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THE RECTILINEAR OSCILLATIONS OF A SPHEROID IN A MICROPOLAR FLUID

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Abstract—The paper examines the oscillatory flow of incompressible micropolar fluid arising from the harmonic oscillation of a spheroid rectilinearly along its axis of symmetry under the assumption of small amplitude of oscillation. The velocity and microrotation are obtained and the drag experienced by the spheroid is determined. The drag parameters K and K' are numerically evaluated.

INTRODUCTION

THE CLASS of micropolar fluids introduced by Eringen[1] is a subclass of simple microfluids the study of which was inaugurated earlier by Eringen himself[2]. These fluids exhibit some microscopic effects arising from local structure and micromotions of the fluid elements and they can sustain couple stress. The field equations of micropolar fluids are presentable in terms of the velocity vector and the microrotation vector.

In this paper we examine the oscillatory flow of incompressible micropolar fluid arising from the harmonic oscillation of a spheroid rectilinearly along its axis of symmetry. The oscillation amplitude is assumed small and omission of second order terms is assumed valid. Analytical expressions are obtained in an infinite series form for the velocity, microrotation, surface and couple stress components. The drag experienced by the spheroid is determined and is expressed in terms of two parameters K and K' whose variations are noted by numerical computation for different sets of micropolarity, frequency and the geometric parameters.

2. BASIC EQUATIONS

The field equations of incompressible micropolar fluid dynamics are[1]

$$\operatorname{div} \mathbf{q} = 0 \tag{2.1}$$

$$\rho \frac{d\mathbf{q}}{dt} = \rho \mathbf{f} - \operatorname{grad} p + k \operatorname{curl} \boldsymbol{\nu} - (\mu + k) \operatorname{curl} \operatorname{curl} \mathbf{q} + (\lambda_1 + 2\mu + k) \operatorname{grad} \operatorname{div} \mathbf{q} \tag{2.2}$$

$$\rho j \frac{d\boldsymbol{\nu}}{dt} = \rho \boldsymbol{\iota} - 2k\boldsymbol{\nu} + k \operatorname{curl} \mathbf{q} - \gamma \operatorname{curl} \operatorname{curl} \boldsymbol{\nu} + (\alpha + \beta + \gamma) \operatorname{grad} \operatorname{div} \boldsymbol{\nu}. \tag{2.3}$$

In the above, the scalar quantities ρ and j are, respectively, the density and gyration parameters and are assumed constant. The vectors $\mathbf{q}$, $\boldsymbol{\nu}$, $\mathbf{f}$, $\boldsymbol{\iota}$ are the velocity, microrotation, body force per unit mass and body couple per unit mass. The material constants λ_1, μ, k and α, β, γ denote the viscosity and gyroviscosity coefficients and these are subject to the inequalities

$$k \geq 0; \quad 2\mu + k \geq 0; \quad 3\lambda_1 + 2\mu + k \geq 0;$$

$$\gamma \geq 0; \quad |\beta| \leq \gamma; \quad 3\alpha + \beta + \gamma \geq 0. \tag{2.4}$$

The stress tensor t_{ij} and the couple stress tensor m_{ij} are given by

$$t_{ij} = (-p + \lambda_1 \operatorname{div} \mathbf{q})\delta_{ij} + (2\mu + k)d_{ij} + k\epsilon_{ijm}(\omega_m - \nu_m), \tag{2.5}$$

$$m_{ij} = \alpha(\operatorname{div} \boldsymbol{\nu})\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i}. \tag{2.6}$$

In (2.5) and (2.6), ν_i and $2\omega_i$ are the components of the microrotation vector and yorticity vector, respectively, d_{ij} are the components of the rate of strain and a comma denotes covariant differentiation.

Let (ξ, η, ϕ) denote an axially symmetric system of coordinates and let $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\phi$ be the corresponding base vectors. The spheroid oscillates harmonically along its axis of symmetry and the speed of oscillation is $U \exp(i\omega t)$. The flow generated by this oscillation is axially symmetric and all the flow field functions are independent of the coordinate variable ϕ. We may choose the velocity and microrotation of the flow in the form

$$\mathbf{g} = \mathbf{Q}(\xi, \eta)\, \mathrm{e}^{i\omega t} = \{u(\xi, \eta)\, \mathbf{e}_\xi + v(\xi, \eta)\, \mathbf{e}_\eta\}\, \mathrm{e}^{i\omega t} \tag{2.7}$$

and

$$\boldsymbol{\nu} = (C(\xi, \eta)\, \mathbf{e}_\phi)\, \mathrm{e}^{i\omega t}. \tag{2.8}$$

Ignoring the body force and body couple terms $\mathbf{f}$, $\boldsymbol{\iota}$ and retaining only the linear terms in the eqns (2.2) and (2.3) the basic equations of the flow can be written in the form

$$\operatorname{div} \mathbf{q} = 0, \tag{2.9}$$

$$\rho \frac{\partial \mathbf{q}}{\partial t} = -\operatorname{grad} p + k \operatorname{curl} \boldsymbol{\nu} - (\mu + k) \operatorname{curl} \operatorname{curl} \mathbf{q}, \tag{2.10}$$

$$\rho j \frac{\partial \boldsymbol{\nu}}{\partial t} = 2k\boldsymbol{\nu} + k \operatorname{curl} \mathbf{q} - \gamma \operatorname{curl} \operatorname{curl} \boldsymbol{\nu} + (\alpha + \beta + \gamma) \operatorname{grad} (\operatorname{div} \boldsymbol{\nu}). \tag{2.11}$$

If h_1, h_2, h_3 are the scale factors of the coordinate systems (ξ, η, ϕ), we may write the velocity components in the form

$$h_2 h_3 u = -\frac{\partial \Psi}{\partial \eta}, \quad h_3 h_1 v = \frac{\partial \Psi}{\partial \xi} \tag{2.12}$$

where $\Psi(\xi, \eta)\, \mathrm{e}^{i\omega t}$ is the Stokes' stream function of the flow. Let

$$p = p(\xi, \eta)\, \mathrm{e}^{i\omega t}. \tag{2.13}$$

From (2.12) we have

$$\operatorname{curl} \mathbf{q} = \left\{\frac{1}{h_3} (E^2 \Psi)\, \mathrm{e}^{i\omega t}\right\} \mathbf{e}_\phi \tag{2.14}$$

in which the Stokes' stream function operator E^2 is given by

$$E^2 = \frac{h_3}{h_1 h_2} \left\{\frac{\partial}{\partial \xi}\left(\frac{h_2}{h_1 h_3} \frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_1}{h_2 h_3} \frac{\partial}{\partial \eta}\right)\right\} \tag{2.15}$$

and

$$\operatorname{curl} \operatorname{curl} \mathbf{q} = \frac{\mathrm{e}^{i\omega t}}{h_1 h_2 h_3} \left\{\left(h_1 \frac{\partial}{\partial \eta} E^2 \Psi\right) \mathbf{e}_\xi - \left(h_2 \frac{\partial}{\partial \xi} E^2 \Psi\right) \mathbf{e}_\eta\right\}. \tag{2.16}$$

