of the oscillating body and the regularity conditions far away from the body. It is to be noted that the boundary and regularity conditions are to be implemented on $F = F_0 + F_1$ and not on F_0 and F_1 individually. The implementation of these conditions will be discussed at appropriate places later in the paper.

Oscillations parallel to major axis

We define the elliptic coordinates (α, β) by the relation

$$x + iy = c \cosh(\alpha + i\beta) \tag{29}$$

and assume that the cross-section of the cylinder is given by $\alpha = \alpha_0$. The scale factors for the frame are

$$h_\alpha = h_\beta = h = c(\cosh^2\alpha - \cos^2\beta)^{1/2}. \tag{30}$$

Assuming that the cylinder is performing harmonic oscillations parallel to the major axis, the condition of no-slip on the boundary and the regularity condition far away from the body lead to

$$\begin{aligned} u(\alpha, \beta, t) &= \frac{u^c}{h}\sinh\alpha\cdot\cos\beta\exp(i\sigma t) \\ v(\alpha, \beta, t) &= -\frac{u^c}{h}\cosh\alpha\cdot\sin\beta\exp(i\sigma t) \end{aligned} \tag{31}$$

on $\alpha = \alpha_0$ and

$$u, v \to 0 \quad \text{as} \quad \alpha \to \infty. \tag{32}$$

These are equivalent to the conditions

$$\left.\begin{aligned} F(\alpha, \beta) &= -\,u^c\sinh\alpha\cdot\sin\beta \\ \frac{\partial F}{\partial\alpha} &= -\,u^c\cosh\alpha\cdot\sin\beta \end{aligned}\right\} \quad \text{on } \alpha = \alpha_0 \tag{33}$$

and

$$F(\alpha, \beta), \frac{\partial F}{\partial\alpha}(\alpha, \beta) \to 0 \quad \text{as} \quad \alpha \to \infty. \tag{34}$$

The conditions (33) and (34) suggest the solution of (21) in the form

$$F_0(\alpha, \beta) = \sum_{n=1}^{\infty} C_n \exp(-n\alpha)\sin n\beta. \tag{35}$$

The Eq. (22) can be rewritten in the form

$$\left\{\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} - \frac{a^2c^2}{2}(\cosh 2\alpha - \cos 2\beta)\right\} F_1 = 0. \tag{36}$$

Writting

$$F_1(\alpha, \beta) = R(\alpha)\,S(\beta) \tag{37}$$

we find that R and S satisfy the Mathieu differential equations [12, 13]

$$R''(\alpha) - \left\{\lambda + \frac{a^2c^2}{2}\cosh 2\alpha\right\} R(\alpha) = 0 \tag{38}$$

$$S''(\beta) + \left\{\lambda + \frac{a^2c^2}{2}\cos 2\beta\right\} S(\beta) = 0 \tag{39}$$

where λ is a constant of separation. The Eq. (39) has the periodic solution

$$ce_m\left(\beta, -\frac{a^2 c^2}{4}\right), \quad se_m\left(\beta, -\frac{a^2 c^2}{4}\right) \tag{40}$$

corresponding to a discrete set of values of λ which are functions of $a^2 c^2/4$. For the present problem we need only the solutions se_m and these correspond to the characteristic numbers $\lambda = a_{2m+1}$, $\lambda = b_{2m+2}$. These functions are given by the Fourier sine series expansions

$$se_{2m+1}\left(\beta, -\frac{a^2 c^2}{4}\right) = \sum_{r=0}^{\infty} (-1)^{m+r} A_{2r+1}^{(2m+1)} \sin(2r+1)\beta \tag{41}$$

$$se_{2m+2}\left(\beta, -\frac{a^2 c^2}{4}\right) = \sum_{r=0}^{\infty} (-1)^{m+r} B_{2r+2}^{(2m+2)} \sin(2r+2)\beta \tag{42}$$

In these expansions, the coefficients A and B are functions of the parameter $\left(-\frac{a^2 c^2}{4}\right)$.

The solutions of the modified Mathieu Eq. (38) corresponding to the solution given in (41), (42) and which vanish as $\alpha \to \infty$ are given by [12]

$$\begin{aligned} Gek_{2m+1}&\left(\alpha, -\frac{a^2 c^2}{4}\right) \\ &= \frac{P'_{2m+1}}{\pi A_1^{(2m+1)}} \sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)} \left\{ I_r\left(\frac{ace^{-\alpha}}{2}\right) K_{r+1}\left(\frac{ace^{\alpha}}{2}\right) \right. \\ &\quad \left. + I_{r+1}\left(\frac{ace^{-\alpha}}{2}\right) K_r\left(\frac{ace^{\alpha}}{2}\right) \right\} \end{aligned} \tag{43}$$

$$\begin{aligned} Gek_{2m+2}&\left(\alpha, -\frac{a^2 c^2}{4}\right) \\ &= \frac{S'_{2m+2}}{\pi B_2^{(2m+2)}} \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} \left\{ I_r\left(\frac{ace^{-\alpha}}{2}\right) K_{r+2}\left(\frac{ace^{\alpha}}{2}\right) \right. \\ &\quad \left. - I_{r+2}\left(\frac{ace^{-\alpha}}{2}\right) K_r\left(\frac{ace^{\alpha}}{2}\right) \right\} \end{aligned} \tag{44}$$

in which I and K are modified Bessel functions.

