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THE SLOW STATIONARY FLOW OF INCOMPRESSIBLE MICROPOLAR FLUID PAST A SPHEROID

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Abstract—The paper examines the slow stationary flow of incompressible micropolar fluid past a spheroid (prolate and oblate) adopting the Stokesian approximation, so that the inertial terms in the momentum equation and the bilinear terms in the balance of first stress moments are neglected. The flow over the space outside the body is analyzed and the velocity, microrotation, stress and couple stress are obtained analytically in infinite series form. The drag on the body is determined and it is observed that there is no couple exerted on the body. Numerical studies are undertaken to see the variation of the drag with respect to the geometric as well as the physical flow parameters. These have been presented in the form of figures. Micropolarity of the fluid has an augmenting effect on the drag. In an Appendix, an alternative method of determining the drag is indicated.

1. INTRODUCTION

THE THEORY of micropolar fluids initiated by Eringen[1] is a subclass of the theory of simple microfluids[2] initiated earlier by Eringen himself. In the micropolar fluid theory, apart from the classical field of velocity, there are two additional field variables, viz. the microrotation vector $\boldsymbol{\nu}$ and the gyration parameter j, introduced to explain the kinematics of micromotions. The microrotation vector represents the rotation of the rigid particles in a small volume element about the centroid of the element in an average sense. This is local in character and is in addition to the usual rigid body motion of the entire volume element. The theory departs from the classical Navier–Stokes model of viscous fluids in the following two aspects: (i) sustenance of couple stress in the fluids; (ii) the nonsymmetry of the stress tensor.

The field equations of micropolar fluid dynamics are

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{q}) = 0, \tag{1.1}$$

$$\rho \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \rho \mathbf{f} - \operatorname{grad} p + k \operatorname{curl} \bar{\nu} - (\mu + k) \operatorname{curl} \operatorname{curl} \mathbf{q} + (\lambda_1 + 2\mu + k) \operatorname{grad} (\operatorname{div} \mathbf{q}), \tag{1.2}$$

$$\rho j \frac{\mathrm{d}\bar{\nu}}{\mathrm{d}t} = \rho \mathbf{l} - 2k\bar{\nu} + k \operatorname{curl} \mathbf{q} - \gamma \operatorname{curl} \operatorname{curl} \bar{\nu} + (\alpha + \beta + \gamma) \operatorname{grad} (\operatorname{div} \bar{\nu}) \tag{1.3}$$

in which $\mathbf{q}$, $\bar{\nu}$, $\mathbf{f}$, $\mathbf{l}$ are, respectively, the velocity, microrotation, body force and body couple vectors per unit mass and p is the pressure. The constants ρ and j are density and gyration parameters while $\{\lambda_1, \mu, k\}$ and $\{\alpha, \beta, \gamma\}$ are material constants which are governed by the inequalities

$$k \geq 0; \quad 2\mu + k \geq 0; \quad 3\lambda_1 + 2\mu + k \geq 0; \quad \gamma \geq 0; \quad 3\alpha + \beta + \gamma \geq 0; \quad |\beta| \leq \gamma. \tag{1.4}$$

$\{\lambda_1, \mu, k\}$ may be called the viscosity coefficients while $\{\alpha, \beta, \gamma\}$ the gyroviscosity coefficients.

The stress tensor t_{ij} and the couple-stress tensor $m_{ij} = -\epsilon_{jpq}\lambda_i^{pq}$ are given by

$$t_{ij} = (-p + \lambda_1 \operatorname{div} \mathbf{q})\delta_{ij} + (2\mu + k)e_{ij} + k\epsilon_{ijm}(\omega^m - \nu^m), \tag{1.5}$$

$$m_{ij} = (\alpha \operatorname{div} \bar{\nu})\delta_{ij} + \beta \nu_{i,j} + \gamma \nu_{j,i}. \tag{1.6}$$

In (1.5) and (1.6) ν_i and $2\omega_i$ are the components of the microrotation vector and vorticity vector, respectively, e_{ij} denote the rate of strain components and comma denotes a covariant differentiation.

In this paper we examine the slow stationary flow of an incompressible micropolar fluid past a spheroid. As is usual with the classical investigations of the problem, as a first step, the inertial terms of the momentum equation and the bilinear terms in the balance of first stress moments are neglected and the flow over the space outside the body is obtained under the above (Stokesian) approximation. The velocity, microrotation, stress and couple stress are all analytically obtained in infinite series form and the drag on the body is determined. We see that the body, however, does not experience any couple. The variation of the drag as well as the drag ratio with respect to the geometric and physical parameters of the flow is examined numerically.

2. FLOW EQUATIONS IN AN AXIALLY SYMMETRIC FRAME

Let $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_z$ be unit base vectors of the cylindrical polar system (r, ϕ, z). The flow past the spheroid has a uniform stream at infinity and the flow of the fluid is in the meridian plane. All physical quantities are independent of ϕ. The velocity and microrotation vectors can, therefore, be presumed in the form

$$\mathbf{q} = u(r, z)\mathbf{e}_r + w(r, z)\mathbf{e}_z, \tag{2.1}$$

$$\bar{\nu} = B(r, z)\mathbf{e}_\phi \tag{2.2}$$

and the stream function $\Psi(r, z)$ can be introduced such that

$$ru = \frac{\partial \Psi}{\partial z}, \qquad rw = -\frac{\partial \Psi}{\partial r}. \tag{2.3}$$

The equations governing the flow are, therefore, given by

$$\frac{\partial p}{\partial r} = -k\frac{\partial B}{\partial z} + (\mu + k)\left(\nabla^2 - \frac{1}{r^2}\right)u, \tag{2.4}$$

$$\frac{\partial p}{\partial z} = k\left(\frac{\partial B}{\partial r} + \frac{B}{r}\right) + (\mu + k)\nabla^2 w, \tag{2.5}$$

$$-2kB + k\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) + \gamma\left(\nabla^2 - \frac{1}{r^2}\right)B = 0 \tag{2.6}$$

where the operator ∇^2 is the Laplacian given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{2.7}$$

Eliminating p between (2.4) and (2.5) and using (2.3) we see that

$$k\left(\nabla^2 - \frac{1}{r^2}\right)B = (\mu + k)\frac{E^4\Psi}{r} \tag{2.8}$$

where the Stokesian stream function operator E^2 is given by

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{2.9}$$

The eqn (2.6) now assumes the form

$$2kB = k\frac{E^2\Psi}{r} + \gamma\left(\nabla^2 - \frac{1}{r^2}\right)B \tag{2.10}$$

so that from (2.8) and (2.10) we have

$$B = \frac{1}{2r}\left[E^2\Psi + \frac{\gamma(\mu + k)}{k^2}E^4\Psi\right]. \tag{2.11}$$

Eliminating B between (2.8) and (2.11) and utilizing the identity

$$\left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{f}{r}\right) = \frac{1}{r}E^2 f \tag{2.12}$$

we see that

$$\left(E^6 - \frac{\lambda^2}{c^2}E^4\right)\Psi = 0 \tag{2.13}$$

where

$$\frac{\lambda^2}{c^2} = \frac{k(2\mu + k)}{\gamma(\mu + k)}. \tag{2.14}$$

Thus, the problem of the slow stationary flow of an incompressible micropolar fluid past a spheroid with a uniform stream parallel to the axis of symmetry at infinity is governed by the system of partial differential equations

$$\left(E^6 - \frac{\lambda^2}{c^2}E^4\right)\Psi = 0 \tag{2.15}$$

and

$$B = \frac{1}{2r}\left[E^2\Psi + \frac{\gamma(\mu + k)}{k^2}E^4\Psi\right]. \tag{2.16}$$

The determination of the velocity and microrotation fields will be possible if the above two equations are supplemented by appropriate conditions relevant to the problem at the boundary and at infinity. We presume that at infinity, the flow is a uniform stream and the microrotation must vanish there. At the boundary Γ of the solid we presume the hyperstick condition of adherence so that

$$\mathbf{q}(\mathbf{x}_\Gamma, t) = \mathbf{q}_\Gamma \tag{2.17}$$

and

$$\bar{\nu}(\mathbf{x}_\Gamma, t) = \nu_\Gamma \tag{2.18}$$

where $\mathbf{x}_\Gamma$ is a point on the solid boundary and $\mathbf{q}$ and $\bar{\nu}_\Gamma$ denote the velocity and microrotation prescribed on it. It may be stated that while other possible forms of boundary conditions for polar fluids have been contemplated, no definite conclusions in this regard and the hyperstick condition of adherence seems to be the most plausible.

The eqns (2.15) and (2.16) indeed characterise the slow flow of an incompressible micropolar fluid past an axially symmetric body of any shape. However, the analysis will be tedious in most of the cases in view of the difficulty in finding the appropriate solutions for the equation

$$\left(E^2 - \frac{\lambda^2}{c^2}\right)\Psi = 0. \tag{2.19}$$

3. PROLATE SPHEROID

Let (ξ, η, ϕ) be prolate spheroidal coordinates such that

$$z + ir = c\cosh(\xi + i\eta) \tag{3.1}$$

and let

$$\cosh\xi = s, \cos\eta = t. \tag{3.2}$$

The solution of eqn (2.15) can be obtained by superposing the solutions of the equations

$$E^4\Psi = 0 \tag{3.3}$$

and

$$\left(E^2 - \frac{\lambda^2}{c^2}\right)\Psi = 0 \tag{3.4}$$

on each other.

For the prolate spheroidal coordinates the Stokes stream function operator E^2 is given by

$$E^2 = \frac{1}{c^2(\cosh^2\xi - \cos^2\eta)}\left[\frac{\partial^2}{\partial\xi^2} - \coth\xi\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\eta^2} - \cot\eta\frac{\partial}{\partial\eta}\right]$$
$$= \frac{1}{c^2(s^2 - t^2)}\left[(s^2 - 1)\frac{\partial^2}{\partial s^2} + (1 - t^2)\frac{\partial^2}{\partial t^2}\right]. \tag{3.5}$$

(i) *Solution of the equation* $E^4\Psi = 0$. The solution of (3.3) is exhibited in the form

$$\Psi = \Psi_0 + \Psi_1 \tag{3.6}$$

where

$$\Psi_0 = -\frac{1}{2}Uc^2(s^2 - 1)(1 - t^2) \tag{3.7}$$

and

$$\Psi_1 = c^2(s^2 - 1)(1 - t^2)\sum_{n=0}^{\infty} G_{n+1}(s)P'_{n+1}(t). \tag{3.8}$$

The function Ψ_0 in (3.7) represents the stream function due to a uniform stream of magnitude U and parallel to the axis of symmetry at infinity. We see that $E^2\Psi_0 = 0$ and hence Ψ_1 has to satisfy the equation

$$E^4\Psi_1 = 0. \tag{3.9}$$

In the function Ψ_1 given in (3.8) the factor $P'_{n+1}(t)$ denotes the derivative of the Legendre polynomial $P_{n+1}(t)$. The choice of the functions $G_{n+1}(s)$ is conditioned by the requirement that Ψ_1 satisfies the eqn (3.9). We restrict the functions $G_{n+1}(s)$ further such that

$$E^2\Psi_1 = c^2(s^2 - 1)(1 - t^2)\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s)P'_{n+1}(t) \tag{3.10}$$

where $Q'_{n+1}(s)$ is the derivative of the Legendre function $Q_{n+1}(s)$ of the second kind. It is easily checked that the r.h.s. expression in (3.10) is a solution of the equation

$$E^2 f = 0 \tag{3.11}$$

and so the restriction on the functions $G_{n+1}(s)$ in (3.8) so as to validate the eqn (3.10) will automatically allow the eqn (3.9) to be satisfied. Applying the operator E^2 on the equation in (3.8) and equating the result with the expression in (3.10) we see that

$$\sum_{n=0}^{\infty}[\{(s^2 - 1)G_{n+1}(s)\}'' - (n+1)(n+2)G_{n+1}(s)]P'_{n+1}(t) = \sum_{n=0}^{\infty} A_{n+1}c^2(s^2 - t^2)Q'_{n+1}(s)P'_{n+1}(t). \tag{3.12}$$

We may write the r.h.s. expression in (3.12) as a combination of the derivatives of Legendre polynomials in t by using the identities

$$(1-t^2)P'_{n+1}(t) = -\frac{(n+1)(n+2)}{(2n+3)(2n+5)}P'_{n+3}(t) + \frac{2(n+1)(n+2)}{(2n+1)(2n+5)}P'_{n+1}(t) - \frac{(n+1)(n+2)}{(2n+1)(2n+3)}P'_{n-1}(t) \tag{3.13}$$

and

$$(s^2-1)Q'_{n+1}(s) = \frac{(n+1)(n+2)}{(2n+3)(2n+5)}Q'_{n+3}(s) - \frac{2(n+1)(n+2)}{(2n+1)(2n+5)}Q'_{n+1}(s) + \frac{(n+1)(n+2)}{(2n+1)(2n+3)}Q'_{n-1}(s). \tag{3.14}$$

The relations (3.13) and (3.14) are valid for $n = 1,2,3,\ldots$. They are also valid for $n = 0$ if in (3.13) the term $P'_{-1}(t)$ is interpreted as zero and in (3.14) the term $Q'_{-1}(s)$ is defined as $-s/(s^2-1)$. After introducing these changes in the r.h.s. of (3.12), we may compare the coefficients of $P'_{n+1}(t)$ on either side and then we are led to the following system of ordinary differential equations for the function $G_{n+1}(s)$

$$(s^2-1)G''_{n+1}(s) + 4sG'_{n+1}(s) - n(n+3)G_{n+1}(s) = g_{n+1}(s) \tag{3.15}$$

where

$$g_{n+1}(s) = c^2\left[\frac{(n+1)(n+2)}{(2n+3)(2n+5)}A_{n+1} - \frac{(n+3)(n+4)}{(2n+5)(2n+7)}A_{n+3}\right]Q'_{n+3}(s)$$
$$-c^2\left[\frac{(n-1)n}{(2n-1)(2n+1)}A_{n-1} - \frac{(n+1)(n+2)}{(2n+1)(2n+3)}A_{n+1}\right]Q'_{n-1}(s). \tag{3.16}$$

The systems in (3.15) and (3.16) are valid for $n = 0, 1, 2, 3, \ldots$ and the term on the r.h.s. of (3.16) involving A_{-1} is to be deleted for obtaining $g_1(s)$.