From (2.8) we see that

$$\operatorname{div} \bar{\nu} = 0, \tag{2.17}$$

$$\operatorname{curl} \boldsymbol{\nu} = \left\{\frac{1}{h_2 h_3} \frac{\partial}{\partial \eta} (h_3 C)\, \mathbf{e}_\xi - \frac{1}{h_1 h_3} \frac{\partial}{\partial \xi} (h_3 C)\, \mathbf{e}_\eta\right\} \mathrm{e}^{i\omega t} \tag{2.18}$$

$$\operatorname{curl} \operatorname{curl} \bar{\nu} = -\left\{\left(\nabla^2 - \frac{1}{h_3^2}\right) C\right\} \mathrm{e}^{i\omega t}\, \mathbf{e}_\phi \tag{2.19}$$

where the Laplacian operator ∇^2 is given by

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial \xi}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial \eta}\right)\right\}. \tag{2.20}$$

From (2.13) it follows that

$$\operatorname{grad} p = \left(\frac{1}{h_1}\frac{\partial P}{\partial \xi}\mathbf{e}_\xi + \frac{1}{h_2}\frac{\partial P}{\partial \eta}\mathbf{e}_\eta\right)e^{i\omega t}. \tag{2.21}$$

The flow eqns (2.9)–(2.11) can be recast in the form below in terms of the two scalar functions Ψ and C.

$$-\frac{i\rho\omega}{h_2 h_3}\frac{\partial \Psi}{\partial \eta} = -\frac{1}{h_1}\frac{\partial P}{\partial \xi} + \frac{k}{h_2 h_3}\frac{\partial}{\partial \eta}(h_3 C) - \frac{(\mu + k)}{h_2 h_3}\frac{\partial}{\partial \eta}(E^2\Psi), \tag{2.22}$$

$$\frac{i\rho\omega}{h_3 h_1}\frac{\partial \Psi}{\partial \xi} = -\frac{1}{h_2}\frac{\partial P}{\partial \eta} - \frac{k}{h_1 h_3}\frac{\partial}{\partial \xi}(h_3 C) + \frac{(\mu + k)}{h_1 h_3}\frac{\partial}{\partial \xi}(E^2\Psi), \tag{2.23}$$

$$i\rho j\omega C = -2kC + \frac{k}{h_3}E^2\Psi + \gamma\left(\nabla^2 - \frac{1}{h_3^2}\right)C. \tag{2.24}$$

From (2.22) and (2.23) we can eliminate the pressure term and the resulting equation is

$$\{(\mu + k)E^4 - i\rho\omega E^2\}\Psi - kE^2(h_3 C) = 0. \tag{2.25}$$

The eqn (2.24) can be written also in the form

$$(2k + i\rho j\omega)h_3 C = \gamma E^2(h_3 C) + kE^2\Psi. \tag{2.26}$$

From (2.25) and (2.26) we can eliminate the function C and obtain the following differential equation for the stream function Ψ.

$$\{\gamma(\mu + k)E^6 - [k(2\mu + k) + i\rho\omega(\gamma + j\mu + jk)]E^4 + i\rho\omega(2k + i\rho j\omega)E^2\}\Psi = 0. \tag{2.27}$$

The function C is expressible in terms of Ψ in the form

$$k(2k + i\rho j\omega)(h_3 C) = \{\gamma(\mu + k)E^4 + (k^2 - i\rho\omega\gamma)E^2\}\Psi. \tag{2.28}$$

The problem thus reduces to the determination of the two scalar functions $\Psi(\xi, \eta)$ and $C(\xi, \eta)$ which are governed by the eqns (2.27) and (2.28) subject to the following conditions.

(i) Far away from the oscillating body there is practically no flow and the functions Ψ, C tend to zero.

(ii) At the boundary of the oscillating body we have the hyperstick or the superadherence condition and the velocity of a fluid element on the body equals that of the oscillating body while the microrotation of the fluid element is zero.

The eqn (2.27) for the determination of Ψ can also be cast into the form

$$E^2(E^2 - \alpha^2)(E^2 - \beta^2)\Psi = 0 \tag{2.29}$$

where α^2 and β^2 are constants to be determined from the relations

$$\alpha^2 + \beta^2 = \frac{k(2\mu + k) + i\rho\omega(\gamma + j\mu + jk)}{\gamma(\mu + k)}, \tag{2.30}$$

$$\alpha^2\beta^2 = \frac{i\rho\omega(2k + i\rho j\omega)}{\gamma(\mu + k)}. \tag{2.31}$$

In view of (2.29) we can build up the solution Ψ by superposition of the solutions of

$$E^2\Psi = 0, \tag{2.32}$$

$$(E^2 - \alpha^2)\Psi = 0, \tag{2.33}$$

$$(E^2 - \beta^2)\Psi = 0. \tag{2.34}$$

3. PROLATE SPHEROID

Let a prolate spheroid (focal distance $= 2c$) oscillate harmonically along its axis of symmetry and let the velocity of the spheroid be $U\,e^{i\omega t}$ in the above direction. We select (ξ, η, ϕ) to represent the prolate spheroidal coordinate system whose scale factors are given by

$$h_1 = h_2 = c\sqrt{(s^2 - t^2)}, \quad h_3 = c\sqrt{((s^2 - 1)(1 - t^2))} \tag{3.1}$$

where

$$s = \cosh \xi, \quad t = \cos \eta. \tag{3.2}$$

The Stokesian stream function operator E^2 is then given by

$$E^2 = \frac{1}{c^2(s^2 - t^2)}\left\{(s^2 - 1)\frac{\partial^2}{\partial s^2} + (1 - t^2)\frac{\partial^2}{\partial t^2}\right\}. \tag{3.3}$$

Let Ψ_0, Ψ_1, Ψ_2 denote, respectively, the solutions of eqns (2.32)–(2.34) which are regular far away from the spheroid. We may choose Ψ_0 in the form

$$\Psi_0 = h_3 \sum_{n=1}^{\infty} A_n Q_n^{(1)}(s) P_n^{(1)}(t) \tag{3.4}$$

where $\{A_n\}$ is an infinite set of constants and the symbols $P_n^{(1)}$ and $Q_n^{(1)}$ represent the Associated Legendre functions. The solutions Ψ_1 and Ψ_2 can similarly be represented in terms of the radial spheroidal wave functions R and angular spheroidal wave functions S with appropriate parameters. To ensure the regularity of these functions on the axis of symmetry we have to restrict the angular wave functions S to the first kind. Further, to ensure the regularity of the solution far from the body, we *select* the parameters α and β from the solutions of (2.30) and (2.31) so as to have positive real parts and the radial wave functions R to be of the third kind [3]. We then have

$$\Psi_1 = h_3 \sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(i\alpha c, s) S_{1n}^{(1)}(i\alpha c, t) \tag{3.5}$$