Now the function $F_1(\alpha, \beta)$ is given by

$$F_1(\alpha, \beta) = \sum_{n=1}^{\infty} D_n \, Gek_n\left(\alpha, -\frac{a^2 c^2}{4}\right) se_n\left(\beta, -\frac{a^2 c^2}{4}\right) \tag{45}$$

and the solution of Eq. (19) is obtained by superposing the solutions F_0 and F_1 given in (35) and (45) respectively. Thus we have

$$F(\alpha, \beta) = \sum_{n=1}^{\infty} C_n \exp(-n\alpha) \sin n\beta + \sum_{n=1}^{\infty} D_n \, Gek_n\left(\alpha, -\frac{a^2 c^2}{4}\right) se_n\left(\beta, -\frac{a^2 c^2}{4}\right). \tag{46}$$

The constants $\{C_n\}$ and $\{D_n\}$ in (46) can be determined using the boundary conditions (33). This determination is facilitated by writing

$$Gek_m\left(\alpha, -\frac{a^2 c^2}{4}\right) se_m\left(\beta, -\frac{a^2 c^2}{4}\right) = \sum_{n=1}^{\infty} F_{mn}(\alpha) \sin n\beta \tag{47}$$

and using the expansion of se_{2m+1}, se_{2m+2} given in (41), (47), we notice that

$$\begin{aligned} F_{2m+1,n}(\alpha) &= 0 \quad n = 2, 4, 6, \ldots \\ &= (-1)^{m+r} A_{2r+1}^{(2m+1)} \, Gek_{2m+1}\left(\alpha, -\frac{a^2 c^2}{4}\right) \\ & \quad n = 2r+1, \quad r = 0, 1, 2, \ldots \end{aligned} \tag{48}$$

and

$$\begin{aligned} F_{2m+2,n}(\alpha) &= 0 \quad n = 1, 3, 5, \ldots \\ &= (-1)^{m+r} B_{2r+2}^{(2m+2)} \, Gek_{2m+2}\left(\alpha, -\frac{a^2 c^2}{4}\right) \\ & \quad n = 2r+2, \quad r = 0, 1, 2, \ldots . \end{aligned} \tag{49}$$

Thus

$$F(\alpha, \beta) = \sum_{n=1}^{\infty} \left\{ C_n \exp(-n\alpha) + \sum_{m=1}^{\infty} D_m F_{mn}(\alpha) \right\} \sin n\beta . \tag{50}$$

Using the boundary conditions (33), we get the following system of linear algebraic equations in $\{C_n\}$, $\{D_n\}$:

$$C_n \exp(-n\alpha_0) + \sum_{m=1}^{\infty} D_m F_{mn}(\alpha_0) = -Uc \sinh \alpha_0 \, \delta_{n1} \tag{51}$$

$$-n C_n \exp(-n\alpha_0) + \sum_{m=1}^{\infty} D_m F'_{mn}(\alpha_0) = -Uc \cosh \alpha_0 \, \delta_{n1} . \tag{52}$$

Eliminating C_n from (51), (52), we get the following infinite nonhomogeneous system of linear equations in the unknowns $\{D_m\}$;

$$\sum_{m=1}^{\infty} \{ n F_{mn}(\alpha_0) + F'_{mn}(\alpha_0) \} D_m = -Uc \exp(\alpha_0) \, \delta_{n1} . \tag{53}$$

This leads to the determination of the constants $\{D_m\}$ and employing later either (51) or (52), it is possible to determine $\{C_n\}$.

Pressure distribution

The pressure p is given by

$$p(\alpha, \beta, t) = P(\alpha, \beta) \exp(i \sigma t) \tag{54}$$

and P can be determined from (17) and (18). Using (7) in (17) and (18) and noting that F is given by (50), solving the equations we get

$$P(\alpha, \beta) = -\mu a^2 \Sigma C_n \exp(-n\alpha) \cos n\beta . \tag{55}$$

Drag on the cylinder

The components of the stress tensor can be obtained from the relation

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu e_{ij} . \tag{56}$$

The nonvanishing components of the stress are $\tau_{\alpha\alpha}$, $\tau_{\alpha\beta}$, $\tau_{\beta\alpha}$, $\tau_{\beta\beta}$ and τ_{zz}. The stress vector is

$$\tau_{\alpha\alpha} \bar{e}_\alpha + \tau_{\alpha\beta} \bar{e}_\beta + \tau_{\alpha z} \bar{e}_z \tag{57}$$

and the drag per length L of the cylinder is given by

$$D = c L \int_0^{2\pi} \{\tau_{\alpha\alpha} \sinh\alpha \cdot \cos\beta - \tau_{\alpha\beta} \cosh\alpha \cdot \sin\beta\}_{\alpha=\alpha_0} \, d\beta . \tag{58}$$

Evaluating the stress components on the boundary $\alpha = \alpha_0$ and integrating the expression in (58), we have

$$D = \pi \mu a^2 c L \exp(i\sigma t) \{C_1 + U c \sinh\alpha_0 \cdot \cosh\alpha_0\} . \tag{59}$$