The function $G_{n+1}(s)$ is found by integrating the differential eqn (3.15) by the method of variation of parameters and we have

$$G_{n+1}(s) = \alpha_{n+1}P'_{n+1}(s) + \beta_{n+1}Q'_{n+1}(s) - \frac{P'_{n+1}(s)}{(n+1)(n+2)}\int_{s_0}^{s}(s^2-1)Q'_{n+1}(s)g_{n+1}(s)\,ds$$
$$+\frac{Q'_{n+1}(s)}{(n+1)(n+2)}\int_{s_0}^{s}(s^2-1)P'_{n+1}(s)g_{n+1}(s)\,ds \tag{3.17}$$

for $n = 0,1,2,3,\ldots$. In the integrals in the above equation the lower limit s_0 is the value specifying the boundary of the prolate spheroid past which the flow is being examined and the domain of the flow is thus restricted to the range $s_0 < s$. As $s \to \infty$, the flow has to be a uniform stream and to ensure this we have to choose $\alpha_{n+1} = 0$ in (3.17). Hence we have

$$G_{n+1}(s) = B_{n+1}Q'_{n+1}(s) - \frac{P'_{n+1}(s)}{(n+1)(n+2)}\int_{s_0}^{s}(s^2-1)Q'_{n+1}(s)g_{n+1}(s)\,ds$$
$$+\frac{Q'_{n+1}(s)}{(n+1)(n+2)}\int_{s_0}^{s}(s^2-1)P'_{n+1}(s)g_{n+1}(s)\,ds. \tag{3.18}$$

The functions $g_{n+1}(s)$ involve one set of constants $\{A_n\}$ (eqn (3.16)) and hence the functions $G_{n+1}(s)$ involve two sets of constants, viz. $\{A_{n+1}\}$ and $\{B_{n+1}\}$.

(ii) *Solution of the equation* $(E^2 - (\lambda^2/c^2))\Psi = 0$. The solution Ψ_2 of this equation is taken in the form

$$\Psi_2 = c\sqrt{((s^2-1)(1-t^2))}R(s)S(t). \tag{3.19}$$

Substituting this in the eqn (3.4) we notice that $R(s)$ and $S(t)$ satisfy the differential equations

$$(s^2-11)R''(s)+2sR'(s)-\left(\Lambda+\lambda^2 s^2+\frac{1}{s^2-1}\right)R=0 \tag{3.20}$$

and

$$(1-t^2)S''(t)-2tS'(t)+\left(\Lambda+\lambda^2 t^2-\frac{1}{1-t^2}\right)S=0 \tag{3.21}$$

where Λ is the separation parameter.

These are spheroidal wave differential equations and have the sets of solutions $\{R_{1n}(i\lambda, s)\}$ and $\{S_{1n}(i\lambda, t)\}$, respectively. The functions R_{1n} are radial spheroidal wave functions and S_{1n} are angular spheroidal wave functions. (Notation as in[3]). The function $R_{1n}(i\lambda, s)$ has the representation

$$R_{1n}(i\lambda, s)=\left\{\sum_{r=0,1}^{\infty}{}' (r+1)(r+2)d_r^{1n}(i\lambda)\right\}^{-1}\left(\frac{s^2-1}{s^3}\right)^{1/2}\left(\frac{\pi}{2\lambda}\right)^{1/2}$$

$$\sum_{r=0,1}^{\infty}{}' i^{r-n+1/2}(r+1)(r+2)d_r^{1n}(i\lambda)C_{r+3/2}(i\lambda s) \tag{3.22}$$

where $C_{r+3/2}(i\lambda s)$ denotes a cylinder function.

To ensure regularity of the solution Ψ_2 at infinity, we have to restrict the radial wave function to $R_{1n}^{(3)}(i\lambda, s)$ which arises from (3.22) by taking the cylinder function $C_{r+3/2}(i\lambda s)$ as the Hankel function of the first kind, viz. $H_{r+3/2}^{(1)}(i\lambda s)$. The Hankel function is expressible in terms of the modified Bessel function of the second kind in the form

$$H_{r+3/2}^{(1)}(i\lambda s)=(2/\pi)\exp(-(r+5/2)i\pi/2)K_{r+3/2}(\lambda s) \tag{3.23}$$

and we have, therefore

$$R_{1n}^{(3)}(i\lambda, s)=\left\{i^{n+2}\sum_{r=0,1}^{\infty}{}' (r+1)(r+2)d_r^{1n}(i\lambda)\right\}^{-1}\left(\frac{2}{\pi\lambda}\right)^{1/2}\left(\frac{s^2-1}{s^3}\right)^{1/2}$$

$$\sum_{r=0,1}^{\infty}{}' (r+1)(r+2)d_r^{1n}(i\lambda)K_{r+3/2}(\lambda s). \tag{3.24}$$

To ensure regularity of the solution in the flow region, it is also necessary to restrict S_{1n} to the angular wave function of the first kind $S_{1n}^{(1)}(i\lambda, t)$ which has the expansion

$$S_{1n}^{(1)}(i\lambda, t)=\sum_{r=0,1}^{\infty}{}' d_r^{1n}(i\lambda)P_{r+1}^{(1)}(t) \tag{3.25}$$

and

$$P_{r+1}^{(1)}(t)=\sqrt{(1-t^2)}\frac{d}{dt}P_{r+1}(t) \tag{3.26}$$

denotes the associated Legendre function of the first kind.

The coefficients $d_r^{1n}(i\lambda)$ in the above expansions are constants depending on the parameter $i\lambda$ and the suffix r has the value 1,3,5,... or 0,2,4,... depending upon the odd or even nature of $n+1$.

We have, therefore, the solution Ψ_2 of (3.4) in the form

$$\Psi_2=c\sqrt{((s^2-1)(1-t^2))}\sum_{n=1}^{\infty}C_n R_{1n}^{(3)}(i\lambda, s)S_{1n}^{(1)}(i\lambda, t) \tag{3.27}$$

where $\{C_n\}$ are constants.

(iii) *Stream function and microrotation.* The stream function Ψ satisfying the eqn (2.15) is given by

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 \tag{3.28}$$

and, therefore, we have

$$\Psi = -\frac{1}{2}Ur^2 + r^2 \sum_{n=0}^{\infty} G_{n+1}(s)P'_{n+1}(t) + r \sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(i\lambda, s)S^{(1)}_{1n}(i\lambda, t) \tag{3.29}$$

in which

$$r = c\surd((s^2-1)(1-t^2)) \tag{3.30}$$

and $G_{n+1}(s)$ is given in (3.18). It can now be seen that

$$E^2\Psi = r^2 \sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s)P'_{n+1}(t) + \frac{\lambda^2}{c^2} r \sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(i\lambda, s)S^{(1)}_{1n}(i\lambda, t) \tag{3.31}$$

and

$$E^4\Psi = \frac{\lambda^4}{c^4} r \sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(i\lambda, s)S^{(1)}_{1n}(i\lambda, t). \tag{3.32}$$

From eqns (2.16), (3.31) and (3.32) it follows that the only nonvanishing component of the microrotation vector, viz. $B(s, t)$ is given by

$$B(s,t) = \frac{r}{2}\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s)P'_{n+1}(t) + \frac{(\mu+k)}{k}\frac{\lambda^2}{c^2} \sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(i\lambda, s)S^{(1)}_{1n}(i\lambda, t). \tag{3.33}$$

The expression for Ψ in (3.29) involves three infinite sets of unknown constants, viz. $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ and these have to be determined by utilizing the conditions on the boundary $s = s_0$ of the spheroid. The super adherence or the hyperstick condition yields the following equations

$$\left.\begin{aligned} \Psi(s,t) &= 0 \\ \frac{\partial\Psi}{\partial s}(s,t) &= 0 \\ B(s,t) &= 0 \end{aligned}\right\} \quad \text{on} \quad s = s_0. \tag{3.34}$$

It is also true that $(\partial\Psi/\partial t) = 0$ on $s = s_0$. But this is not independent of the conditions in (3.34). In principle it should be possible to determine the constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ by invoking the conditions (3.34). However, it does not seem to be possible to find explicit evaluation for these constants and one has to resort to determination by numerical computation for specific values of the various parameters of the problem.

The eqns (3.34) can be put in the form

$$c\surd((s_0^2-1)) \sum_{n=0}^{\infty} B_{n+1}Q'_{n+1}(s_0)P^{(1)}_{n+1}(t) + \sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(i\lambda, s_0)S^{(1)}_{1n}(i\lambda, t) = \frac{1}{2}Uc\surd((s_0^2-1)(1-t^2)), \tag{3.35}$$

$$-c\sum_{n=0}^{\infty} [s_0Q'_{n+1}(s_0) - (n+1)(n+2)Q_{n+1}(s_0)]B_{n+1}P^{(1)}_{n+1}(t) + \surd(s_0^2-1) \sum_{n=1}^{\infty} C_n\left\{\frac{d}{ds}R^{(3)}_{1n}(i\lambda, s)\right\}_{s=s_0} S^{(1)}_{1n}(i\lambda, t) = \frac{Uc}{2}s_0\surd(1-t^2), \tag{3.36}$$

$$c\surd((s_0^2-1)) \sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s_0)P^{(1)}_{n+1}(t) + \frac{2(2\mu+k)}{\gamma} \sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(i\lambda, s_0)S^{(1)}_{1n}(i\lambda, t) = 0. \tag{3.37}$$

The functions $P_n^{(1)}(t)$ form an orthogonal set on the interval $-1 \le t \le 1$ and so it is possible to obtain from eqns (3.35) and (3.36) two distinct expressions for B_{n+1} which are both linear forms in the vector $\{C_m\}$ apart from an additional term independent of the constants $\{C_m\}$. The functions $S_{1n}^{(1)}(t)$ also constitute an orthogonal set on the interval $-1 \le t \le 1$ and it is, therefore, possible from each of (3.35) and (3.36) to express the constants C_n as linear forms involving the vector $\{B_m\}$, here again with an additional independent term. However, the eqn (3.37) does not contain the constants B_m and it is, therefore, preferable to obtain the expressions for B_n in terms of the constants C_m as suggested in the first alternative above. We may then eliminate the constant B_n between these two relations and obtain a nonhomogeneous linear algebraic system of equations for the unknowns $\{C_n\}$. This is an infinite system and exact evaluation of the constants C_n in explicit form is not possible. The constants C_n can, however, be evaluated by a numerical method after deciding the stage of truncation of the infinite system. The constants B_n are already seen to be expressible in terms of the constants $\{C_m\}$ and we may, therefore, determine these also numerically after the determination of the constants $\{C_m\}$. The constants A_n are also expressible in terms of C_m's from (3.37) and can thus be evaluated once the C_m's are determined.

From (3.35)–(3.37) we obtain the following three systems of equations for $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$.

$$c\sqrt{((s_0^2-1))}B_{n+1}Q'_{n+1}(s_0)+\sum_{m=1}^{\infty} C_m d_n^{1m}(i\lambda)R_{1m}^{(3)}(i\lambda, s_0)=\frac{1}{2}Uc\sqrt{((s_0^2-1))}\delta_{0n}, \tag{3.38}$$

$$c\{(n+1)(n+2)Q_{n+1}(s_0)-s_0Q'_{n+1}(s_0)\}B_{n+1}+\sum_{m=1}^{\infty} C_m d_n^{1m}(i\lambda)\sqrt{((s_0^2-1))} \times\left\{\frac{d}{ds}R_{1m}^{(3)}(i\lambda, s)\right\}_{s=s_0}=\frac{1}{2}Ucs_0\delta_{0n}, \tag{3.39}$$

$$c\sqrt{((s_0^2-1))}Q'_{n+1}(s_0)A_{n+1}+\frac{2(2\mu+k)}{\gamma}\sum_{m=1}^{\infty} C_m d_n^{1m}(i\lambda)R_{1m}^{(3)}(i\lambda, s_0)=0. \tag{3.40}$$

From (3.38) and (3.39) we can eliminate B_{n+1} and then arrive at the following system of nonhomogeneous linear algebraic equations for the determination of the constants $\{C_n\}$

$$\sum_{m=1}^{\infty} D_{nm}C_m=\alpha_n \quad n=0,1,2,3,\ldots \tag{3.41}$$

where

$$D_{nm}=d_n^{1m}(i\lambda)\left\{(s_0^2-1)Q'_{n+1}(s_0)\left(\frac{d}{ds}R_{1m}^{(3)}(i\lambda, s)\right)_{s=s_0} -[(n+1)(n+2)Q_{n+1}(s_0)-s_0Q'_{n+1}(s_0)]R_{1m}^{(3)}(i\lambda, s_0)\right\} \tag{3.42}$$

and

$$\alpha_n=-\frac{Uc}{\sqrt{((s_0^2-1))}}\delta_{0n}. \tag{3.43}$$

The constants B_{n+1} and A_{n+1} are then determined from (3.38) and (3.40), respectively. The coefficients D_{nm} in the system (3.41) depend on the two parameters λ and s_0 and numerical determination of the constants C_m will, therefore, be possible only when these parameters are assigned specific values. The constants B_m may then be determined from the eqn (3.38). However, to determine the constants A_{n+1} from (3.40), we have to specify an additional parameter, viz. $(2\mu+k)/(\gamma)c^2$.