$$\Psi_2 = h_3 \sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\beta c, s) S_{1n}^{(1)}(i\beta c, t) \tag{3.6}$$

where R and S are the prolate spheroidal wave functions. From [3] (pp. 753–756), we have

$$R_{1n}^{(3)}(i\alpha c, s) = \left\{ \sum_{r=0,1}^{\infty}{}' (r+1)(r+2) d_r^{1n}(i\alpha c) \right\}^{-1} \left(\sqrt{\frac{\pi}{2\alpha c}}\right) \left(\frac{s^2 - 1}{s^3}\right)^{1/2}$$
$$\times \sum_{r=0,1}^{\infty}{}' i^{r-n+(1/2)} (r+1)(r+2) d_r^{1n}(i\alpha c) H_{r+3/2}^{(1)}(i\alpha c s), \tag{3.7}$$

and

$$S_{1n}^{(1)}(i\alpha c, t) = \sum_{r=0,1}^{\infty}{}' d_r^{1n}(i\alpha c) P_{r+1}^{(1)}(t), \tag{3.8}$$

where $H_{r+(3/2)}^{(1)}(i\alpha c s)$ denotes Hankel function of the first kind.

The Hankel function is expressible in terms of the modified Bessel function of the second kind in the form[4] (p. 204)

$$H^{(1)}_{r+(3/2)}(i\alpha cs) = \frac{2}{\pi}\exp\left(-\left(r+\frac{5}{2}\right)\frac{\pi i}{2}\right)K_{r+(3/2)}(\alpha cs) \tag{3.9}$$

and hence

$$R^{(3)}_{1n}(i\alpha c, s) = \left\{i^{n+2}\sum_{r=0,1}^{\infty}{}'(r+1)(r+2)d^{1n}_r(i\alpha c)\right\}^{-1}\left(\frac{2(s^2-1)}{\pi\alpha cs^3}\right)^{1/2}$$
$$\times\sum_{r=0,1}^{\infty}{}'(r+1)(r+2)d^{1n}_r(i\alpha c)K_{r+(3/2)}(\alpha cs). \tag{3.10}$$

The stream function for the flow is, therefore, given by

$$\Psi = h_3\left\{\sum_{n=1}^{\infty}A_nQ^{(1)}_n(s)P^{(1)}_n(t) + \sum_{n=1}^{\infty}B_nR^{(3)}_{1n}(i\alpha c, s)S^{(1)}_{1n}(i\alpha c, t) + \sum_{n=1}^{\infty}C_nR^{(3)}_{1n}(i\beta c, s)S^{(1)}_{1n}(i\beta c, t)\right\} \tag{3.11}$$

and this involves three infinite sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$. The velocity components u and v and the solitary microrotation component C are all determined from the above expression Ψ on using the relations (2.12) and (2.28). We can easily arrive at the following expressions for u, v and C

$$u = \frac{1}{c\sqrt{(s^2-t^2)}}\left\{\sum_{n=1}^{\infty}A_nQ^{(1)}_n(s)\frac{d}{dt}(\sqrt{(1-t^2)}P^{(1)}_n(t))\right.$$
$$+\sum_{n=1}^{\infty}B_nR^{(3)}_{1n}(i\alpha c, s)\frac{d}{dt}(\sqrt{(1-t^2)}S^{(1)}_{1n}(i\alpha c, t))$$
$$\left.+\sum_{n=1}^{\infty}C_nR^{(3)}_{1n}(i\beta c, s)\frac{d}{dt}(\sqrt{(1-t^2)}S^{(1)}_{1n}(i\beta c, t))\right\}, \tag{3.12}$$

$$v = \frac{1}{c\sqrt{(s^2-t^2)}}\left\{\sum_{n=1}^{\infty}A_n\frac{d}{ds}(\sqrt{(s^2-1)}Q^{(1)}_n(s))P^{(1)}_n(t)\right.$$
$$+\sum_{n=1}^{\infty}B_n\frac{d}{ds}(\sqrt{(s^2-1)}R^{(3)}_{1n}(i\alpha c, s))S^{(1)}_{1n}(i\alpha c, t)$$
$$\left.+\sum_{n=1}^{\infty}C_n\frac{d}{ds}(\sqrt{(s^2-1)}R^{(3)}_{1n}(i\beta c, s))S^{(1)}_{1n}(i\beta c, t)\right\}, \tag{3.13}$$

and

$$C = \frac{(\mu+k)\alpha^2 - i\rho\omega}{k}\sum_{n=1}^{\infty}B_nR^{(3)}_{1n}(i\alpha c, s)S^{(1)}_{1n}(i\alpha c, t)$$
$$+\frac{(\mu+k)\beta^2 - i\rho\omega}{k}\sum_{n=1}^{\infty}C_nR^{(3)}_{1n}(i\beta c, s)S^{(1)}_{1n}(i\beta c, t). \tag{3.14}$$

The three infinite sets of constants in the above functions u, v and C have to be determined by invoking the boundary conditions. Let the oscillating spheroid be given by $s = s_0$. In view of the hyperstick boundary conditions, the velocity $\mathbf{q}$ reduces to

$$\frac{U\,e^{i\omega t}(t\sqrt{(s^2-1)}\,\mathbf{e}_\xi - s\sqrt{(1-t^2)}\,\mathbf{e}_\eta)}{\sqrt{(s^2-t^2)}} \tag{3.15}$$

on $s = s_0$ and $C = 0$ and thus for $|t| \leq 1$

$$u(s_0, t) = \frac{U\surd(s_0^2 - 1)}{\surd(s_0^2 - t^2)} t, \tag{3.16}$$

$$v(s_0, t) = \frac{U\surd(1 - t^2)}{\surd(s_0^2 - t_0^2)} s_0, \tag{3.17}$$

$$C(s_0, t) = 0. \tag{3.18}$$

Determination of the constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$

From the boundary conditions (3.16)–(3.18) we have the following three equations valid for $|t| \leq 1$

$$-\sum_{n=1}^{\infty} n(n+1)A_n Q_n^{(1)}(s_0)P_n(t)$$
$$+\sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(i\alpha c, s_0)\frac{\mathrm{d}}{\mathrm{d}t}(\surd(1 - t^2)S_{1n}^{(1)}(i\alpha c, t))$$
$$+\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\beta c, s_0)\frac{\mathrm{d}}{\mathrm{d}t}(\surd(1 - t^2)S_{1n}^{(1)}(i\beta c, t)) = Uc\surd(s_0^2 - 1)t, \tag{3.19}$$

$$\sum_{n=1}^{\infty} n(n+1)A_n Q_n(s_0)P_n^{(1)}(t)$$
$$+\sum_{n=1}^{\infty} B_n \left[\frac{\mathrm{d}}{\mathrm{d}s}(\surd(s^2 - 1)R_{1n}^{(3)}(i\alpha c, s))\right]_{s_0} S_{1n}^{(1)}(i\alpha c, t)$$
$$+\sum_{n=1}^{\infty} C_n \left[\frac{\mathrm{d}}{\mathrm{d}s}(\surd(s^2 - 1)R_{1n}^{(3)}(i\beta c, s))\right]_{s_0} S_{1n}^{(1)}(i\beta c, t)$$
$$= -\ Ucs_0\surd(1 - t^2), \tag{3.20}$$

$$((\mu + k)\alpha^2 - i\rho\omega)\sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(i\alpha c, s_0)S_{1n}^{(1)}(i\alpha c, t)$$
$$+((\mu + k)\beta^2 - i\rho\omega)\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\beta c, s_0)S_{1n}^{(1)}(i\beta c, t) = 0. \tag{3.21}$$