Limiting case

By allowing α_0 to zero, the oscillating elliptic cylinder reduces to a flat plate harmonically oscillating along its edge. The stream function $\psi(\alpha, \beta, t)$ and the constants C_n and D_n can be determined as before. The drag on the flat plate is seen to be

$$\pi \mu a^2 c L \exp(i\sigma t) C_1 . \tag{60}$$

Oscillations parallel to minor axis

This case can be treated similar to the case of oscillations parallel to major axis and we briefly state results below:

We introduce the elliptic coordinates (α, β) by

$$x + i y = c \sinh(\alpha + i\beta) \tag{61}$$

and in this case

$$h = c\{\sinh^2\alpha + \cos^2\beta\}^{1/2}. \tag{62}$$

On the cylinder $\alpha = \alpha_0$

$$\begin{aligned} F(\alpha, \beta) &= -Uc\cosh\alpha\cdot\sin\beta \\ \frac{\partial F}{\partial\alpha}(\alpha, \beta) &= -Uc\sinh\alpha\cdot\sin\beta \end{aligned} \tag{63}$$

The appropriate solution of Eq. (21) is

$$F_0(\alpha, \beta) = \sum_{n=1}^{\infty} C_n^* \exp(-n\alpha)\sin n\beta. \tag{64}$$

The solution $F_1(\alpha, \beta)$ of (22) is taken in the form $R(\alpha)\,S(\beta)$ and we have

$$R''(\alpha) - \left\{\lambda + \frac{a^2c^2}{2}\cosh 2\alpha\right\} R(\alpha) = 0 \tag{65}$$

$$S''(\beta) + \left\{\lambda - \frac{a^2c^2}{2}\cos 2\beta\right\} S(\beta) = 0. \tag{66}$$

The solutions of (66) are $\left\{ce_m\left(\beta, \frac{a^2c^2}{4}\right), se_m\left(\beta, \frac{a^2c^2}{4}\right)\right\}$ corresponding to a discrete set of characteristic values of the separation constant λ.

In this problem we need only the functions se_m corresponding to the characteristic numbers $\lambda = b_{2m+1}$ and $\lambda = b_{2m+2}$. These have the Fourier sine series expansions

$$se_{2m+1}\left(\beta, \frac{a^2c^2}{4}\right) = \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)} \sin(2r+1)\beta \tag{67}$$

$$se_{2m+2}\left(\beta, \frac{a^2c^2}{4}\right) = \sum_{r=0}^{\infty} B_{2r+2}^{(2m+1)} \sin(2r+1)\beta. \tag{68}$$

The solutions of (65) which correspond to the above solutions and which vanish as $\alpha \to \infty$ are the modified Mathieu functions Gek^*_{2m+1}, Gek^*_{2m+2} given by

$$Gek^*_{2m+1}\left(\alpha, -\frac{a^2c^2}{4}\right) = \frac{s'_{2m+1}}{\pi B_1^{(2m+1)}} \cdot \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)} \left\{ I_r\left(\frac{ace^{-\alpha}}{2}\right) K_{r+1}\left(\frac{ace^{\alpha}}{2}\right) + I_{r+1}\left(\frac{ace^{-\alpha}}{2}\right) K_r\left(\frac{ace^{\alpha}}{2}\right) \right\} \tag{69}$$

$$Gek^*_{2m+2}\left(\alpha, -\frac{a^2c^2}{4}\right) = \frac{s'_{2m+2}}{\pi B_2^{(2m+2)}} \cdot \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} \left\{ I_r\left(\frac{ace^{-\alpha}}{2}\right) K_{r+2}\left(\frac{ace^{\alpha}}{2}\right) - I_{r+2}\left(\frac{ace^{-\alpha}}{2}\right) K_r\left(\frac{ace^{\alpha}}{2}\right) \right\} \tag{70}$$

The solution $F_1(\alpha, \beta)$ for the Eq. (22) is given by

$$F_1(\alpha, \beta) = \sum_{n=1}^{\infty} D_n^* \, Gek_n^*\left(\alpha, -\frac{a^2c^2}{4}\right) se_n\left(\beta, \frac{a^2c^2}{4}\right) \tag{71}$$

and as before

$$F(\alpha, \beta) = F_0 + F_1 = \sum_{n=1}^{\infty} C_n^* \exp(-n\alpha) \sin n\beta + \sum_{n=1}^{\infty} D_n^* \, Gek_n^*\left(\alpha, -\frac{a^2c^2}{4}\right) se_n\left(\beta, \frac{a^2c^2}{4}\right). \tag{72}$$