It is well known from the theory of spheroidal wave functions that the coefficients d_n^{1m} have to be defined as zero when $n+m+1$ is a positive odd integer. In the system (3.41) we see that the r.h.s. vector $\{\alpha_n\}$ has only one nonvanishing component α_0 corresponding to $n=0$. Hence the subsystem of (3.41) given by

$$\sum_{m=1}^{\infty} D_{2n+1,2m} C_{2m} = 0 \tag{3.44}$$

is a homogeneous subsystem and we may choose the constants C_2, C_4, C_6, etc. to be equal to zero. From (3.38) and (3.40) we now easily see that the constants B_n and A_n are also zero when n is an even positive integer. We may, therefore, rewrite the expressions for stream function Ψ and microrotation component B in the form

$$\Psi = -\frac{1}{2}Ur^2 + r^2 \sum_{n=0}^{\infty} G_{2n+1}(s)P'_{2n+1}(t) + r\sum_{n=0}^{\infty} C_{2n+1}R^{(3)}_{1,2n+1}(i\lambda, s)S^{(1)}_{1,2n+1}(i\lambda, t) \tag{3.45}$$

and

$$B = \frac{r}{2}\sum_{n=0}^{\infty} A_{2n+1}Q'_{2n+1}(s)P'_{2n+1}(t) + \frac{\mu+k}{k}\frac{\lambda^2}{c^2}\sum_{n=0}^{\infty} C_{2n+1}R^{(3)}_{1,2n+1}(i\lambda, s)S^{(1)}_{1,2n+1}(i\lambda, t). \tag{3.46}$$

(iv) *Pressure distribution.* The equations of motion (2.4) and (2.5) are written with reference to the cylindrical frame of coordinates. In the prolate spheroidal frame, we may write them in the form

$$\frac{\partial p}{\partial s} = \frac{(2\mu+k)}{2c(s^2-1)}\frac{\partial}{\partial t}(E^2\Psi) - \frac{\gamma(\mu+k)}{2kc}\frac{1}{(s^2-1)}\frac{\partial}{\partial t}(E^4\Psi) \tag{3.47}$$

and

$$\frac{\partial p}{\partial t} = -\frac{(2\mu+k)}{2c(1-t^2)}\frac{\partial}{\partial s}(E^2\Psi) + \frac{\gamma(\mu+k)}{2kc(1-t^2)}\frac{\partial}{\partial s}(E^4\Psi). \tag{3.48}$$

Using the expression for Ψ given in (3.45) we find that

$$\frac{\partial p}{\partial s} = -(2\mu+k)c\sum_{n=0}^{\infty}(n+1)(2n+1)A_{2n+1}Q'_{2n+1}(s)P_{2n+1}(t) \tag{3.49}$$

and

$$\frac{\partial p}{\partial t} = -(2\mu+k)c\sum_{n=0}^{\infty}(n+1)(2n+1)A_{2n+1}Q_{2n+1}(s)P'_{2n+1}(t). \tag{3.50}$$

Either of the above two equations is readily integrable and we have

$$p = -(2\mu+k)c\sum_{n=0}^{\infty}(n+1)(2n+1)A_{2n+1}Q_{2n+1}(s)P_{2n+1}(t). \tag{3.51}$$

(v) *Rate of deformation.* The velocity vector $\mathbf{q}$ can be written in the form

$$\mathbf{q} = q_\xi \mathbf{e}_\xi + q_\eta \mathbf{e}_\eta \tag{3.52}$$

where

$$\begin{aligned} q_\xi &= \frac{1}{c^2\sqrt{((s^2-t^2)(s^2-1))}}\frac{\partial \Psi}{\partial t}, \\ q_\eta &= \frac{1}{c^2\sqrt{((s^2-t^2)(1-t^2))}}\frac{\partial \Psi}{\partial s}. \end{aligned} \tag{3.53}$$

The rate of strain components are given by

$$e_{\xi\xi}=\frac{1}{c^3(s^2-t^2)}\left\{\Psi_{st}+\frac{t\Psi_s}{s^2-t^2}-\frac{s(2s^2-1-t^2)}{(s^2-t^2)(s^2-1)}\Psi_t\right\}$$

$$e_{\eta\eta}=\frac{1}{c^3(s^2-t^2)}\left\{-\Psi_{st}+\frac{s\Psi_t}{s^2-t^2}+\frac{t(2t^2-1-s^2)}{(s^2-t^2)(1-t^2)}\Psi_s\right\}$$

$$e_{\phi\phi}=\frac{1}{c^3(s^2-t^2)}\left\{\frac{s\Psi_t}{s^2-1}+\frac{t\Psi_s}{1-t^2}\right\}$$

$$e_{\xi\eta}=e_{\eta\xi}=\frac{(s^2-1)\Psi_{ss}-(1-t^2)\Psi_{tt}}{2c^3(s^2-t^2)\sqrt{((s^2-1)(1-t^2))}}-\frac{s\sqrt{((s^2-1))}\Psi_s}{c^3(s^2-t^2)^2\sqrt{((1-t^2))}}-\frac{t\sqrt{((1-t^2))}\Psi_t}{c^3(s^2-t^2)^2\sqrt{((s^2-1))}}$$

$$e_{\xi\phi}=e_{\phi\xi}=e_{\eta\phi}=e_{\phi\eta}=0. \tag{3.54}$$

The spin = (1/2)curl **q** has only one nonzero component ω_ϕ in the direction of the vector $\mathbf{e}_\phi$ and this is given by

$$\omega_\phi=\frac{1}{2c\sqrt{((s^2-1)(1-t^2))}}E^2\Psi. \tag{3.55}$$

The surface stress t_{ij} for the micropolar fluid is given by eqn (1.5) and we find that the only nonvanishing components of t_{ij} are $t_{\xi\xi}$, $t_{\eta\eta}$, $t_{\phi\phi}$, $t_{\xi\eta}$, $t_{\eta\xi}$. These are given by

$$\begin{aligned}t_{\xi\xi}&=-p+(2\mu+k)e_{\xi\xi},\\ t_{\eta\eta}&=-p+(2\mu+k)e_{\eta\eta},\\ t_{\phi\phi}&=-p+(2\mu+k)e_{\phi\phi},\\ t_{\xi\eta}&=(2\mu+k)e_{\xi\eta}+k(\omega_\phi-B),\\ t_{\eta\xi}&=(2\mu+k)e_{\xi\eta}-k(\omega_\phi-B).\end{aligned} \tag{3.56}$$

The stress vector **t** on the boundary of the body is given by

$$\mathbf{t}=t_{\xi\xi}\mathbf{e}_\xi+t_{\xi\eta}\mathbf{e}_\eta. \tag{3.57}$$

We find that

$$(t_{\xi\xi})_{s=s_0}=-p(s_0,t) \tag{3.58}$$

and

$$(t_{\xi\eta})_{s=s_0}=(\mu+k)\left\{\frac{E^2\Psi}{c\sqrt{((s^2-1)(1-t^2))}}\right\}_{s=s_0}. \tag{3.59}$$

The stress vector has the component

$$(\text{Stress})_{\text{axial}}=\frac{1}{\sqrt{((s_0^2-t^2))}}\{t\sqrt{((s^2-1))}t_{\xi\xi}-s\sqrt{((1-t^2))}t_{\xi\eta}\}_{s=s_0} \tag{3.60}$$

in the direction of axis of symmetry and

$$(\text{Stress})_{\text{radial}}=\frac{1}{\sqrt{((s_0^2-t^2))}}\{s\sqrt{((1-t^2))}t_{\xi\xi}+t\sqrt{((s^2-1))}t_{\xi\eta}\}_{s=s_0} \tag{3.61}$$

in the radial direction in the meridian plane. The resultants of these two vector components over the entire surface of the body are obtained by integration and it is seen that the radial component integrates to zero. Thus, the resultant of the stress vector on the body is a force in the direction of the axis of symmetry and this gives the drag on the body. The drag D can be written in the form

$$D = 2\pi c^2\sqrt{((s_0^2-1))}\int_{-1}^{1}\{t\sqrt{((s^2-1))}t_{\xi\xi} - s\sqrt{((1-t^2))}t_{\xi\eta}\}_{s=s_0}\,dt \tag{3.62}$$

and this simplifies to

$$2\pi c^2\sqrt{((s_0^2-1))}\Big\{(2\mu+k)c\sqrt{((s_0^2-1))}Q_1(s_0)\frac{2}{3}A_1 - (\mu+k)cs_0\sqrt{((s_0^2-1))}Q_1'(s_0)\frac{4}{3}A_1$$

$$-(\mu+k)\frac{4\lambda^2 s_0}{3c^2}\sum_{n=0}^{\infty}C_{2n+1}d_0^{1,2n+1}(i\lambda)R_{1,2n+1}^{(3)}(i\lambda,s_0)\Big\}. \tag{3.63}$$

Using the eqn (3.40) we may eliminate the series involving the constants C_{2n+1} in the above expression for the drag and after further simplification we see that the drag due to the surface stress is given by

$$D = \frac{4}{3}\pi c^3(2\mu+k)A_1. \tag{3.64}$$

The drag on an axially symmetric body in the Stokes' flow of micropolar fluid has been expressed through an elegant formula by Ramkissoon and Majumdar[6]. The drag on the prolate spheroid seen above is also recoverable from the above formula and this is shown in the appendix.

Writing

$$D_0 = 4\pi(2\mu+k)Uc \tag{3.65}$$

and

$$\tilde{A}_1 = (A_1/U)c^2 \tag{3.66}$$

we see that the drag is equal to

$$D_0(\tilde{A}_1/3). \tag{3.67}$$

We may refer to $\tilde{A}_1/3$ as the nondimensional drag and this depends upon the eccentricity of the spheroid, the micropolarity parameter and an additional material constant $(2\mu+k)c^2/\gamma$.

The only nonvanishing shear stress components are $t_{\xi\eta}$ and $t_{\eta\xi}$. The symmetry of the shear stress that obtains in classical nonpolar fluid flow is no longer valid and we have the shear stress difference

$$t_{\xi\eta} - t_{\eta\xi} = -\frac{(2\mu+k)\lambda^2 U}{c}\sum_{n=0}^{\infty}\tilde{C}_{2n+1}R_{1,2n+1}^{(3)}(i\lambda,s)S_{1,2n+1}^{(1)}(i\lambda,t) \tag{3.68}$$

where

$$\tilde{C}_{2n+1} = C_{2n+1}/(Uc). \tag{3.69}$$

(vi) *Couple stress.* The couple stress tensor m_{ij} is given by (1.6) and we see that the only nonvanishing components of this tensor are $m_{\eta\phi}$, $m_{\phi\eta}$, $m_{\xi\phi}$, $m_{\phi\xi}$.

We find that

$$m_{\eta\phi} = -\frac{1}{c\sqrt{((s^2-t^2)(1-t^2))}}\left\{\beta t B + \gamma(1-t^2)\frac{\partial B}{\partial t}\right\}$$

$$m_{\phi\eta} = -\frac{1}{c\sqrt{((s^2-t^2)(1-t^2))}}\left\{\beta(1-t^2)\frac{\partial B}{\partial t} + \gamma t B\right\}$$

$$m_{\xi\phi} = \frac{1}{c\sqrt{((s^2-t^2)(s^2-1))}}\left\{-\beta s B + \gamma(s^2-1)\frac{\partial B}{\partial s}\right\}$$

$$m_{\phi\xi} = \frac{1}{c\sqrt{((s^2-t^2)(s^2-1))}}\left\{\beta(s^2-1)\frac{\partial B}{\partial s} - \gamma s B\right\}. \tag{3.70}$$

The couple vector is $m_{\xi\phi}\mathbf{e}_\phi$ and on the boundary it reduces to

$$\left(\frac{\gamma\sqrt{((s^2-1))}}{c\sqrt{((s^2-t^2))}}\frac{\partial B}{\partial s}\right)_{s=s_0}\mathbf{e}_\phi. \tag{3.71}$$

It is seen that the resultant couple vector due to the couple stresses on the spheroid equals

$$c^2\sqrt{((s^2-1))}\int_{-1}^{1}\sqrt{((s^2-t^2))}\left(\int_{\phi=0}^{2\pi}(m_{\xi\phi})_{s=s_0}\mathbf{e}_\phi\,\mathrm{d}\phi\right)\mathrm{d}t \tag{3.72}$$

and this vanishes since $\int_0^{2\pi}\mathbf{e}_\phi\,\mathrm{d}\phi = 0$.

The moment of the stress vector about the centre of the spheroid is

$$\mathbf{m} = (z\mathbf{e}_z + r\mathbf{e}_r)\times\mathbf{t} \tag{3.73}$$

and the integral of this over the surface of the spheroid is seen to be zero. The scalar moment of the stress vector about the axis of symmetry is $\mathbf{m}\cdot\mathbf{e}_z$ and this is zero every where. Thus, there is no couple exerted on the body in spite of the fluid sustaining a couple strese.

(vii) *Numerical results.* The drag on the spheroid is numerically evaluated for several parametric values by computing the values of the constants C_n from the system of eqns (3.41) by truncating it to a 5 by 5 system. The motivation for this order of truncation is the fact that the coefficients needed for the evaluation of the constants $d_r^{mn}(i\lambda)$ are available only to a limited extent in the published literature[3].

The drag on the prolate spheroid is given by (3.64) in the polar case. The drag in the nonpolar case[4] is

$$8\pi\mu Uc\left[\frac{s_0^2+1}{2}\log\frac{s_0+1}{s_0-1} - s_0\right]^{-1} = 8\pi\mu Uc\left(\frac{\tilde{A}_1}{3}\right)_n. \tag{3.74}$$

The nondimensional drag for the polar fluid is

$$\frac{\tilde{A}_1}{3} = (D)_{\text{polar}}/4\pi(2\mu+k)Uc \tag{3.75}$$

and this for the nonpolar fluid is given by

$$\left(\frac{\tilde{A}_1}{3}\right)_n = \left[\frac{s_0^2+1}{2}\log\frac{s_0+1}{s_0-1} - s_0\right]^{-1}. \tag{3.76}$$

The drag ratio for the prolate spheroid is defined as the ratio of the drag on the spheroid with the drag on a sphere of diameter equal to the minor axis of the meridian ellipse generating the spheroid. The drag on a sphere of radius $c\sqrt{((s_0^2-1))}$ is equal to[5]

$$\frac{3\pi(2\mu+k)Uc\sqrt{((s_0^2-1))}}{\left[1-\dfrac{1}{\Lambda^2(\lambda\sqrt{((s_0^2-1)+1)}}\right]} \tag{3.77}$$

where

$$\Lambda^2 = 2(\mu + k)/k, \tag{3.78}$$

and hence the drag ratio in the polar case is given by

$$\frac{4}{3\sqrt{((s_0^2-1))}}\left[1 - \frac{1}{\Lambda^2(\lambda\sqrt{((s_0^2-1))}+1)}\right]\left(\frac{\tilde{A}_1}{3}\right). \tag{3.79}$$

This for the nonpolar fluid becomes

$$\frac{4}{3\sqrt{((s_0^2-1))}}\left(\frac{\tilde{A}_1}{3}\right)_n. \tag{3.80}$$

The graphs giving the drag and drag ratio for several parameter values show that the magnitudes of the drag and drag ratio increase with s_0 and also with each of the parameters Λ^2 and λ.

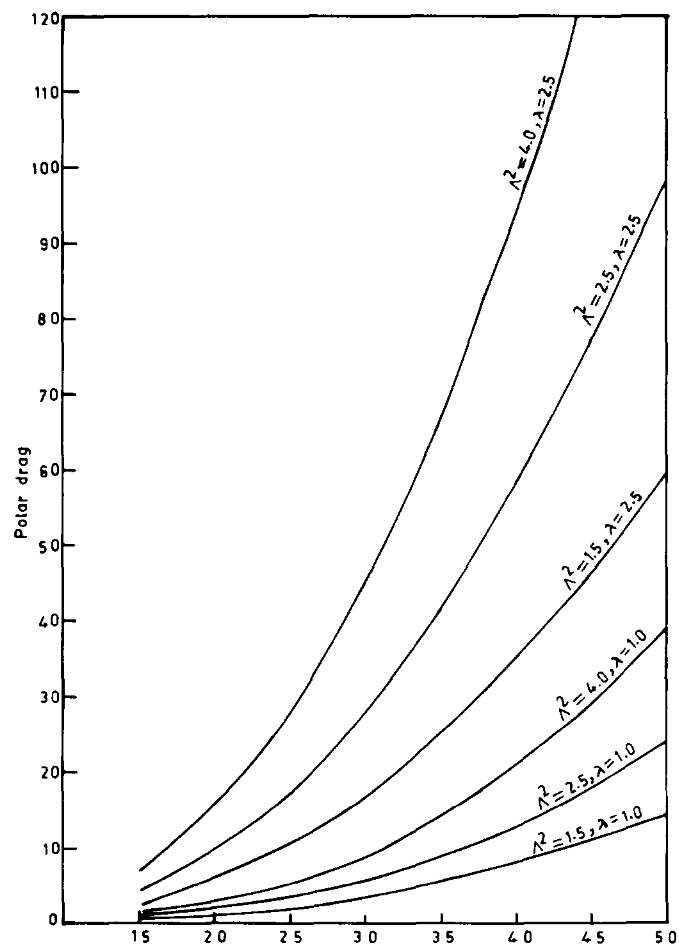

Fig. 1. Variation of polar drag with respect to s_0.