From (3.19) we can isolate the constant A_n by first multiplying the equation by $P_n(t)$ and integrating with respect to t from -1 to 1. The result can be expressed in the form

$$A_n Q_n^{(1)}(s_0) + \sum_{m=1}^{\infty} B_m R_{1m}^{(3)}(i\alpha c, s_0)d_{n-1}^{1m}(i\alpha c)$$
$$+\sum_{m=1}^{\infty} C_m R_{1m}^{(3)}(i\beta c, s_0)d_{n-1}^{1m}(i\beta c) = -\frac{1}{2}Uc\surd(s_0^2 - 1)\delta_{n1}$$
$$n = 1, 2, 3, \ldots. \tag{3.22}$$

From the eqn (3.20) we can similarly deduce the equation

$$n(n+1)A_n Q_n(s_0) + \sum_{m=1}^{\infty} B_m d_{n-1}^{1m}(i\alpha c)\left[\frac{\mathrm{d}}{\mathrm{d}s}(\surd(s^2 - 1)R_{1m}^{(3)}(i\alpha c, s))\right]_{s_0}$$
$$+\sum_{m=1}^{\infty} C_m d_{n-1}^{1m}(i\beta c)\left[\frac{\mathrm{d}}{\mathrm{d}s}(\surd(s^2 - 1)R_{1m}^{(3)}(i\beta c, s))\right]_{s_0} = Ucs_0\delta_{n1}$$
$$n = 1, 2, 3, \ldots. \tag{3.23}$$

From (3.21) we can express the constant B_n in terms of the infinite set $\{C_m\}$ by multiplying it by the function $S_{1n}^{(1)}(i\alpha c, t)$ and integrating with respect to t from -1 to 1. It is known that the functions $\{S_{1n}^{(1)}\}$ are orthogonal over the interval -1 to 1. Indeed we can check the result ([5], p. 151)

$$\int_{-1}^{1} S_{1n}^{(1)}(i\alpha c, t)S_{1m}^{(1)}(i\alpha c, t)\,\mathrm{d}t = N_{nn}(i\alpha c)\delta_{nm} \tag{3.24}$$

where

$$N_{nn}(i\alpha c) = \sum_{r=0,1}^{\infty}{}' \frac{2(r+1)(r+2)}{(2r+3)}(d_r^{1n}(i\alpha c))^2. \tag{3.25}$$

Further we can verify that

$$\int_{-1}^{1} S_{1n}^{(1)}(i\alpha c, t)S_{1m}^{(1)}(i\beta c, t)\,\mathrm{d}t = M_{nm}(i\alpha c, i\beta c) = \sum_{r=0,1}^{\infty}{}' \frac{2(r+1)(r+2)}{(2r+3)} d_r^{1n}(i\alpha c)d_r^{1m}(i\beta c). \tag{3.26}$$

From the eqn (3.21) we can, therefore, deduce the following relation which expresses each of the constants $B_n (n = 1, 2, \ldots)$ as a linear combination of the constants $\{C_m\}$.

$$((\mu + k)\alpha^2 - i\rho\omega)N_{nn}(i\alpha c)B_n R_{1n}^{(3)}(i\alpha c, s_0) + ((\mu + k)\beta^2 - i\rho\omega)\sum_{m=1}^{\infty} C_m R_{1m}^{(3)}(i\beta c, s_0)M_{nm}(i\alpha c, i\beta c) = 0, \tag{3.27}$$

$$\text{for} \quad n = 1, 2, 3, \ldots.$$

From (3.22) and (3.23) we can eliminate the constant A_n and the result is an equation that connects the constants $\{B_n\}$ and $\{C_n\}$. From this equation we can replace each of the constants B_m in terms of the constants $\{C_m\}$ (using (3.27)) and, thus, we have a nonhomogeneous infinite system of linear algebraic equations in which the unknowns are the constants $\{C_m\}$. The constants $\{C_m\}$ have to be determined from this infinite system. Once this is done, the constants $\{B_n\}$ can be determined from (3.27). The constants $\{A_n\}$ can later be determined from (3.22) or (3.23). It is also possible to express the constants A_n directly as a linear combinations of C_m's apart from the term involving the parameter U. Thus we have a feasible procedure for the determination of each of the sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$. It has not been possible to obtain exact analytical expressions for the coefficients A_n, B_n, C_n though it is seen easily later that these constants vanish for even values of n. The details are shown in the appendix.

Pressure distribution. The eqns (2.22) and (2.23) can be simplified by expressing C in terms of Ψ from the eqn (2.28). We find that

$$\frac{\partial P}{\partial s} = i\rho\omega \sum_{n=1}^{\infty} n(n+1)A_n Q_n'(s)P_n(t), \tag{3.28}$$

$$\frac{\partial P}{\partial t} = i\rho\omega \sum_{n=1}^{\infty} n(n+1)A_n Q_n(s)P_n'(t). \tag{3.29}$$

By integrating these two equations we obtain the pressure distribution in the form (an additive constant is ignored).

$$P(s, t) = i\rho\omega \sum_{n=1}^{\infty} n(n+1)A_n Q_n(s)P_n(t). \tag{3.30}$$

Strain rate components. The rate of deformation components are

$$e_{jk} = E_{jk}\, e^{i\omega t} \tag{3.31}$$

where

$$E_{jk} = \frac{1}{2}(Q_{j,k} + Q_{k,j}). \tag{3.32}$$

We find that

$$\begin{aligned}
E_{\xi\xi} &= \frac{1}{c^3(s^2-t^2)}\left\{\Psi_{st} + \frac{t\Psi_s}{(s^2-t^2)} - \frac{s(2s^2-1-t^2)}{(s^2-t^2)(s^2-1)}\Psi_t\right\},\\
E_{\eta\eta} &= \frac{1}{c^3(s^2-t^2)}\left\{-\Psi_{st} + \frac{s\Psi_t}{(s^2-t^2)} + \frac{t(2t^2-1-s^2)}{(s^2-t^2)(1-t^2)}\Psi_s\right\},\\
E_{\phi\phi} &= \frac{1}{c^3(s^2-t^2)}\left\{\frac{s\Psi_t}{s^2-1} + \frac{t\Psi_s}{(1-t^2)}\right\},\\
E_{\xi\eta} = E_{\eta\xi} &= \frac{(s^2-1)\Psi_{ss} - (1-t^2)\Psi_{tt}}{2c^3(s^2-t^2)\sqrt{((s^2-1)(1-t^2))}}\\
&\quad - \frac{s\sqrt{(s^2-1)}\Psi_s}{c^3(s^2-t^2)^2\sqrt{(1-t^2)}} - \frac{t\sqrt{(1-t^2)}\Psi_t}{c^3(s^2-t^2)^2\sqrt{(s^2-1)}},\\
E_{\xi\phi} = E_{\phi\xi} &= E_{\eta\phi} = E_{\phi\eta} = 0.
\end{aligned} \tag{3.33}$$