Here again by introducing the functions $F_{mn}^*(\alpha)$ as before we recast $F(\alpha, \beta)$ as below:

$$F(\alpha, \beta) = \sum_{n=1}^{\infty} \left\{ C_n^* \exp(-n\alpha) + \sum_{m=1}^{\infty} D_m^* F_{mn}^*(\alpha) \right\} \sin n\beta \tag{73}$$

where the functions $F^*_{mn}(\alpha)$ are defined by

$$F^*_{2m+1,n}(\alpha) = 0 \quad n = 2, 4, 6, \ldots$$
$$= B^{(2m+1)}_{2r+1}\, Gek^*_{2m+1}\left(\alpha, -\frac{a^2c^2}{4}\right)$$
$$n = 2r+1; \quad r = 0, 1, 2, \ldots \tag{74}$$

$$F^*_{2m+2,n}(\alpha) = 0 \quad n = 1, 3, 5, \ldots$$
$$= B^{(2m+2)}_{2r+2}\, Gek^*_{2m+2}\left(\alpha, -\frac{a^2c^2}{4}\right)$$
$$n = 2r+2, \quad r = 0, 1, 2, \ldots. \tag{75}$$

Using the boundary conditions in (63), the equations that lead to the determination of $\{C^*_n\}$ and $\{D^*_n\}$ are

$$C^*_n \exp(-n\alpha_0) + \sum_{m=1}^{\infty} D^*_m F^*_{mn}(\alpha_0) = -Uc\cosh\alpha_0\,\delta_{nl} \tag{76}$$

$$-nC^*_n \exp(-n\alpha_0) + \sum_{m=1}^{\infty} D^*_m F^{*\prime}_{mn}(\alpha_0) = -Uc\sinh\alpha_0\,\delta_{nl} \tag{77}$$

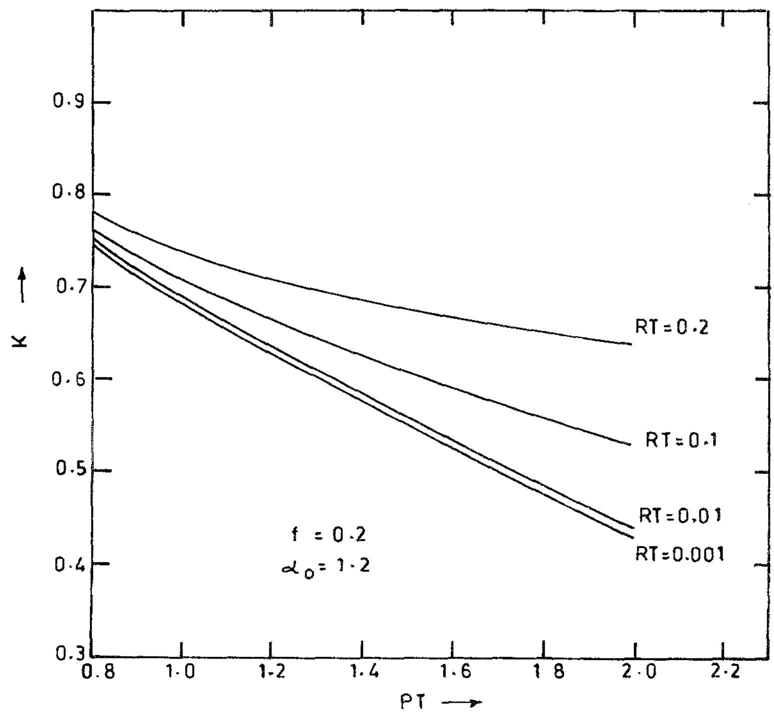

Figure 1
Variation of K (major axis).