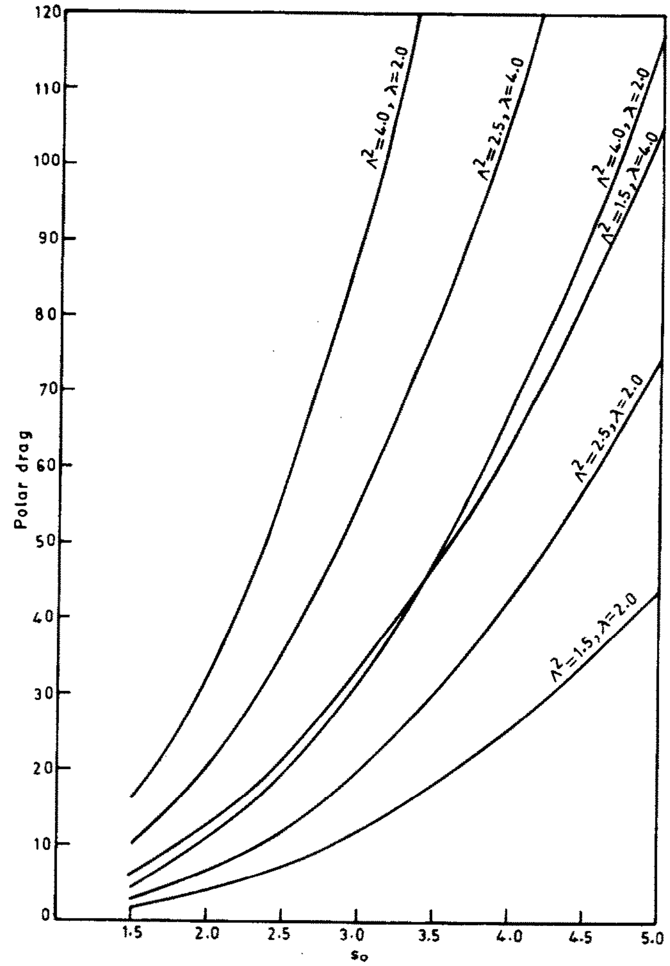

Fig. 2. Variation of polar drag with respect to s_0.

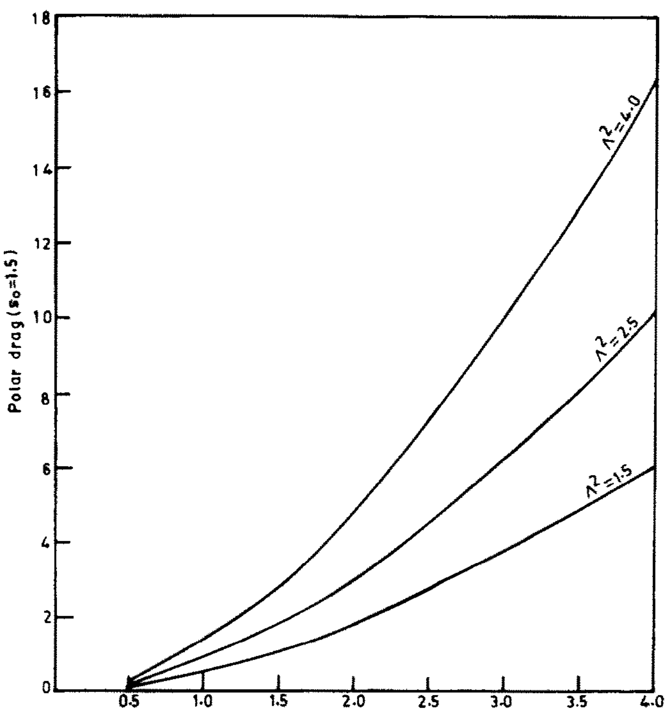

Fig. 3. Variation of polar drag with respect to λ.

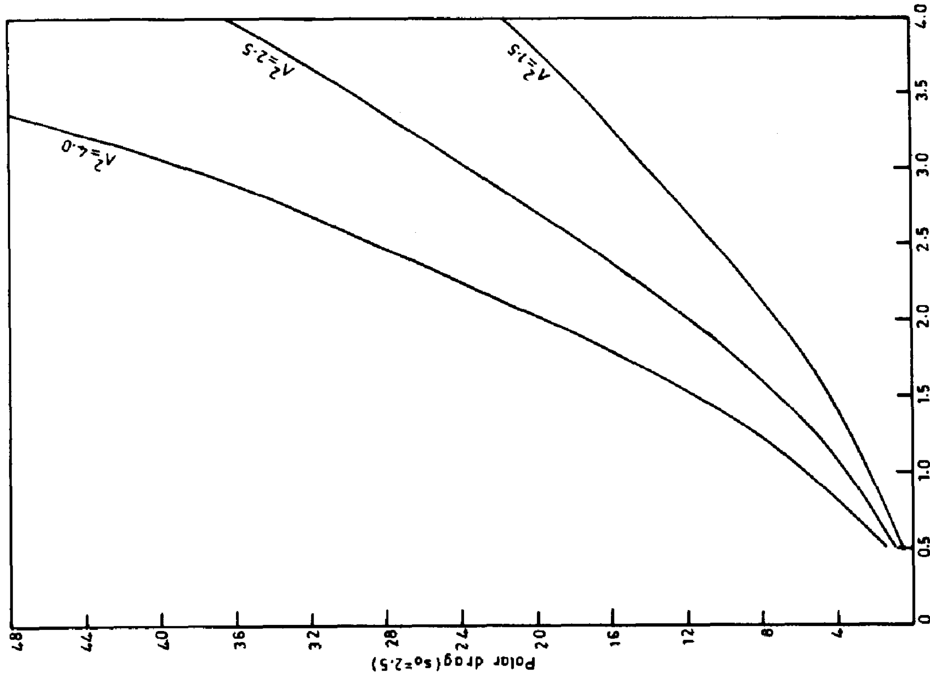

Fig. 5. Variation of polar drag with respect to λ.

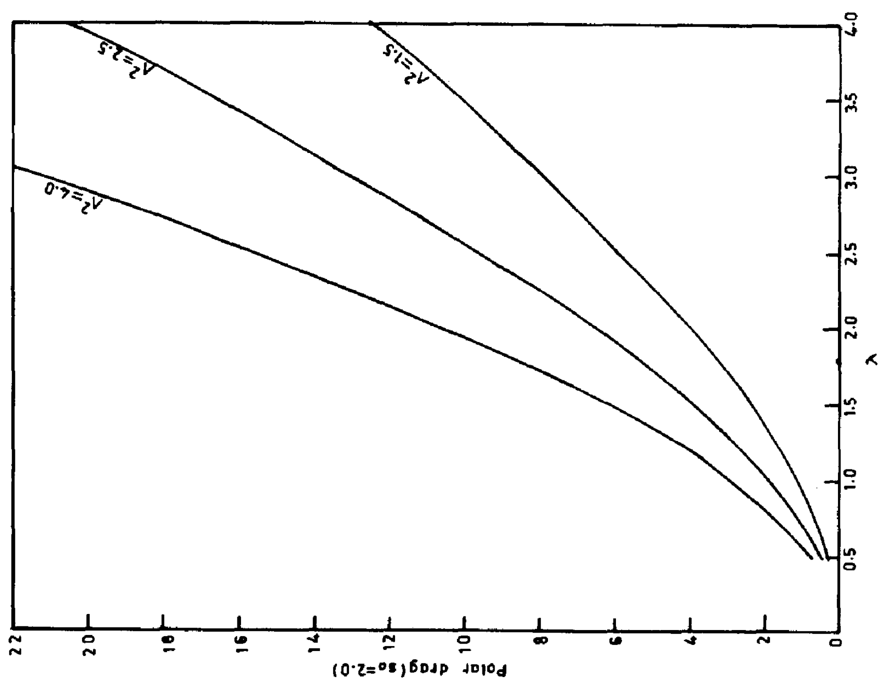

Fig. 4. Variation of polar drag with respect to λ.

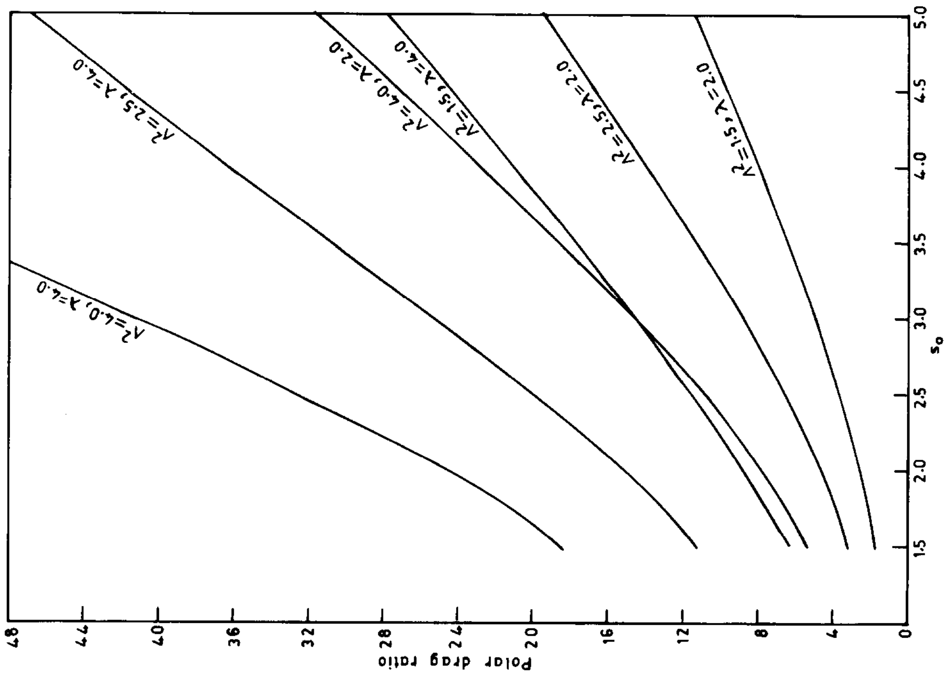

Fig. 7. Variation of polar drag ratio with respect to s_0.

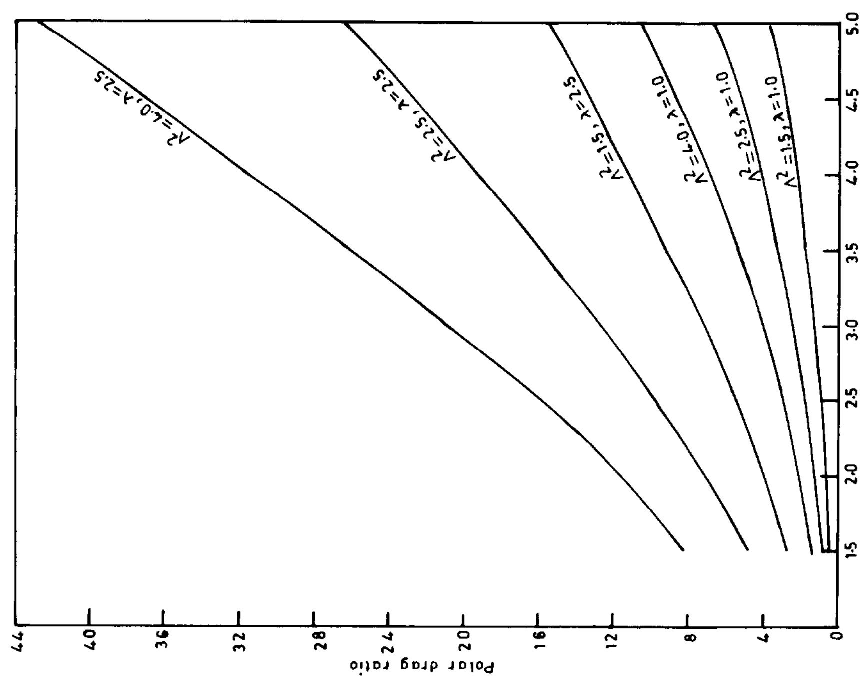

Fig. 6. Variation of polar drag ratio with respect to s_0.

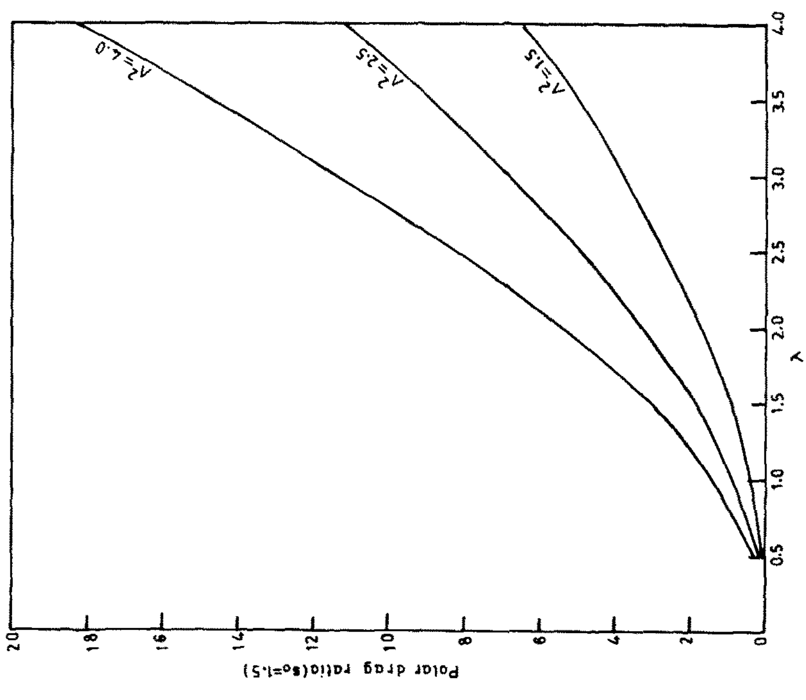

Fig. 8. Variation of polar drag ratio with respect to λ.