The non-zero spin component is given by

$$\omega_\phi = \frac{1}{2h_3}E^2\Psi. \tag{3.34}$$

Stress components. The stress tensor t_{jk} defined in (2.5) can be written in the form

$$t_{jk} = T_{jk}\, e^{i\omega t}. \tag{3.35}$$

We find that

$$\begin{aligned}
T_{\xi\xi} &= -P + (2\mu + k)E_{\xi\xi}\\
T_{\eta\eta} &= -P + (2\mu + k)E_{\eta\eta}\\
T_{\phi\phi} &= -P + (2\mu + k)E_{\phi\phi}\\
T_{\xi\eta} &= (2\mu + k)E_{\xi\eta} + k(\omega_\phi - C)\\
T_{\eta\xi} &= (2\mu + k)E_{\eta\xi} - k(\omega_\phi - C)\\
T_{\phi\xi} &= T_{\xi\phi} = T_{\eta\phi} = T_{\phi\eta} = 0.
\end{aligned} \tag{3.36}$$

The stress vector (**t**) on the surface of the spheroid is

$$(\mathbf{t}) = t_{\xi\xi}\,\mathbf{e}_\xi + t_{\xi\eta}\,\mathbf{e}_\eta + t_{\xi\phi}\,\mathbf{e}_\phi \tag{3.37}$$

and its component along the axis of the spheroid is

$$(t_{\xi\xi}t\sqrt{(s^2-1)} - t_{\xi\eta}s\sqrt{(1-t^2)})/\sqrt{(s^2-t^2)}. \tag{3.38}$$

We have

$$t_{\xi\xi} = \left\{ -P + \frac{(2\mu + k)}{c^3(s^2 - t^2)} \left[\Psi_{st} + \frac{t\Psi_s}{s^2 - t^2} - \frac{s(2s^2 - t^2 - 1)}{(s^2 - t^2)(s^2 - 1)} \Psi_t \right] \right\} e^{i\omega t}, \tag{3.39}$$

$$t_{\xi\eta} = \left\{ \frac{(2\mu + k)}{2c^3(s^2 - t^2)^2\sqrt{((s^2 - 1)(1 - t^2))}} [(s^2 - t^2)((s^2 - 1)\Psi_{ss} - (1 - t^2)\Psi_{tt}) - 2s(s^2 - 1)\Psi_s - 2t(1 - t^2)\Psi_t] + k \left(\frac{E^2\Psi}{2h_3} - C \right) \right\} e^{i\omega t}. \tag{3.40}$$

On the boundary we have the adherence condition, (3.16)–(3.18), and this can also be expressed in the following way

$$\left.\begin{aligned} \Psi_s &= -Uc^2 s(1 - t^2), \\ \Psi_t &= Uc^2 t(s^2 - 1), \\ C &= 0. \end{aligned}\right\} \quad \text{on } s = s_0. \tag{3.41}$$

From these it follows that on the boundary we can write

$$T_{\xi\xi} = -P(s_0, t), \tag{3.42}$$

and

$$T_{\xi\eta} = \frac{\mu + k}{h_3} E^2\Psi|_{s = s_0}. \tag{3.43}$$

It is known that on the boundary $C(s_0, t) = 0$ and hence from (3.21)

$$(\mu + k)E^2\Psi = i\rho\omega(\Psi_1 + \Psi_2) \tag{3.44}$$

and the component of the stress vector along the axis of the spheroid is, therefore, given by

$$-\frac{c(s_0^2 - 1)tP(s_0, t) + i\rho\omega s_0(\Psi_1 + \Psi_2)_{s_0}}{c\sqrt{((s_0^2 - t^2)(s_0^2 - 1))}} e^{i\omega t}. \tag{3.45}$$

The radial component of the stress vector is

$$\frac{s\sqrt{(1 - t^2)}t_{\xi\xi} + t\sqrt{(s^2 - 1)}t_{\xi\eta}}{\sqrt{(s_0^2 - t^2)}}$$

and the resultant of this force over the entire body is seen to be zero on integration. Thus the body experiences a drag only in the direction of the axis of symmetry. The drag on the spheroid is obtained from (3.45) by integration.

If the drag is denoted by

$$D = D_0 e^{i\omega t} \tag{3.46}$$

we have

$$\begin{aligned} D_0 &= 2\pi c^2\sqrt{(s_0^2 - 1)} \int_{-1}^{1} (t\sqrt{(s^2 - 1)}T_{\xi\xi} - s\sqrt{(1 - t^2)}T_{\xi\eta})_{s_0}\, dt \\ &= 2\pi c^2\sqrt{(s_0^2 - 1)}(I_1 - I_2), \end{aligned} \tag{3.47}$$

where

$$I_1 = \int_{-1}^{1} (t\sqrt{(s^2-1)}T_{\xi\xi})_{s_0}\,dt, \tag{3.48}$$

and

$$I_2 = \int_{-1}^{1} (s\sqrt{(1-t^2)}T_{\xi\eta})_{s_0}\,dt. \tag{3.49}$$

Using (3.30) and (3.42) in (3.48) we find that

$$I_1 = -\frac{4}{3}A_1 i\rho\omega\sqrt{(s_0^2-1)}Q_1(s_0). \tag{3.50}$$

From (3.43), (3.44) and (3.49) we see that

$$I_2 = i\rho\omega s_0\frac{4}{3}\left[\sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(i\alpha c, s_0)d_0^{1n}(i\alpha c) + \sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\beta c, s_0)d_0^{1n}(i\beta c)\right] \tag{3.51}$$

and this simplifies to

$$-\frac{4}{3}i\rho\omega s_0\sqrt{(s_0^2-1)}\left[\frac{1}{2}Uc + Q_1'(s_0)A_1\right] \tag{3.52}$$

on using the eqn (3.22). We have, therefore

$$D_0 = \frac{8\pi\rho\omega c^2 i}{3}\left[\frac{Ucs_0}{2}(s_0^2-1) - A_1\right] \tag{3.53}$$

and the drag on the spheroid is $D_0\,e^{i\omega t}$.