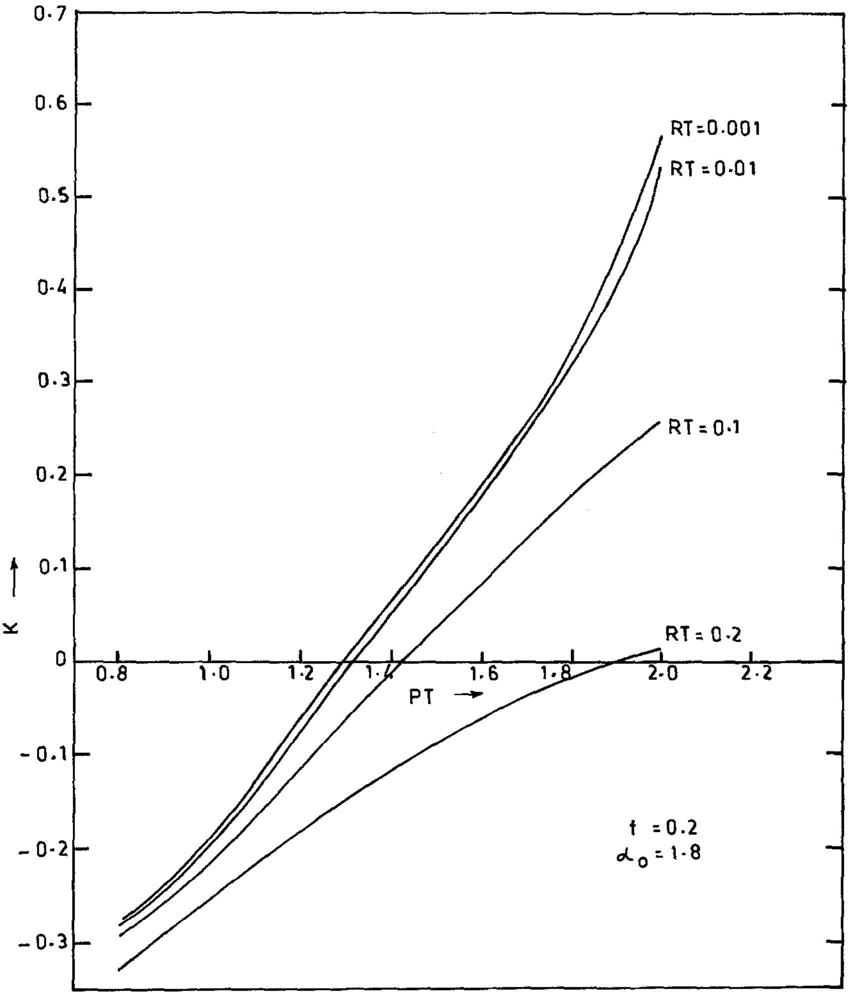

Figure 2
Variation of K (major axis).

Eliminating C_n^* from (76), (77), we get

$$\sum_{m=1}^{\infty} \{n F_{mn}^*(\alpha_0) + F_{mn}^{*\prime}(\alpha_0)\} D_m^* = - U c \exp(\alpha_0) \delta_{n1} \tag{78}$$

and in principle we can determine D_m^* and hence later C_n^* using either of (76) or (77). Thus the stream function is completely determinable.

Pressure Distribution: The pressure

$p(\alpha, \beta, t) = p(\alpha, \beta) \exp(i \sigma t)$ is seen to be

$$p(\alpha, \beta, t) = - \mu a^2 \{\Sigma C_n^* \exp(-n\alpha) \cos n\beta\} \exp(i \sigma t) \tag{79}$$

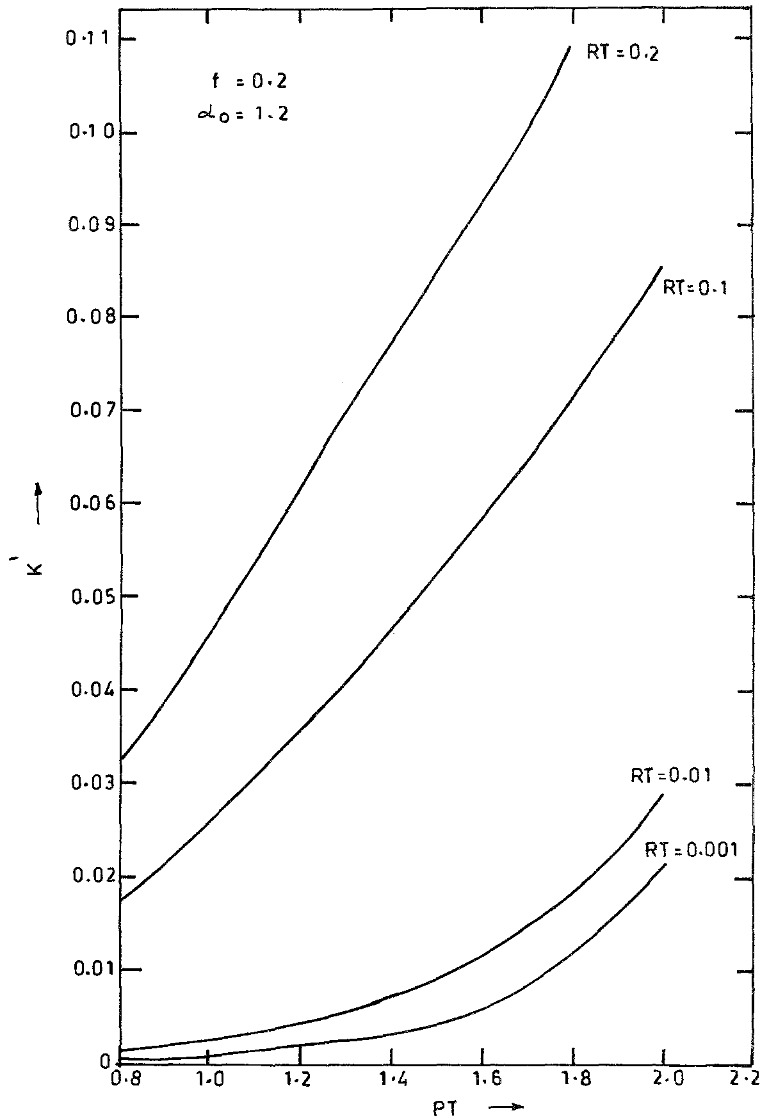

Figure 3
Variation of K' (major axis).