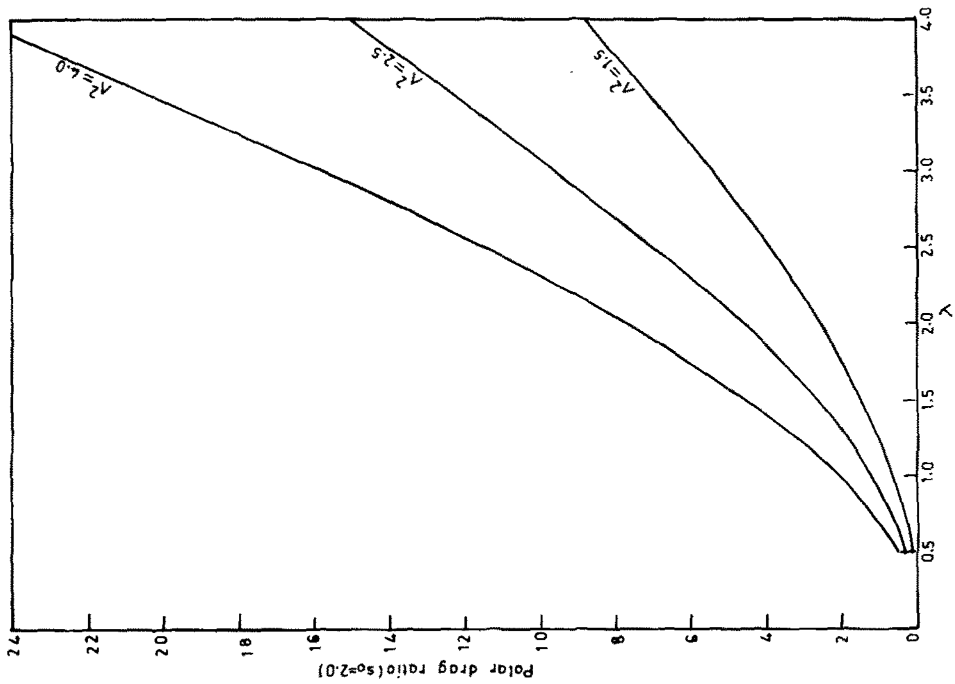

Fig. 9. Variation of polar drag ratio with respect to λ.

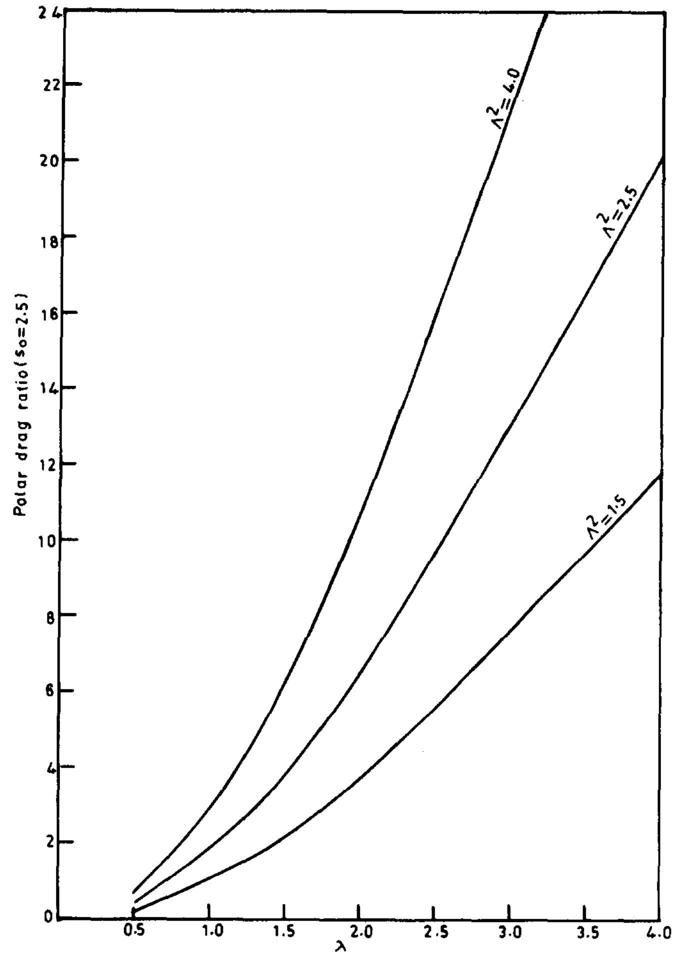

Fig. 10. Variation of polar drag ratio with respect to λ.

4. OBLATE SPHEROID

Let (ξ, η, ϕ) be oblate spheroidal coordinates such that

$$z + ir = c \sinh(\xi + i\eta) \tag{4.1}$$

and let

$$\sinh \xi = \tau, \quad \cos \eta = t. \tag{4.2}$$

We can build up a solution of the eqn (2.15) by the superposition of the solutions of the equations

$$E^4\Psi = 0 \tag{4.3}$$

and

$$\left(E^2 - \frac{\lambda^2}{c^2}\right)\Psi = 0. \tag{4.4}$$

In the oblate spheroidal coordinate system, the Stokesian stream function operator E^2 has the representation

$$\begin{aligned} E^2 &= \frac{1}{c^2(\sinh^2 \xi + \cos^2 \eta)}\left[\frac{\partial^2}{\partial \xi^2} - \tanh \xi \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \eta^2} - \cot \eta \frac{\partial}{\partial \eta}\right] \\ &= \frac{1}{c^2(\tau^2 + t^2)}\left[(\tau^2 + 1)\frac{\partial^2}{\partial \tau^2} + (1 - t^2)\frac{\partial^2}{\partial t^2}\right]. \end{aligned} \tag{4.5}$$

(i) *Solution of* $E^4\Psi = 0$. The solution of (4.3) is exhibited in the form

$$\Psi = \Psi_0 + \Psi_1 \tag{4.6}$$

where

$$\Psi_0 = -\frac{1}{2}Uc^2(\tau^2+1)(1-t^2) \tag{4.7}$$

and

$$\Psi_1 = c^2(\tau^2+1)(1-t^2)\sum_{n=0}^{\infty} H_{n+1}(i\tau)P'_{n+1}(t). \tag{4.8}$$

In (4.8) the function $P'_{n+1}(t)$ denotes the derivative of the Legendre polynomial $P_{n+1}(t)$ and the function $H_{n+1}(i\tau)$ is to be defined later. The function Ψ_0 in (4.7) represents the stream function due to a uniform stream at infinity of magnitude U in the direction of the axis of symmetry. Since Ψ_1 has to satisfy the eqn (4.3) we select the functions $H_{n+1}(i\tau)$ in (4.8) such that

$$E^2\Psi_1 = c^2(\tau^2+1)(1-t^2)\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(i\tau)P'_{n+1}(t) \tag{4.9}$$

where $Q'_{n+1}(i\tau)$ is the derivative of the Legendre function of the second kind. It is easily seen that the r.h.s. expression in (4.9) satisfies the equation

$$E^2 f = 0 \tag{4.10}$$

and the choice of the functions $H_{n+1}(i\tau)$ in (4.8) must be so as to validate (4.9). From the equation for Ψ_1 in (4.8) we can calculate $E^2\Psi_1$. On using the two identities

$$(\tau^2+1)Q'_n(i\tau) = -\frac{n(n+1)}{(2n-1)(2n+1)}Q'_{n-2}(i\tau) + \frac{2n(n+1)}{(2n-1)(2n+3)}Q'_n(i\tau) - \frac{n(n+1)}{(2n+1)(2n+3)}Q'_{n+2}(i\tau) \tag{4.11}$$

and

$$(1-t^2)P'_n(t) = -\frac{n(n+1)}{(2n-1)(2n+1)}P'_{n-2}(t) + \frac{2n(n+1)}{(2n-1)(2n+3)}P'_n(t) - \frac{n(n+1)}{(2n+1)(2n+3)}P'_{n+2}(t) \tag{4.12}$$

the expression for $E^2\Psi_1$ can be recast and identifying this with the expression in (4.9) we have the equation

$$(\tau^2+1)\frac{d^2}{d\tau^2}H_{n+1}(i\tau) + 4\tau\frac{d}{d\tau}H_{n+1}(i\tau) - n(n+3)H_{n+1}(i\tau) = h_{n+1}(i\tau) \tag{4.13}$$

where

$$\begin{aligned} h_{n+1}(i\tau) = &-c^2\left[\frac{(n+1)(n+2)}{(2n+3)(2n+5)}A_{n+1} - \frac{(n+3)(n+4)}{(2n+5)(2n+7)}A_{n+3}\right]Q'_{n+3}(i\tau) \\ &+ c^2\left[\frac{(n-1)n}{(2n-1)(2n+1)}A_{n-1} - \frac{(n+1)(n+2)}{(2n+1)(2n+3)}A_{n+1}\right]Q'_{n-1}(i\tau) \end{aligned}$$

$$\text{for} \quad n = 0,1,2,3,\ldots. \tag{4.14}$$

For $n = 0$, we have the term $h_1(i\tau)$ from the r.h.s. of (4.14) by deleting the term involving the coefficient A_{-1} and noting that

$$Q'_{-1}(i\tau) = \frac{i\tau}{\tau^2+1}. \tag{4.15}$$

The differential eqns (4.13) constitute a linear system and $H_{n+1}(i\tau)$ can be determined by integration using the method of variation of parameters. We have

$$H_{n+1}(i\tau) = \alpha_{n+1}P'_{n+1}(i\tau) + \beta_{n+1}Q'_{n+1}(i\tau) + \frac{iP'_{n+1}(i\tau)}{(n+1)(n+2)}\int_{\tau_0}^{\tau}(\tau^2+1)Q'_{n+1}(i\tau)h_{n+1}(i\tau)\,d\tau - \frac{iQ'_{n+1}(i\tau)}{(n+1)(n+2)}\int_{\tau_0}^{\tau}(\tau^2+1)P'_{n+1}(i\tau)h_{n+1}(i\tau)\,d\tau. \quad (4.16)$$

The lower limit τ_0 in the above integrals is positive and specifies the surface of the oblate spheroidal body which is nondegenerate. We stipulate that $\Psi_1 \to 0$ as $\tau \to \infty$ and this requires that $\alpha_{n+1} = 0$ in (4.16). Thus, we have

$$H_{n+1}(i\tau) = B_{n+1}Q'_{n+1}(i\tau) + \frac{iP'_{n+1}(i\tau)}{(n+1)(n+2)}\int_{\tau_0}^{\tau}(\tau^2+1)Q'_{n+1}(i\tau)h_{n+1}(i\tau)\,dt - \frac{iQ'_{n+1}(i\tau)}{(n+1)(n+2)}\int_{\tau_0}^{\tau}(\tau^2+1)P'_{n+1}(i\tau)h_{n+1}(i\tau)\,d\tau. \quad (4.17)$$

It is noted that these functions involve two sets of constants, viz. $\{B_{n+1}\}$ and $\{A_{n+1}\}$ (through $h_{n+1}(i\tau)$).

(ii) *Solution of* $(E^2 - (\lambda^2/c^2))\Psi = 0$. We can choose a solution of this equation in separable form using the radial and angular oblate spheroidal wave functions $R_{1n}(i\lambda, \tau)$ and $S_{1n}(i\lambda, t)$[3] and we have then

$$\Psi = c\sqrt{((\tau^2+1)(1-t^2))}\sum_{n=1}^{\infty} C_n R_{1n}(i\lambda, \tau) S_{1n}(i\lambda, t). \quad (4.18)$$

It is well known that the oblate spheroidal angular wave function $S_{1n}(i\lambda, t)$ can be represented as a prolate spheroidal wave function by changing the parameter from $i\lambda$ to λ. Also, the oblate radial wave function $R_{1n}(i\lambda, \tau)$ can be represented as a prolate spheroidal radial wave function by changing the parameter from $i\lambda$ to λ and the variable from τ to $i\tau$. Hence, the solution Ψ_2 of the eqn (4.4) may, therefore, be chosen in the form

$$\Psi_2 = c\sqrt{((\tau^2+1)(1-t^2))}\sum_{n=1}^{\infty} C_n R_{1n}(\lambda, i\tau) S_{1n}(\lambda, t) \quad (4.19)$$

where the functions R and S now specify the *prolate spheroidal* functions. The requirement of regularity of solution on the boundary restricts the angular function to the type $S_{1n}^{(1)}(\lambda, t)$. Since the function Ψ_2 has to vanish at infinity, we select the radial function in (4.19) to the specific type $R_{1n}^{(3)}(\lambda, i\tau)$. (Ref. [3]). We have, therefore, the solution Ψ_2 of (4.4) in the form

$$\Psi_2 = c\sqrt{((\tau^2+1)(1-t^2))}\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(\lambda, i\tau) S_{1n}^{(1)}(\lambda, t) \quad (4.20)$$

where $\{C_n\}$ are constants. The functions $R_{1n}^{(3)}(\lambda, i\tau)$ and $S_{1n}^{(1)}(\lambda, t)$ are given by the expansions

$$R_{1n}^{(3)}(\lambda, i\tau) = \left[i^{n+2}\sum_{r=0,1}^{\infty}{}'(r+1)(r+2)d_r^{1n}(\lambda)\right]^{-1}\left(\frac{2}{\pi\lambda}\right)^{1/2}\left(\frac{\tau^2+1}{\tau^3}\right)^{1/2} \sum_{r=0,1}^{\infty}{}'(r+1)(r+2)d_r^{1n}(\lambda)K_{r+3/2}(\lambda\tau) \quad (4.21)$$

and

$$S_{1n}^{(1)}(\lambda, t) = \sum_{r=0,1}^{\infty}{}' d_r^{1n}(\lambda)P_{r+1}^{(1)}(t) \quad (4.22)$$

where $K_{r+3/2}(\lambda\tau)$ is the modified Bessel function of the second kind and $P^{(1)}_{r+1}(t)$ is the associated Legendre function of the first kind.

(iii) *The stream function and microrotation.* The stream function satisfying the eqn (2.15) is given by

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 \tag{4.23}$$

and, therefore, we have

$$\Psi = -\frac{1}{2}Ur^2 + r^2\sum_{n=0}^{\infty} H_{n+1}(i\tau)P'_{n+1}(t) + r\sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(\lambda, i\tau)S^{(1)}_{1n}(\lambda, t) \tag{4.24}$$

where

$$r = c\sqrt{((\tau^2+1)(1-t^2))} \tag{4.25}$$

and $H_{n+1}(i\tau)$ is given in (4.17).

The only component B of the microrotation vector given in (2.16) is given by

$$B(\tau, t) = \frac{r}{2}\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(i\tau)P'_{n+1}(t) + \frac{\mu+k}{k}\frac{\lambda^2}{c^2}\sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(\lambda, i\tau)S^{(1)}_{1n}(\lambda, t). \tag{4.26}$$

The expression for Ψ in (4.24) consists of three infinite sets of constants, viz. $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ and these are to be determined by the boundary conditions on the spheroid $\tau = \tau_0$. Using the hyperstick condition of adherence we have on the boundary $\tau = \tau_0$

$$\begin{aligned} \Psi(\tau, t) &= 0 \\ \frac{\partial\Psi}{\partial\tau}(\tau, t) &= 0 \\ B(\tau, t) &= 0 \end{aligned} \tag{4.27}$$

and it is also true that

$$\frac{\partial\Psi}{\partial t}(\tau, t) = 0 \tag{4.28}$$

on $\tau = \tau_0$.