We can write the drag in the form

$$\frac{8\pi}{3}\rho U c^3\omega \exp\left(i\left(\omega t + \frac{\pi}{2}\right)\right)\left[\frac{1}{2}s_0(s_0^2-1) - \frac{A_1}{Uc}\right] \tag{3.54}$$

$$= MU\omega\, e^{i\omega t}(-K' - iK) \tag{3.55}$$

where M is the mass of the fluid displaced by the spheroid and

$$-K' - iK = i\left[1 - \frac{2A_1}{Ucs_0(s_0^2-1)}\right]. \tag{3.56}$$

The drag parameters K and K' depend on the imposed frequency of oscillations, the eccentricity of the spheroid as well as the micropolarity of the fluid.

Couple stress. The couple stress tensor m_{ij} is defined in (2.6) and its only nonvanishing components are

$$(m_{\eta\phi},\ m_{\phi\eta},\ m_{\xi\phi},\ m_{\phi\xi}) = (M_{\eta\phi},\ M_{\phi\eta},\ M_{\xi\phi},\ M_{\phi\xi})\,e^{i\omega t}. \tag{3.57}$$

It is seen that

$$M_{\eta\phi} = \frac{1}{c\sqrt{((s^2-t^2)(1-t^2))}}\left(\beta t C + \gamma(1-t^2)\frac{\partial C}{\partial t}\right), \tag{3.58}$$

$$M_{\phi\eta} = -\frac{1}{c\sqrt{((s^2-t^2)(1-t^2))}}\left(\beta(1-t^2)\frac{\partial C}{\partial t} + \gamma t C\right), \tag{3.59}$$

$$M_{\xi\phi} = -\frac{1}{c\sqrt{((s^2-t^2)(s^2-1))}}\left(-\beta sC + \gamma(s^2-1)\frac{\partial C}{\partial s}\right), \tag{3.60}$$

$$M_{\phi\xi} = \frac{1}{c\sqrt{((s^2-t^2)(s^2-1))}}\left(\beta(s^2-1)\frac{\partial C}{\partial s} - \gamma sC\right). \tag{3.61}$$

The couple vector is $m_{\xi\phi}\bar{e}_\phi$ and on the spheroid $s = s_0$ this reduces to

$$\left(\frac{-\gamma\sqrt{(s^2-1)}}{c\sqrt{(s^2-t^2)}}\frac{\partial C}{\partial s}\right)_{s=s_0}(e^{i\omega t})\bar{e}_\phi. \tag{3.62}$$

The resultant couple vector due to the couple stress on the spheroid is seen to be zero on integrating the expression in (3.62) over the surface of the spheroid.

The moment of the stress vector $\bar{t}$ about the centre of the spheroid is

$$\bar{m}_0 = \bar{p} \times \bar{t} \tag{3.63}$$

where $\bar{p}$ is the radius vector and the integral of $\bar{m}_0$ over the surface of the spheroid is seen to be zero. Scalar product of $\bar{m}_0$ with the unit axial vector (along the axis of symmetry) is seen to be zero. Thus, there is no exertion of couple on the body even though the fluid sustains couple stress.

Numerical results

The drag on the spheroid given in (3.46) involves only the single constant A_1, cf. (3.53), and the drag parameters K and K' are defined in (3.56) in terms of the constant A_1. These are numerically evaluated for several parameter combination involving the size of the spheroid, imposed frequency ω and micropolarity constants by computing the values of the constants C_n from eqn (A1.7) in the Appendix A1, by truncating it to a 5 by 5 system. This choice of the order of truncation is motivated by the extent to which the coefficients needed for the evaluation of the constants $d_r^{mn}(i\alpha c)$, $d_r^{mn}(i\beta c)$ are available in the published literature[3]. The parameters relevant for the problem are eccentricity ($= 1/s_0$) of the spheroid, the frequency parameter

$$PT = \frac{\rho\omega c^2}{\mu + k}, \quad PL = \frac{K(2\mu + k)}{\gamma(\mu + k)}c^2 \quad \text{and} \quad PJ = j\frac{(\mu + k)}{\gamma}.$$

The Tables 1 and 2 and the Figs. 1–6 show the variation of the drag parameters K, K' in the polar case.

Nonpolar case

The rectilinear oscillations of a spheroid in classical viscous fluid governed by the Navier-Stokes equations of motion have earlier been analyzed by Kanwal[5]. The solution in this case consists of a sum of two infinite series for the stream function and this analysis is identifiable

Table 1. Variation of K and K' (polar case)
$PL = 2.0$, $PJ = 0.5$, $PT = 0.4$, $\alpha^2 = 2.5$

s_0	K	K'
1.5	0.19158607(3)	0.22544936(3)
1.8	0.13766592(2)	0.15172150(2)
2.0	0.26427353(2)	0.28878296(2)
2.4	0.63503551(0)	0.21737045(0)
2.8	0.37272177(1)	0.37468023(1)
3.5	0.35189238(1)	0.34906301(1)
4.0	0.32340479(1)	0.31557665(1)
4.5	0.32116308(1)	0.31228218(1)

Table 2
$s_0 = 1.8$, $PL = 2.0$, $PJ = 1.5$, $\alpha^2 = 4.0$

PT	K	K'
0.4	0.25392199(1)	0.15264654(1)
0.8	0.63206396(1)	0.94602752(0)
1.0	0.58885489(1)	0.39613285(1)
1.2	0.59381361(1)	0.48087186(0)
1.6	0.38934174(2)	0.43797302(2)

with that in the polar case described above by passage to the limit in the following sense

$$k \to 0, \quad \frac{k}{\mu + k} \to 0, \quad \frac{k}{\gamma} \to 0, \quad \frac{j}{\gamma} \to 0, \quad \alpha^2 \to \frac{i\rho\omega}{\mu}.$$

The expression for the drag is *formally* the same as in the polar case given above in (3.46) and (3.53) and the parameters K and K' can be defined as in (3.56). Numerical evaluation of these parameters has been included for the sake of completeness. The Figs. 7–10 show the variations of K and K' in the nonpolar case.

4. OBLATE SPHERIOD

An oblate spheroid (focal distance $= 2c$) oscillates harmonically along its axis of symmetry and its velocity in the above direction equals $U\,e^{i\omega t}$. We select the coordinates (ξ, η, ϕ) from the oblate spheroidal system with the scale factors

$$h_1 = h_2 = c\sqrt{(\tau^2 + t^2)}, \quad h_3 = c\sqrt{((\tau^2 + 1)(1 - t^2))}, \tag{4.1}$$

where

$$\tau = \sinh \xi, \quad t = \cos \eta. \tag{4.2}$$

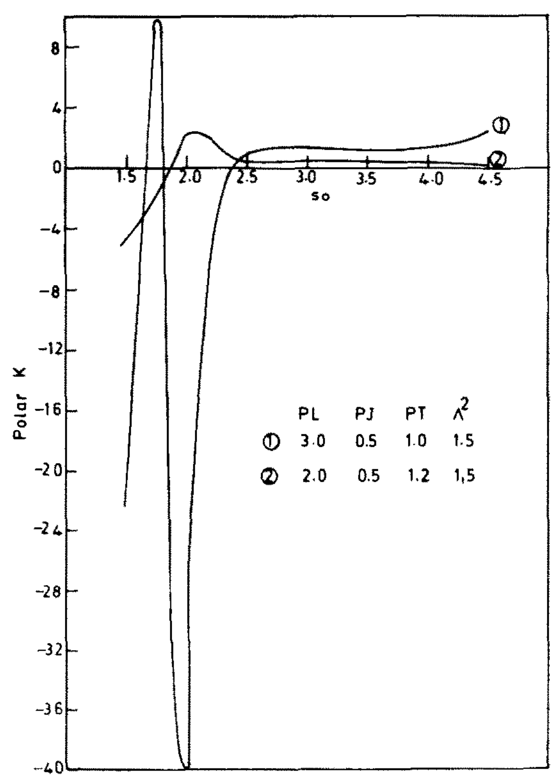

Fig. 1(a).