Drag on the cylinder

The drag on the cylinder per length L is given by

$$D = c\,L \int_0^{2\pi} \{\tau_{\alpha\alpha} \cosh\alpha \cdot \cos\beta - \tau_{\alpha\beta} \sinh\alpha \cdot \sin\beta\}_{\alpha=\alpha_0}\, d\beta \tag{80}$$

and this reduces to

$$\pi\,\mu\,a^2\,c\,L \exp(i\,\sigma\,t)\,\{C_1^* + U\,c\,\sinh\alpha_0 \cosh\alpha_0\} \tag{81}$$

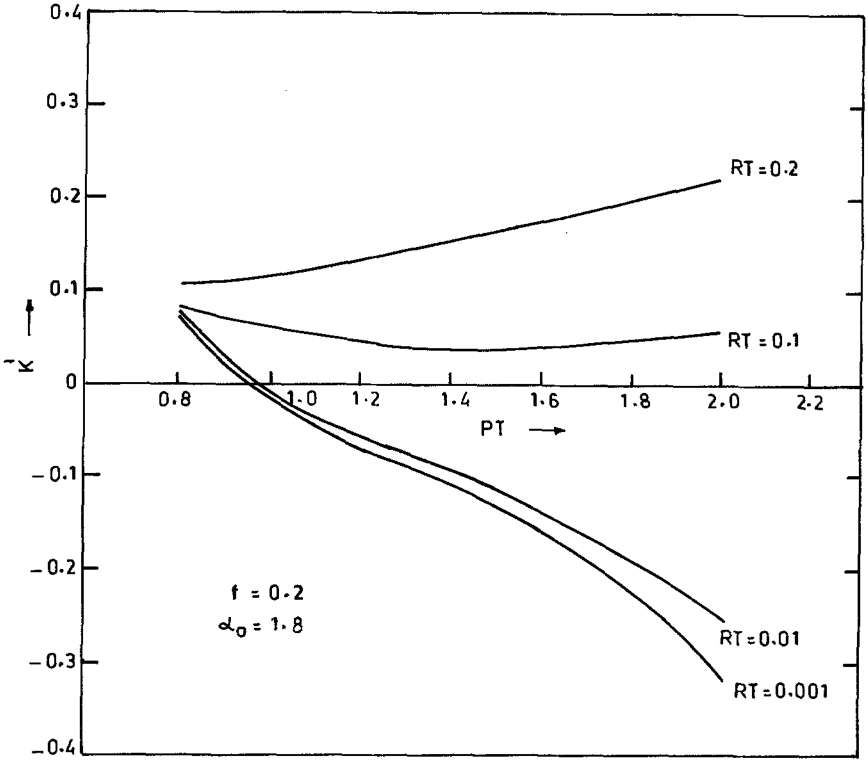

Figure 4
Variation of K' (major axis).

Limiting case

By allowing α_0 to zero we can recover the case of a flat plate performing harmonic oscillations transverse to its plane. Taking $\alpha_0 = 0$ and solving the system of Eqs. (76), (77) we can determine C_n^*, D_n^* and write the stream function. The drag in this case is obtained as

$$\pi \mu a^2 c L C_1^* \exp(i \sigma t). \tag{82}$$

Numerical work

In both the cases of the oscillations of the elliptic cylinder parallel to the major axis and minor axis the drag on the body can be expressed as

$$D = -M U \left(\frac{\mu a^2}{\varrho}\right)(iK + K') \exp(i \sigma t) \tag{83}$$

where M is the mass of the fluid displaced by the cylinder of length L and is given by

$$M = \pi \varrho L c^2 \cosh \alpha_0 \sinh \alpha_0. \tag{84}$$

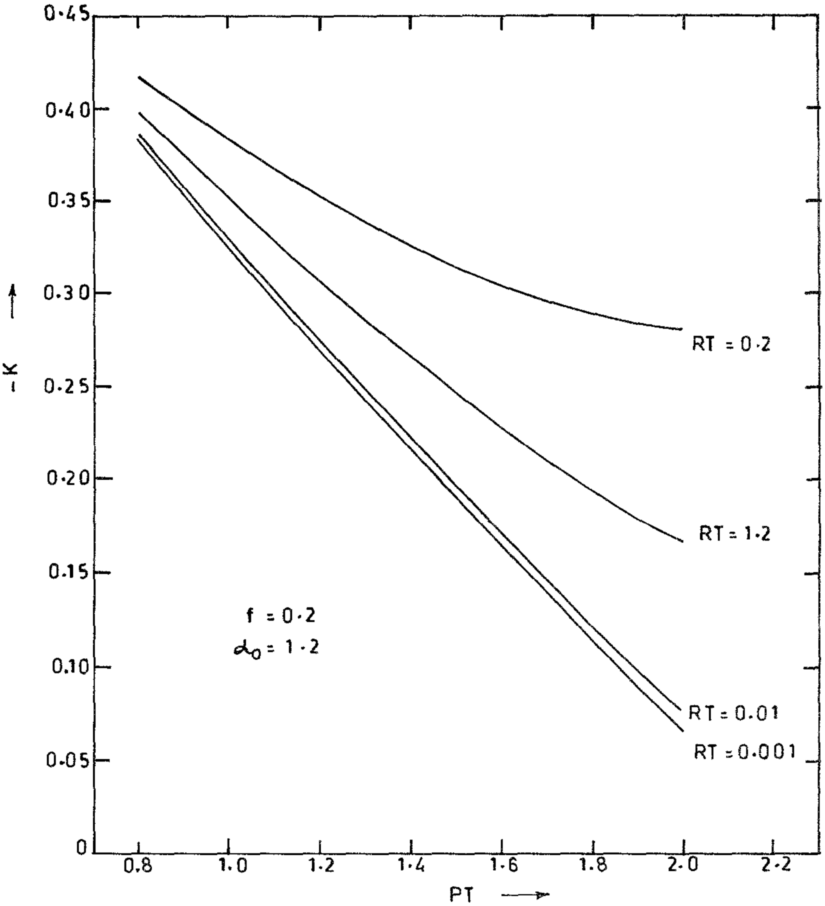

Figure 5
Variation of K (minor axis).