The three sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ have to be determined by invoking the above boundary conditions. However, it does not seem to be possible to have explicit evaluation of these constants since each of the three sets is an infinite set and the boundary conditions lead to an infinite linear but coupled system of equations involving them. One has, therefore, to resort to the evaluation of these constants by numerical computation for specific values of the parameters in the problem. The three conditions in (4.27) can be put in the form

$$c\sqrt{((\tau_0^2+1))}\sum_{n=0}^{\infty} B_{n+1}Q'_{n+1}(i\tau_0)P^{(1)}_{n+1}(t) + \sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(\lambda, i\tau_0)S^{(1)}_{1n}(\lambda, t) = \frac{1}{2}Uc\sqrt{((\tau_0^2+1)1(1-t^2))}, \tag{4.29}$$

$$\begin{aligned} c\sum_{n=0}^{\infty} \{\tau_0 Q'_{n+1}(i\tau_0) + (\tau_0^2+1)iQ''_{n+1}(i\tau_0)\}B_{n+1}P^{(1)}_{n+1}(t) \\ + \sqrt{((\tau_0^2+1))}\sum_{n=1}^{\infty} C_n\left\{\frac{d}{d\tau}R^{(3)}_{1n}(\lambda, i\tau)\right\}_{\tau=\tau_0} S^{(1)}_{1n}(\lambda, t) = \frac{1}{2}Uc\tau_0\sqrt{((1-t^2))}, \end{aligned} \tag{4.30}$$

$$c\sqrt{((\tau_0^2+1))}\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(i\tau_0)P^{(1)}_{n+1}(t) + \frac{2(2\mu+k)}{\gamma}\sum_{n=1}^{\infty} C_n R^{(3)}_{1n}(\lambda, i\tau_0)S^{(1)}_{1n}(\lambda, t) = 0. \tag{4.31}$$

The functions $P_n^{(1)}(t)$ constitute an orthogonal set on the interval $-1 \le t \le 1$ and it is, therefore, possible to obtain from (4.29) and (4.30) distinct expressions for B_{n+1} involving the set of constants $\{C_m\}$. Eliminating B_{n+1} from these two resulting expressions, we are led to a nonhomogeneous linear system of algebraic equations for the set $\{C_m\}$. These can, however, be evaluated by a numerical method after deciding the stage of truncation of the infinite system. The constants $\{B_n\}$ can then be evaluated since each of the B_n's is expressible in terms of the set $\{C_m\}$. From eqn (4.31) it is possible to express A_n in terms of the C_m's and we can, therefore, evaluate the constants $\{A_n\}$.

From (4.29) to (4.31) we derive the following three equations by using the orthogonality of the function $P_n^{(1)}(t)$

$$c\sqrt{((\tau_0^2+1))}Q'_{n+1}(i\tau_0)B_{n+1} + \sum_{m=1}^{\infty} C_m d_n^{1m}(\lambda)R_{1m}^{(3)}(\lambda, i\tau_0) = \frac{Uc}{2}\sqrt{((\tau_0^2+1))}\delta_{on}, \tag{4.32}$$

$$c\{\tau_0 Q'_{n+1}(i\tau_0) + (\tau_0^2+1)iQ''_{n+1}(i\tau_0)\}B_{n+1} + \sum_{m=1}^{\infty} C_m \sqrt{((\tau_0^2+1))}\left(\frac{d}{d\tau}R_{1m}^{(3)}(\lambda, i\tau)\right)_{\tau=\tau_0} \times d_n^{1m}(\lambda) = \frac{1}{2}Uc\tau_0\delta_{0n}, \tag{4.33}$$

$$c\sqrt{((\tau_0^2+1))}Q'_{n+1}(i\tau_0)A_{n+1} + \frac{2(2\mu+k)}{\gamma}\sum_{m=1}^{\infty} C_m d_n^{1m}(\lambda)R_{1m}^{(3)}(\lambda, i\tau_0) = 0. \tag{4.34}$$

Eliminating B_{n+1} from (4.32) and (4.33) we have the following linear nonhomogeneous system of algebraic equations for the unknown constants C_m

$$\sum_{m=1}^{\infty} D_{nm}C_m = \alpha_n \quad n = 0, 1, 2, \ldots \tag{4.35}$$

where

$$D_{nm} = d_n^{1m}(\lambda)\Big[\{\tau_0 Q'_{n+1}(i\tau_0) + (n+1)(n+2)iQ_{n+1}(i\tau_0)\}R_{1m}^{(3)}(\lambda, i\tau_0) + (\tau_0^2+1)Q'_{n+1}(i\tau_0)\left(\frac{d}{d\tau}R_{1m}^{(3)}(\lambda, i\tau)\right)_{\tau=\tau_0}\Big] \tag{4.36}$$

and

$$\alpha_n = -\frac{iUc}{\sqrt{((\tau_0^2+1))}}\delta_{0n} \tag{4.37}$$

for $n = 0, 1, 2, 3, \ldots$.

As in the case of prolate spheroid here again we have to resort to numerical determination of the coefficients A_n, B_n, C_n. Further it is also seen that the system (4.35) can be partitioned into the two subsystems

$$\left.\begin{aligned} &\sum_{m=1}^{\infty} D_{2n,m}C_m = \alpha_{2n}, \\ &\sum_{m=1}^{\infty} D_{2n+1,m}C_m = \alpha_{2n+1} \end{aligned}\right\} n = 0, 1, 2, \ldots \tag{4.38}$$

and the second subsystem turns out to be a homogeneous system for the same reason as in the case of prolate spheroid. We may, therefore, presume that the coefficients C_{2m} are all zero. From this we can also see that the coefficients B_{2n}, A_{2n} are also trivial. The constants C_{2n+1} are determined from the first subsystem

$$\sum_{m=0}^{\infty} D_{2n,2m+1}C_{2m+1} = \alpha_{2n} \quad n = 0, 1, 2, \ldots \tag{4.39}$$

The stream function $\Psi(\tau, t)$ and the microrotation component $B(\tau, t)$ are, therefore, expressed in the following form

$$\Psi(\tau, t) = -\frac{1}{2}Ur^2 + r^2 \sum_{n=0}^{\infty} H_{2n+1}(i\tau)P'_{2n+1}(t) + r \sum_{n=0}^{\infty} C_{2n+1}R^{(3)}_{1,2n+1}(\lambda, i\tau)S^{(1)}_{1,2n+1}(\lambda, t), \tag{4.40}$$

$$B(\tau, t) = \frac{r}{2}\sum_{n=0}^{\infty} A_{2n+1}Q'_{2n+1}(i\tau)P'_{2n+1}(t) + \frac{\mu + k}{k}\frac{\lambda^2}{c^2}\sum_{n=0}^{\infty} C_{2n+1}R^{(3)}_{1,2n+1}(\lambda, i\tau)S^{(1)}_{1,2n+1}(\lambda, t). \tag{4.41}$$

(iv) *Pressure distribution.* The equations of motion (2.4) and (2.5) can be written in the form below using oblate spheroidal coordinates (τ, t).

$$\frac{\partial p}{\partial \tau} = \frac{(2\mu + k)}{2c(\tau^2+1)}\frac{\partial}{\partial t}(E^2\Psi) - \frac{\gamma(\mu + k)}{2kc(\tau^2+1)}\frac{\partial}{\partial t}(E^4\Psi), \tag{4.42}$$

$$\frac{\partial p}{\partial t} = -\frac{(2\mu + k)}{2c(1-t^2)}\frac{\partial}{\partial \tau}(E^2\Psi) + \frac{\gamma(\mu + k)}{2kc(1-t^2)}\frac{\partial}{\partial \tau}(E^4\Psi). \tag{4.43}$$

These can be written in the form below on substituting for Ψ in (4.40)

$$\frac{\partial p}{\partial \tau} = -(2\mu + k)c \sum_{n=0}^{\infty} (2n+1)(n+1)A_{2n+1}Q'_{2n+1}(i\tau)P_{2n+1}(t) \tag{4.44}$$

$$\frac{\partial p}{\partial t} = (2\mu + k)c \sum_{n=0}^{\infty} (n+1)(2n+1)A_{2n+1}Q_{2n+1}(i\tau)P'_{2n+1}(t). \tag{4.45}$$

We find from either of these on integration the following expression for the pressure distribution

$$p(\tau, t) = (2\mu + k)ic \sum_{n=0}^{\infty} (n+1)(2n+1)A_{2n+1}Q_{2n+1}(i\tau)P_{2n+1}(t). \tag{4.46}$$

(v) *Surface stress.* The velocity vector $\mathbf{q}$ can be written in the form

$$\mathbf{q} = q_\xi \mathbf{e}_\xi + q_\eta \mathbf{e}_\eta \tag{4.47}$$

where

$$q_\xi = \frac{1}{c^2\sqrt{((\tau^2+t^2)(\tau^2+1))}}\frac{\partial\Psi}{\partial t}$$

$$q_\eta = \frac{1}{c^2\sqrt{((\tau^2+t^2)(1-t^2))}}\frac{\partial\Psi}{\partial \tau}. \tag{4.48}$$

The rate of strain components are given by

$$\begin{aligned}
e_{\xi\xi} &= \frac{1}{c^3(\tau^2+t^2)}\left[\Psi_{\tau t} - \frac{(2\tau^2+t^2+1)\tau}{(\tau^2+t^2)(\tau^2+1)}\Psi_t - \frac{t}{\tau^2+t^2}\Psi_\tau\right]\\
e_{\eta\eta} &= \frac{1}{c^3(\tau^2+t^2)}\left[-\Psi_{\tau t} + \frac{(1-\tau^2-2t^2)t}{(\tau^2+t^2)(1-t^2)}\Psi_\tau + \frac{\tau\Psi_t}{\tau^2+t^2}\right]\\
e_{\phi\phi} &= \frac{1}{c^3(\tau^2+t^2)}\left[\frac{\tau\Psi_t}{\tau^2+1} + \frac{t\Psi_\tau}{1-t^2}\right]\\
e_{\xi\eta} = e_{\eta\xi} &= \frac{(\tau^2+1)\Psi_{\tau\tau} - (1-t^2)\Psi_{tt}}{2c^3(\tau^2+t^2)\sqrt{((\tau^2+1)(1-t^2))}}\\
&\quad + \frac{t\sqrt{((1-t^2))}\Psi_t}{c^3(\tau^2+t^2)^2\sqrt{((\tau^2+1))}} - \frac{\tau\sqrt{((\tau^2+1))}\Psi_\tau}{c^3(\tau^2+t^2)^2\sqrt{((1-t^2))}}\\
e_{\eta\phi} &= e_{\phi\eta} = e_{\xi\phi} = e_{\phi\xi} = 0.
\end{aligned} \tag{4.49}$$

The spin = (1/2) curl **q** has only one nonzero component ω_ϕ in the direction of the vector $\mathbf{e}_\phi$ and this is given by

$$\omega_\phi = \frac{1}{2c\sqrt{((\tau^2+1)(1-t^2))}} E^2\Psi. \tag{4.50}$$

The only nonvanishing components of t_{ij} (defined in (1.5)) are $t_{\xi\xi}$, $t_{\eta\eta}$, $t_{\phi\phi}$, $t_{\xi\eta}$ and $t_{\eta\xi}$.

To evaluate the drag on the spheroid we note that the stress vector **t** on the boundary of the body is given by

$$\mathbf{t} = t_{\xi\xi}\mathbf{e}_\xi + t_{\xi\eta}\mathbf{e}_\eta. \tag{4.51}$$

We find that

$$(t_{\xi\xi})_{\tau_0} = -p(\tau_0, t) \tag{4.52}$$

and

$$(t_{\xi\eta})_{\tau_0} = \left\{\frac{(\mu+k)E^2\Psi}{c\sqrt{((\tau^2+1)(1-t^2))}}\right\}_{\tau=\tau_0}. \tag{4.53}$$

The stress vector has the component

$$(\text{Stress})_{\text{axial}} = \frac{1}{\sqrt{((\tau^2+t^2))}}\{t\sqrt{((\tau^2+1))}t_{\xi\xi} - \tau\sqrt{((1-t^2))}t_{\xi\eta}\} \tag{4.54}$$

along the direction of axis of symmetry and

$$(\text{Stress})_{\text{radial}} = \frac{1}{\sqrt{((\tau^2+t^2))}}\{\tau\sqrt{((1-t^2))}t_{\xi\xi} + t\sqrt{((\tau^2+1))}t_{\xi\eta}\} \tag{4.55}$$

in the radial direction in the meridian plane. The resultants of these two vector components over the entire surface of the body are obtained by integration and it is seen that the radial component integrates to zero. Thus, the resultant of the stress vector on the body is a force in the direction of the axis of symmetry and this gives the drag on the body. The drag D can be written in the form

$$D = 2\pi c^2\sqrt{((\tau_0^2+1))}\int_{-1}^{1}\{t\sqrt{((\tau^2+1))}t_{\xi\xi} - \tau\sqrt{((1-t^2))}t_{\xi\eta}\}_{\tau_0}\,dt \tag{4.56}$$

and this simplifies to

$$2\pi c^3(2\mu+k)(\tau_0^2+1)\left\{-\frac{2}{3}A_1 iQ_1(i\tau_0) - \frac{\tau_0}{2}\int_{-1}^{1}(1-t^2)\sum_{n=0}^{\infty}A_{2n+1}Q'_{2n+1}(i\tau_0)P'_{2n+1}(t)\,dt\right\} \tag{4.57}$$

and this is seen to be

$$\frac{4}{3}\pi c^3(2\mu+k)A_1 i. \tag{4.58}$$

This expression for the drag can also be recovered from the elegant formula of Ramkissoon and Majumdar[6]. This is shown in the Appendix.

Writing

$$D_0 = 4\pi(2\mu+k)Uc \tag{4.59}$$

we see that the drag is

$$D_0\cdot(\tilde{A}_1/3) \tag{4.60}$$

where

$$\tilde{A}_1 = A_1 i/(U/c^2). \tag{4.61}$$

We may refer to $(\tilde{A}_1/3)$ as the nondimensional drag and this depends upon the eccentricity of the oblate spheroid, the micropolarity parameter λ and an additional material constant $(2\mu+k)c^2/\gamma$.