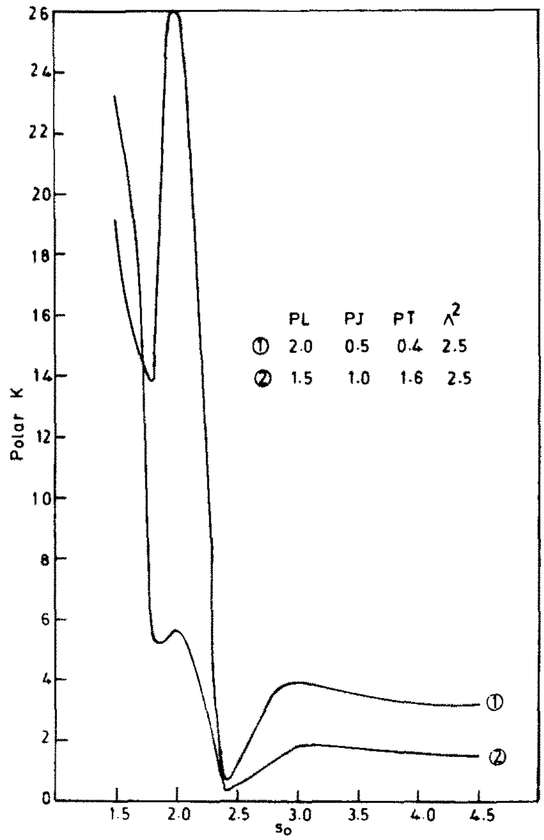

Fig. 1(b).

Fig. 1. Variation of polar K with respect to s_0.

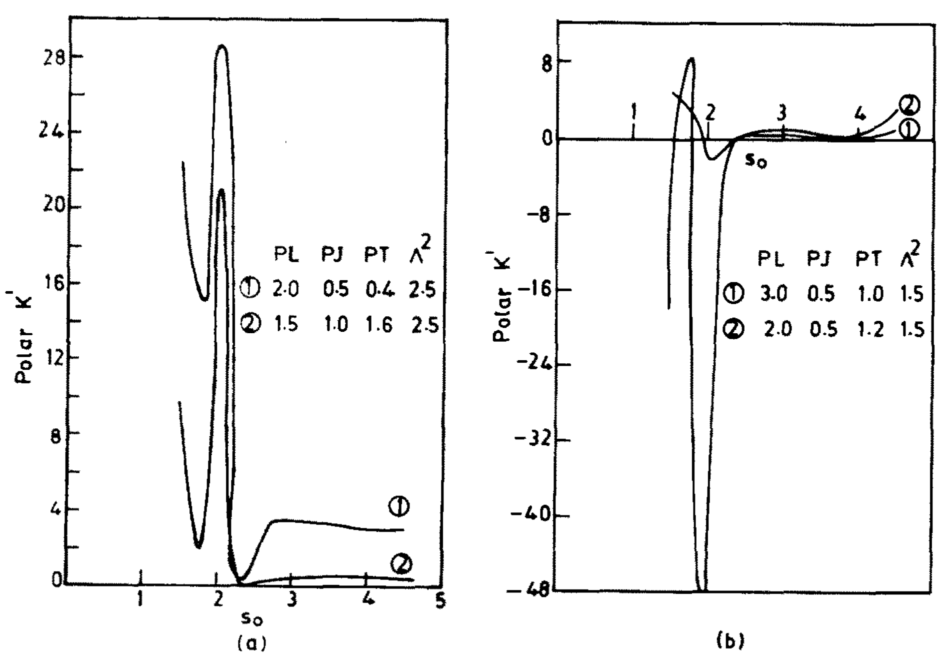

Fig. 2. Variation of polar K' with respect to s_0.

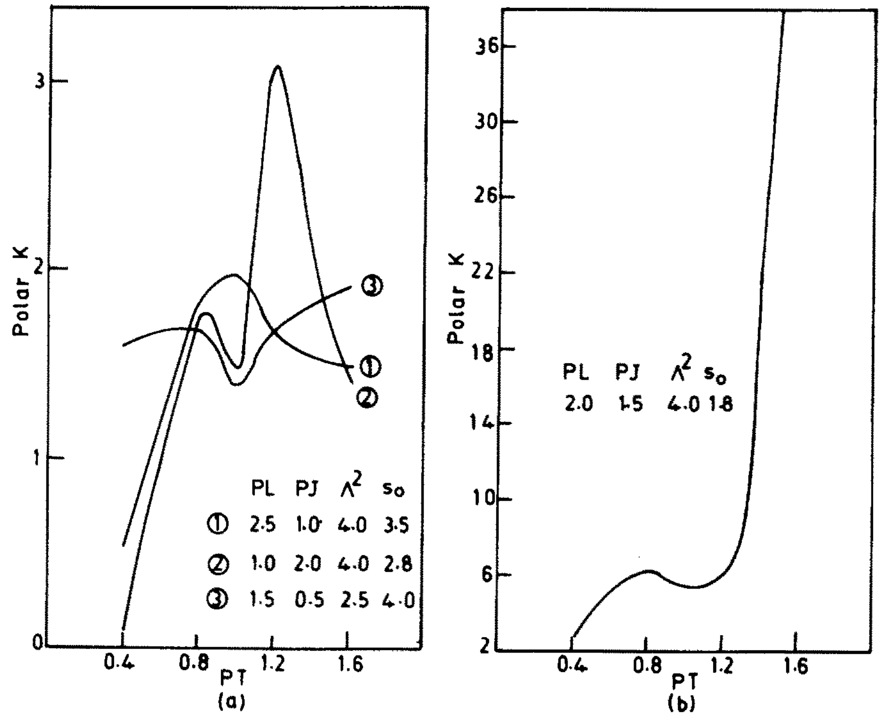

Fig. 3. Variation of polar K with respect to PT.