The variation of the drag parameters K and K' introduced in (83) is studied numerically.

The numerical determination of K and K' involves the determination of C_n, D_n and C_n^*, D_n^*. The system of Eqs. (53) involving D_m is an infinite one and we have to necessarily truncate the system suitable to get solution. The elements of the coefficient matrix involve modified Mathieu functions which have infinite series expansions involving the modified Bessel functions I_r and K_r as well as the coefficients A_n^m, B_n^m [12, 13].

The eigenvalue parameter λ present in (38), (39) has an infinite series expansion in powers of (ca. /2) and the leading terms are available in the classic work of Mch Lachlan [12]. We have evaluated λ using all the terms given therein. The functions $F_{mn}(\alpha)$ and $F'_{mn}(\alpha)$ are needed for odd values of m and n only and accordingly these are evaluated for $m, n = 0, 1, 3, 5, 7, 9$. Any attempt to truncate

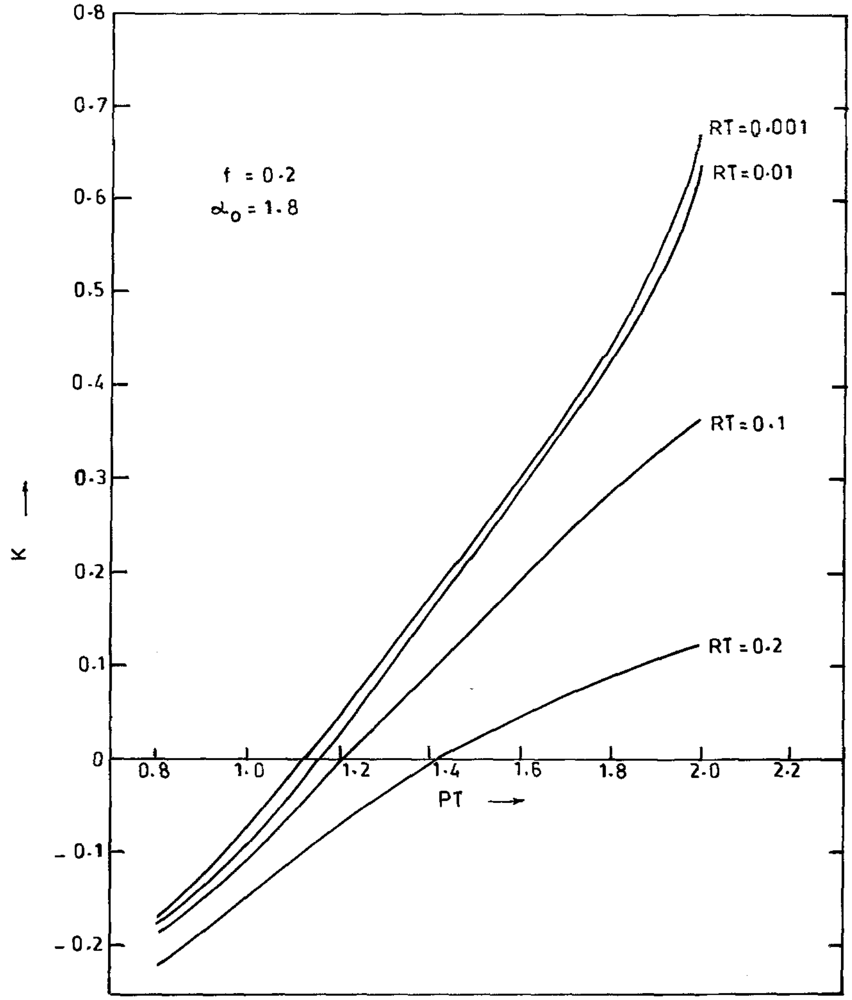

Figure 6
Variation of K (minor axis).

the system at a higher order stage will involve the numerical evaluation of an enlarged set of constants A_n^m, B_n^m and functions $Ge k_n$.

The case of minor axis is also dealt with on the same lines.

The figures show the variation of K and K' for various frequency parameter values $PT = \varrho \sigma c^2/\mu$, relaxation time parameter $RT = \sigma \tau$ and mass concentration $f = 0.2$.