The only nonvanishing shear stress components are $t_{\xi\eta}$ and $t_{\eta\xi}$. These two are no longer equal in view of the polar nature of the fluid and the shear stress difference has the expression

$$t_{\xi\eta}-t_{\eta\xi}=-\frac{(2\mu+k)\lambda^2U}{c}\sum_{n=0}^{\infty}\tilde{C}_{2n+1}R^{(3)}_{1,2n+1}(\lambda,i\tau)S^{(1)}_{1n}(\lambda,t) \tag{4.62}$$

where

$$\tilde{C}_{2n+1}=\frac{C_{2n+1}}{Uc}. \tag{4.63}$$

(vi) *Couple stress.* The couple stress tensor m_{ij} is given by the eqn (1.6) and we see that the only nonvanishing components of this tensor are

$$m_{\eta\phi},\ m_{\phi\eta},\ m_{\xi\phi},\ m_{\phi\xi}.$$

We find that

$$\begin{aligned}
m_{\eta\phi}&=-\frac{1}{c\sqrt{((\tau^2+t^2))}}\left\{\frac{\beta tB}{\sqrt{((1-t^2))}}+\gamma\sqrt{((1-t^2))}\frac{\partial B}{\partial t}\right\}\\
m_{\phi\eta}&=-\frac{1}{c\sqrt{((\tau^2+t^2))}}\left\{\beta\sqrt{((1-t^2))}\frac{\partial B}{\partial t}+\frac{\gamma tB}{\sqrt{((1-t^2))}}\right\}\\
m_{\xi\phi}&=\frac{1}{c\sqrt{((\tau^2+t^2))}}\left\{\frac{-\beta\tau B}{\sqrt{((\tau^2+1))}}+\gamma\sqrt{((\tau^2+1))}\frac{\partial B}{\partial\tau}\right\}\\
m_{\phi\xi}&=\frac{1}{c\sqrt{((\tau^2+t^2))}}\left\{\beta\sqrt{((\tau^2+1))}\frac{\partial B}{\partial\tau}-\frac{\gamma\tau B}{\sqrt{((\tau^2+1))}}\right\}.
\end{aligned} \tag{4.64}$$

The couple vector is $m_{\xi\phi}\mathbf{e}_\phi$ and on the boundary it reduces to

$$\left(\frac{\tau\sqrt{((\tau^2+1))}}{c\sqrt{((\tau^2+t^2))}}\left(\frac{\partial B}{\partial\tau}\right)\right)_{\tau_0}\mathbf{e}_\phi. \tag{4.65}$$

It is seen that the resultant couple vector due to the couple stresses on the spheroid is

$$c^2\sqrt{((\tau_0^2+1))}\int_{-1}^{1}\sqrt{((\tau_0^2+t^2))}\left(\int_{\phi=0}^{2\pi}(m_{\xi\phi})_{\tau_0}e_\phi\,\mathrm{d}\phi\right)\mathrm{d}t \tag{4.66}$$

and this vanishes since

$$\int_0^{2\pi}e_\phi\,\mathrm{d}\phi=0.$$

The moment of the stress vector about the centre of the spheroid is

$$m=(z\mathbf{e}_z+r\mathbf{e}_r)\times\mathbf{t} \tag{4.67}$$

and the integral of this over the surface of the spheroid is zero. The scalar moment of the stress vector about the axis of symmetry is $\mathbf{m}\cdot\mathbf{e}_z$ and this is zero everywhere. Thus, there is no couple exerted on the body in spite of the fluid sustaining a couple stress.

(vii) *Numerical results.* To extract numerical information and evaluate the drag on the body the system of eqns (4.35) was solved by restricting the system to a 5 by 5 linear system. The drag on the body is (see (4.58))

$$\frac{4\pi(2\mu+k)}{3}iA_1c^3=4\pi(2\mu+k)Uc\frac{\tilde{A}_1}{3} \tag{4.68}$$

and for the nonpolar fluid the drag has the value[4]

$$\frac{8\pi\mu Uc}{\tau_0-(\tau_0^2-1)\cot^{-1}\tau_0}. \tag{4.69}$$

The nondimensional drag for the polar fluid is $(\tilde{A}_1/3)$ and for the nonpolar case it is

$$\left(\frac{\tilde{A}_1}{3}\right)_n=[\tau_0-(\tau_0^2-1)\cot^{-1}\tau_0]^{-1}. \tag{4.70}$$

The drag ratio for the oblate spheroid is defined as the ratio of the drag on the spheroid to the drag on a sphere of radius equal to the semimajor axis of the meridian ellipse. The drag on the sphere of radius $c\sqrt{((\tau_0^2+1))}$ is[5]

$$\frac{3\pi Uc(2\mu+k)\sqrt{((\tau_0^2+1))}}{\left[1-\frac{1}{\Lambda^2(\lambda\sqrt{((\tau_0^2+1))}+1}\right]}. \tag{4.71}$$

The drag ratio for the oblate spheroid for polar fluid is

$$\frac{4}{3\sqrt{((\tau_0^2+1))}}\left[1-\frac{1}{\Lambda^2(\lambda\sqrt{((\tau_0^2+1))}+1}\right]\left(\frac{\tilde{A}_1}{3}\right) \tag{4.72}$$

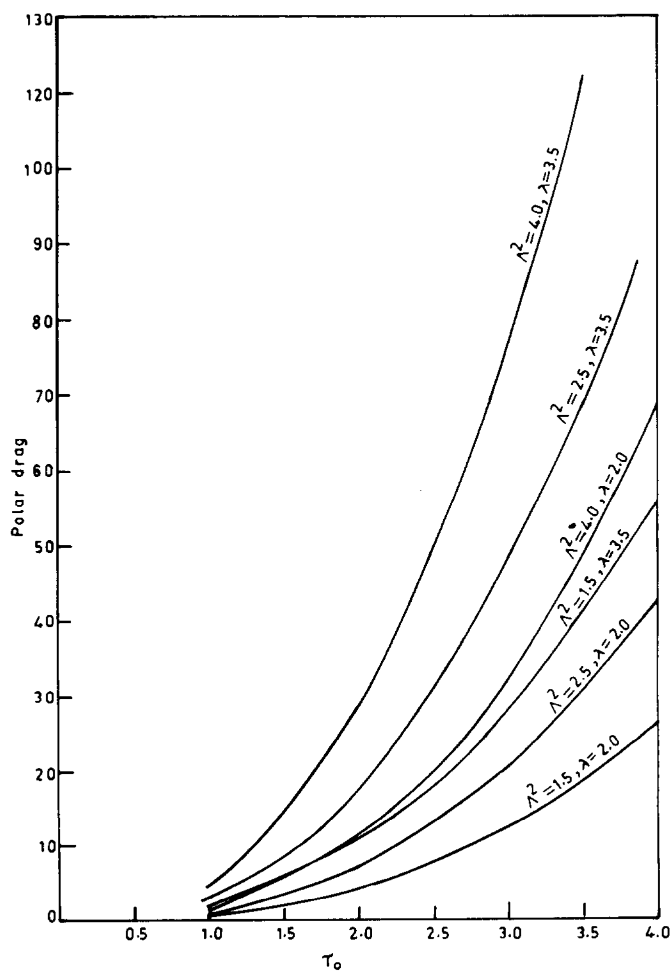

Fig. 11. Variation of polar drag with respect to τ_0.

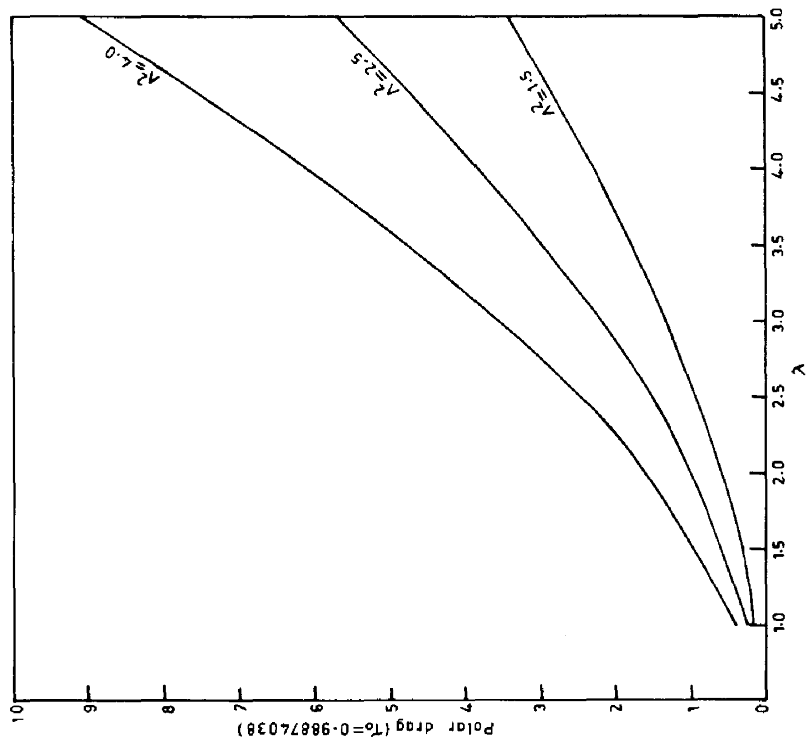

Fig. 13. Variation of polar drag with respect to λ.

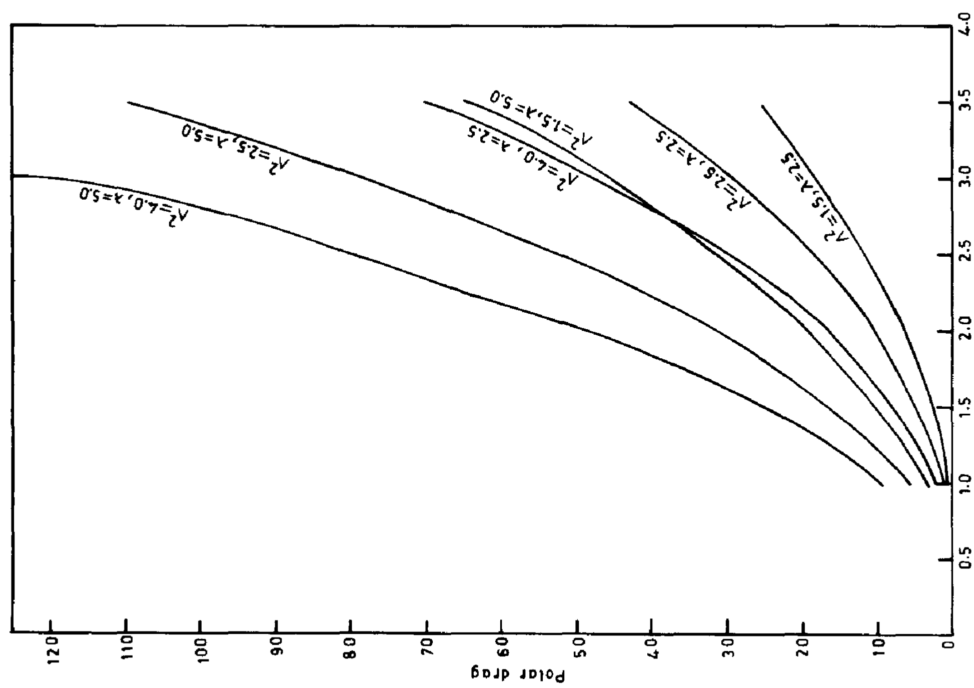

Fig. 12. Variation of polar drag with respect to τ_0.

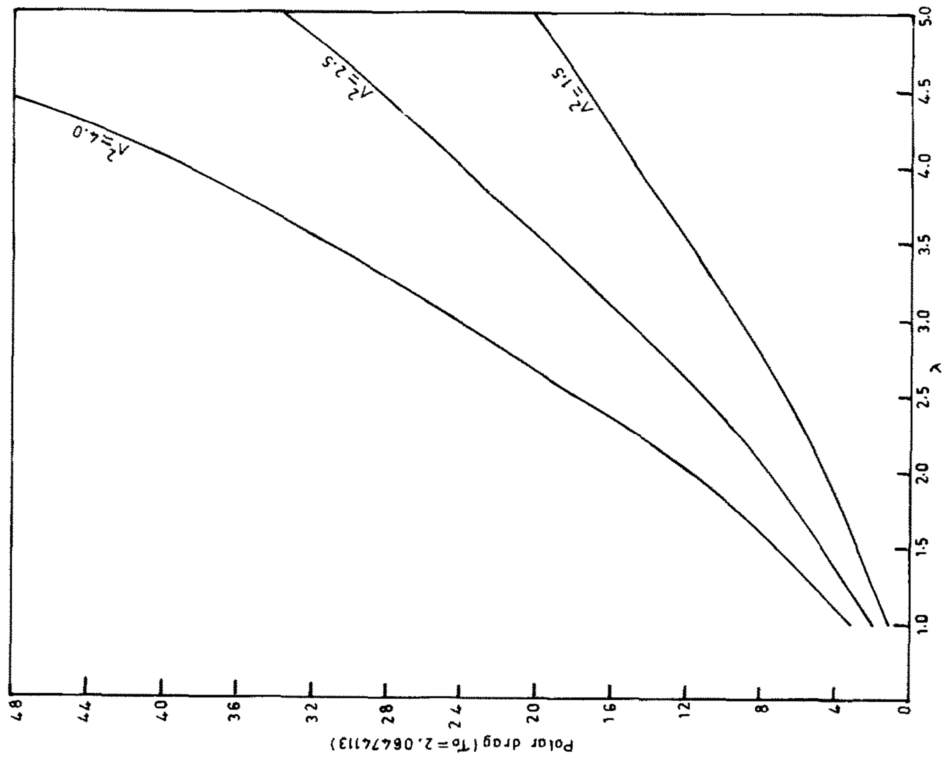

Fig. 15. Variation of polar drag with respect to λ.

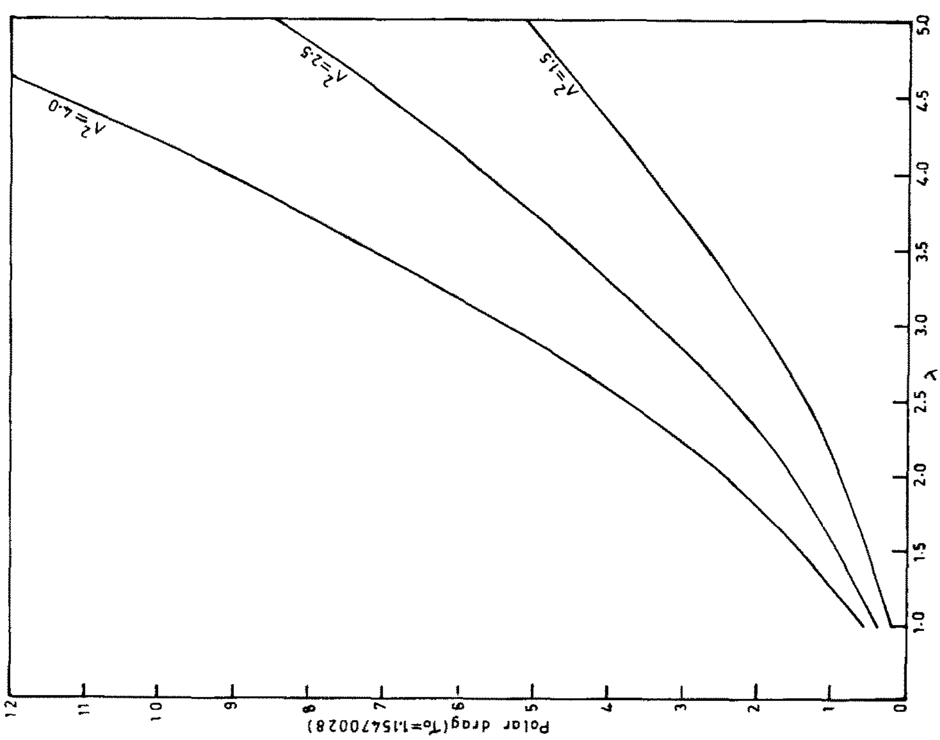

Fig. 14. Variation of polar drag with respect to λ.

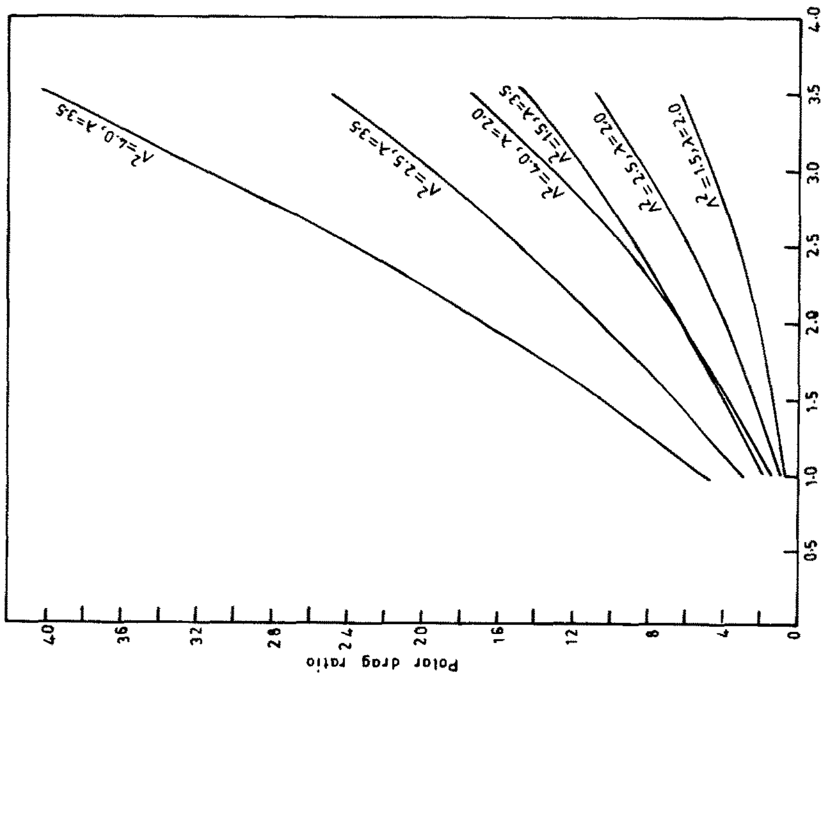

Fig. 16. Variation of polar drag ratio with respect to τ_0.

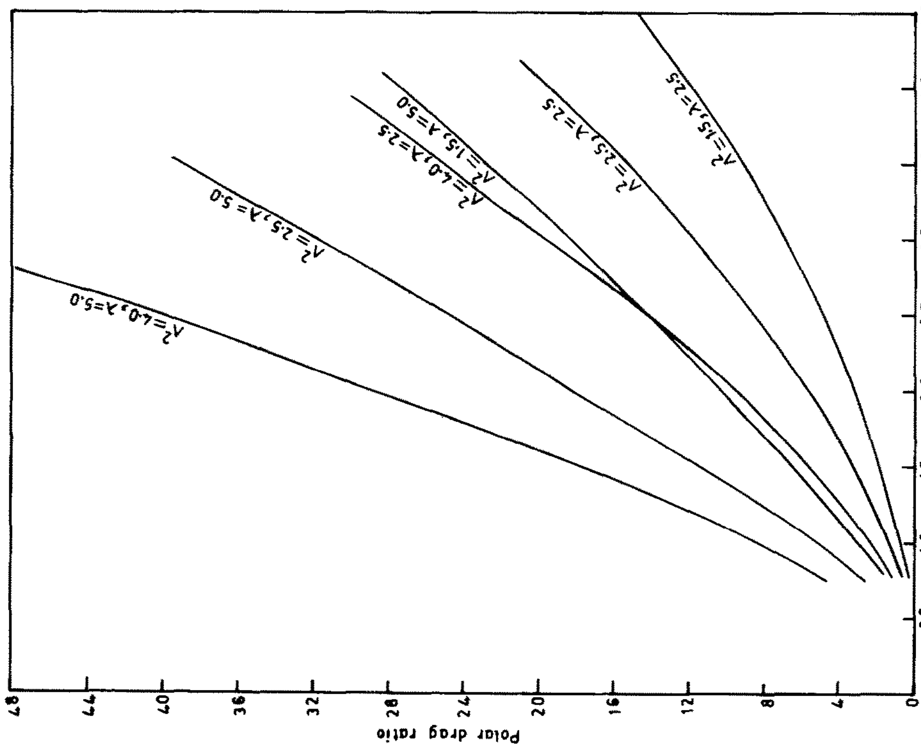

Fig. 17. Variation of polar drag ratio with respect to τ_0.

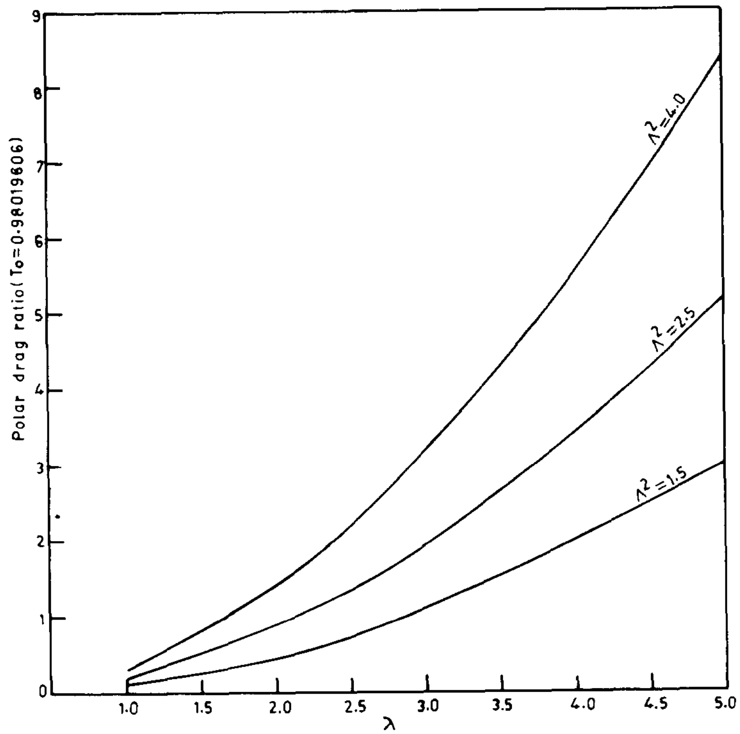

Fig. 18. Variation of polar drag ratio with respect to λ.

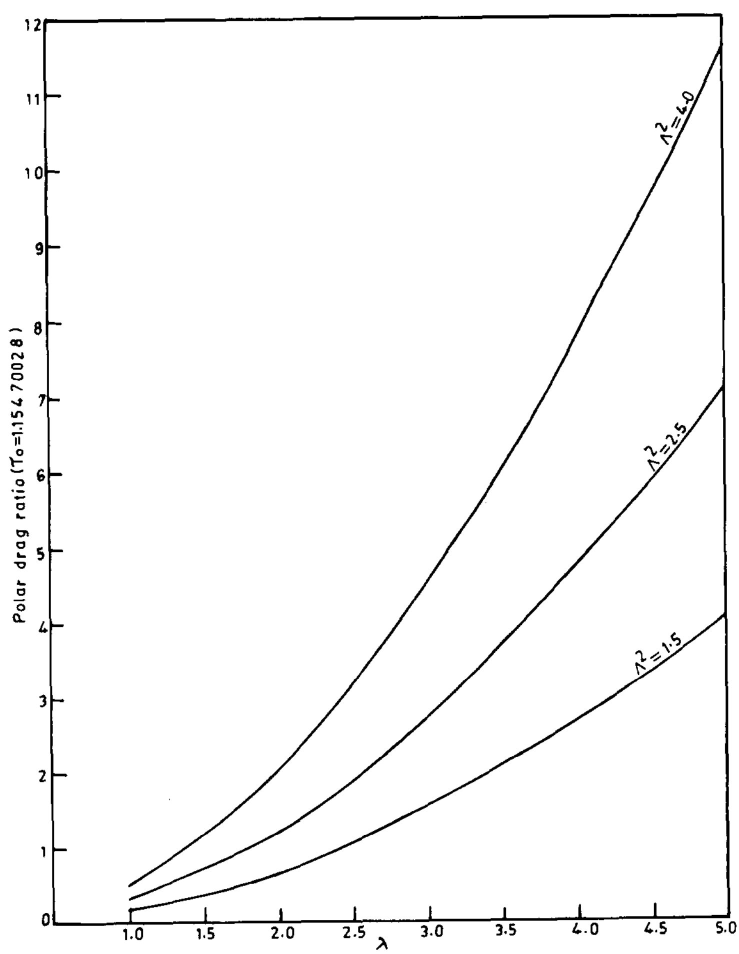

Fig. 19. Variation of polar drag ratio with respect to λ.

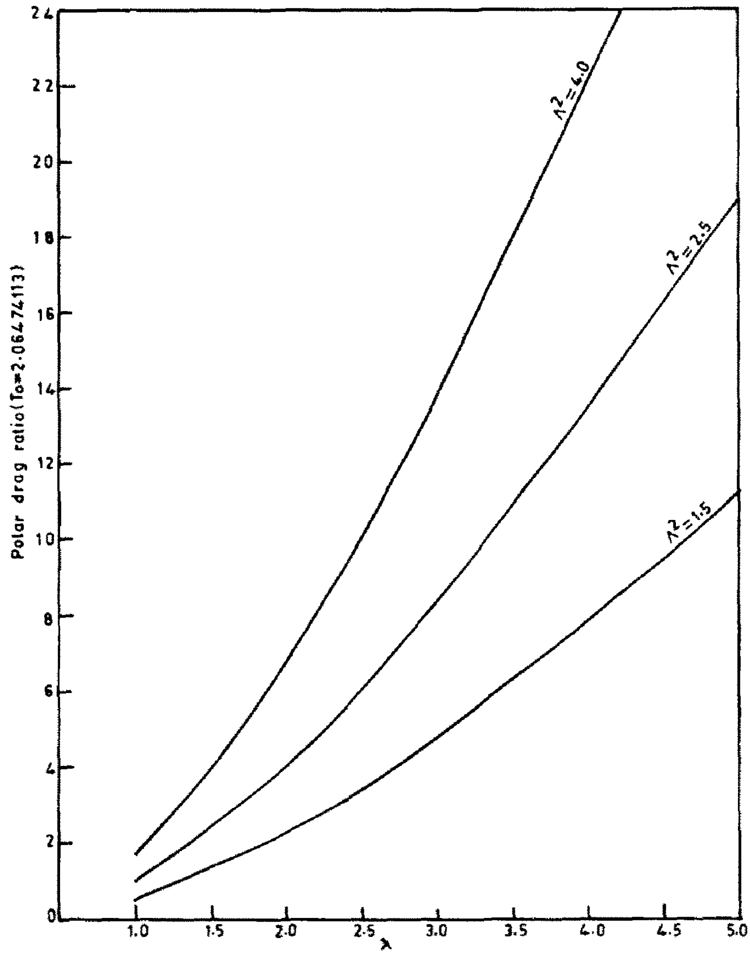

Fig. 20. Variation of polar drag ratio with respect to λ.

and this ratio for the nonpolar fluid equals

$$\frac{4}{3\sqrt{((\tau_0^2+1))}}\left(\frac{\tilde{A}_1}{3}\right)_n. \tag{4.73}$$

The figures giving the drag and the drag ratio for varying values of the parameters λ and Λ^2 show that the drag as well as the drag ratio increase with τ_0 and also with each of the parameters Λ^2 and λ.

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APPENDIX

The drag on an axially symmetric body in the Stokes flow of micropolar fluid has been expressed by an elegant formula by Ramkissoon and Majumdar[6]. This can be expressed in the form

$$D = 4\pi(2\mu+k)\lim_{\rho\to\infty}\frac{\rho(\Psi-\Psi_0)}{r^2} \tag{A1}$$

where

$$\rho = \sqrt{(r^2 + z^2)} \tag{A2}$$

is the polar radial distance of the point from the origin and Ψ, Ψ_0 denote, respectively, the stream function of the flow and its asymptotic value. From the solutions presented earlier, it is clear that for both the prolate and oblate spheroids

$$\Psi - \Psi_0 = \Psi_1 + \Psi_2. \tag{A3}$$

For the prolate spheroid

$$\rho^2 = c^2 s^2\left(1 - \frac{1-t^2}{s^2}\right) \tag{A4}$$

and the expressions for Ψ_1 and Ψ_2 are given in (3.8) and (3.27), respectively. It is easily seen that

$$\operatorname*{lt}_{\rho\to\infty} \frac{\rho\Psi_2}{r^2} = 0. \tag{A5}$$

The contribution to the limit of $(\rho\Psi_1/r^2)$ arises only from the first term $r^2G_1(s)P_1'(t)$ in the expansion of Ψ_1 and after some calculation, we see that•

$$\operatorname*{lt}_{s\to\infty} sG_1(s) = \frac{c^2A_1}{3}. \tag{A6}$$

Thus we have the drag on the prolate spheroid given by

$$D = 4\pi(2\mu + k)c^3A_1/3. \tag{A7}$$

For the oblate spheroid

$$\rho^2 = c^2\tau^2\left(1 + \frac{1-t^2}{2}\right) \tag{A8}$$

and the expressions for Ψ_1 and Ψ_2 are given in (4.8) and (4.20), respectively. It is easily seen that the only contribution to the limit in (A1) arises from the term $r^2H_1(i\tau)P_1'(t)$ in Ψ_1. By straight forward calculation it is seen that

$$\operatorname*{lt}_{\tau\to\infty} \tau H_i(i\tau) = ic^2A_1/3. \tag{A9}$$

Hence the drag on the oblate spheroid equals

$$4\pi(2\mu + k)c^3A_1/3. \tag{A10}$$