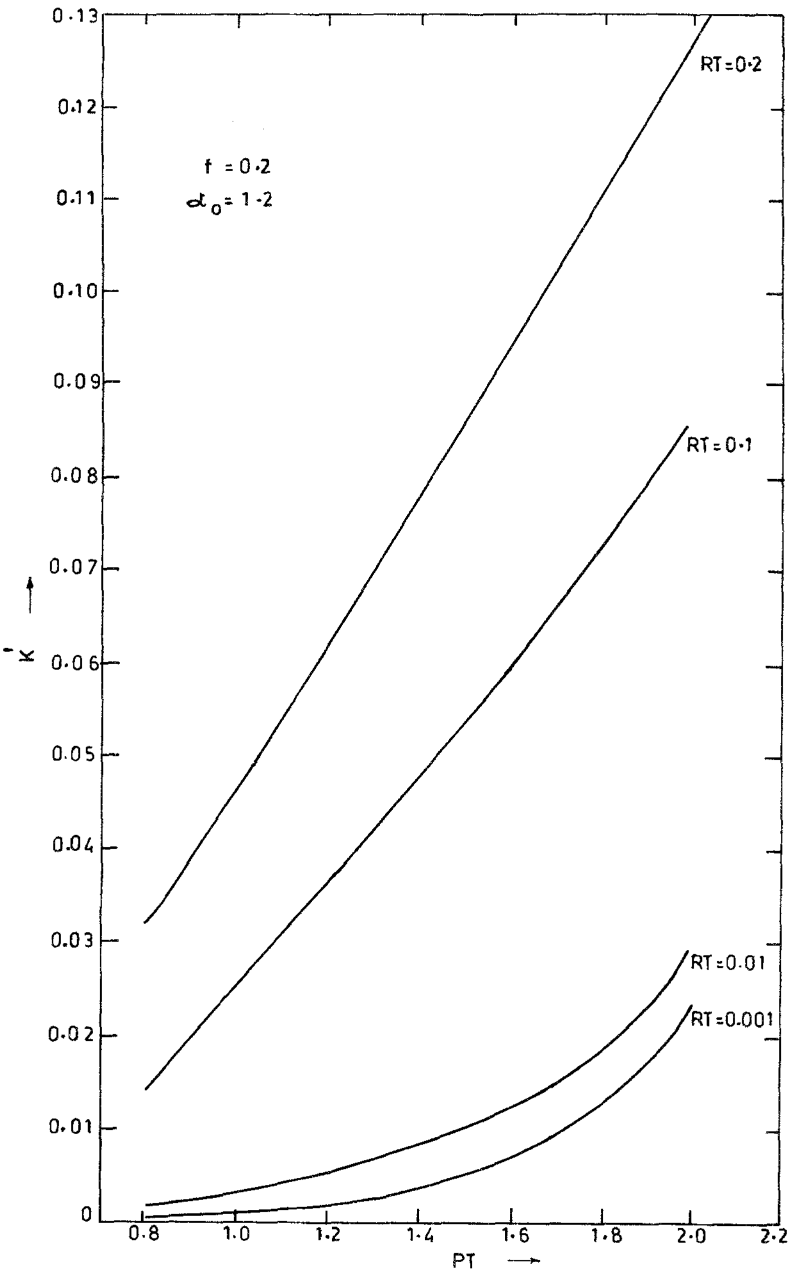

Figure 7
Variation of K' (minor axis).

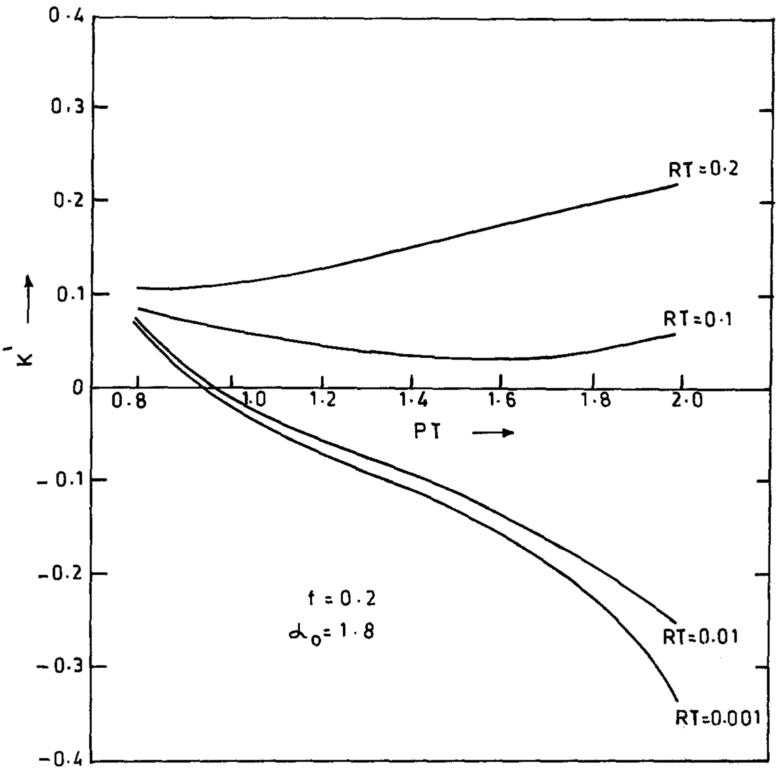

Figure 8
Variation of K' (minor axis).

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Abstract

An elliptic cylinder is performing oscillations parallel to either of the principal axes of the cross-sectional ellipse in a fluid particle suspension. The stream function governing the flow and the velocity components are determined in terms of Mathieu functions. The drag on the cylinder is evaluated and expressed in terms of two parameters K and K'. The effects of the variation of the frequency parameter, eccentricity parameter and relaxation time parameter on the drag parameters K, K' is studied numerically.

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