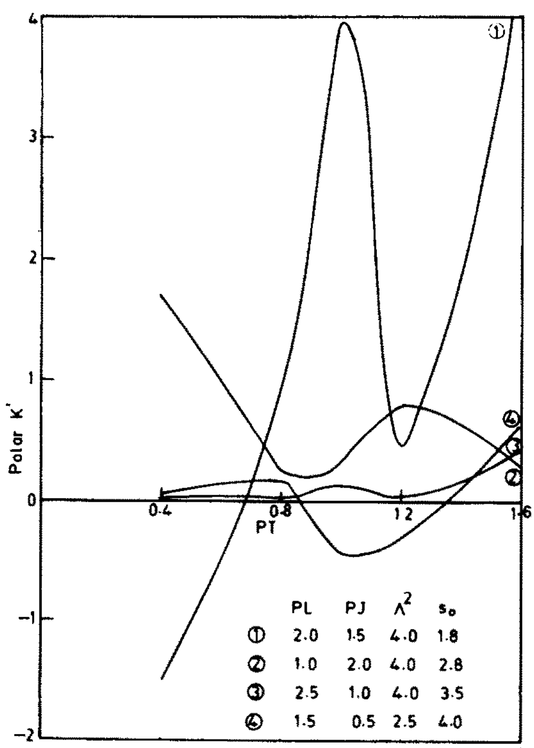

Fig. 4. Variation of polar K' with respect to PT.

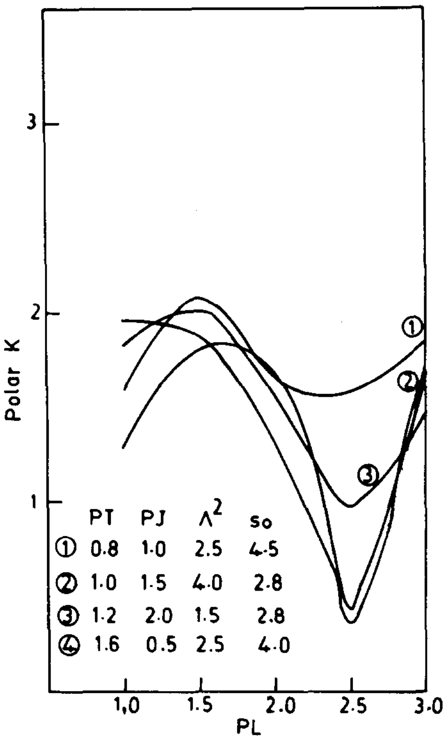

Fig. 5. Variation of polar K with respect to PL.

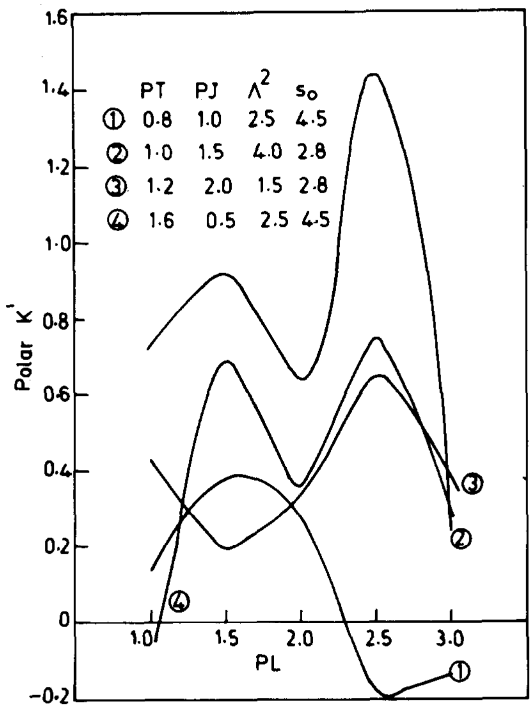

Fig. 6. Variation of polar K' with respect to PL.

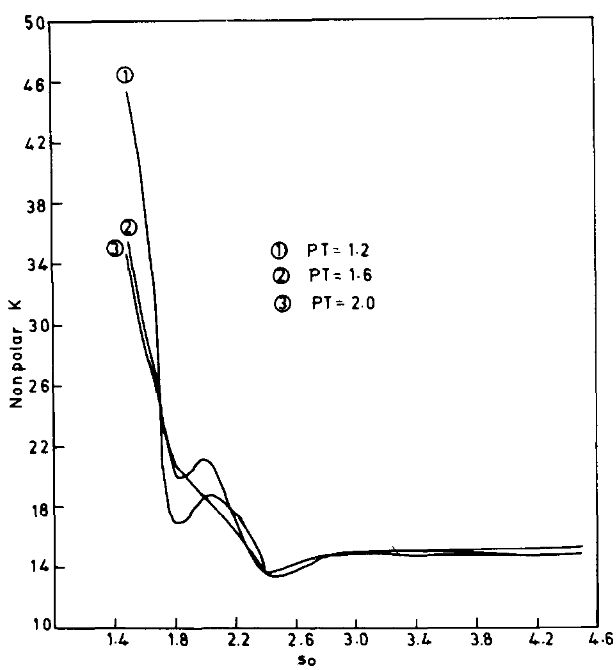

Fig. 7. Variation of nonpolar K with respect to s_0.

The Stokes's stream function operator E^2 is then given by

$$E^2 = \frac{1}{c^2(\tau^2 + t^2)}\left[(\tau^2 + 1)\frac{\partial^2}{\partial \tau^2} + (1 - t^2)\frac{\partial^2}{\partial t^2}\right]. \tag{4.3}$$

Let Ψ_0, Ψ_1, Ψ_2 be solutions of the eqns (2.32)–(2.34), respectively, that are regular far away

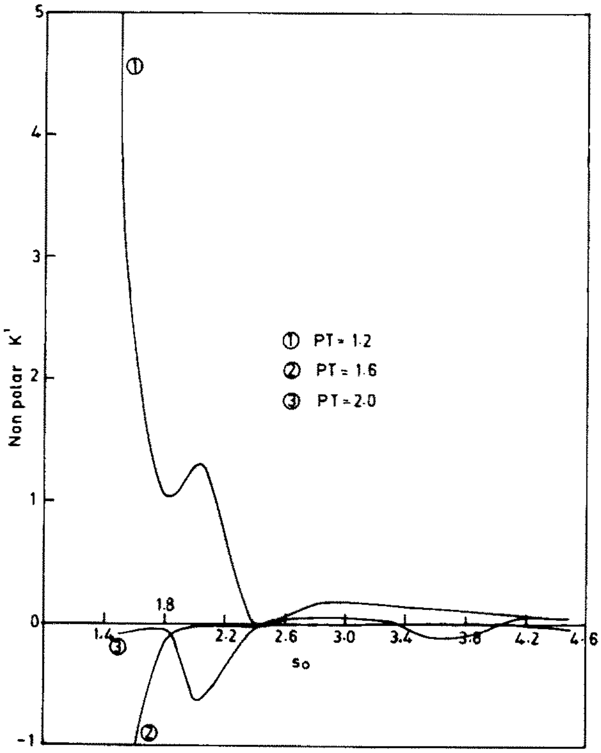

Fig. 8. Variation of nonpolar K' with respect to s_0.

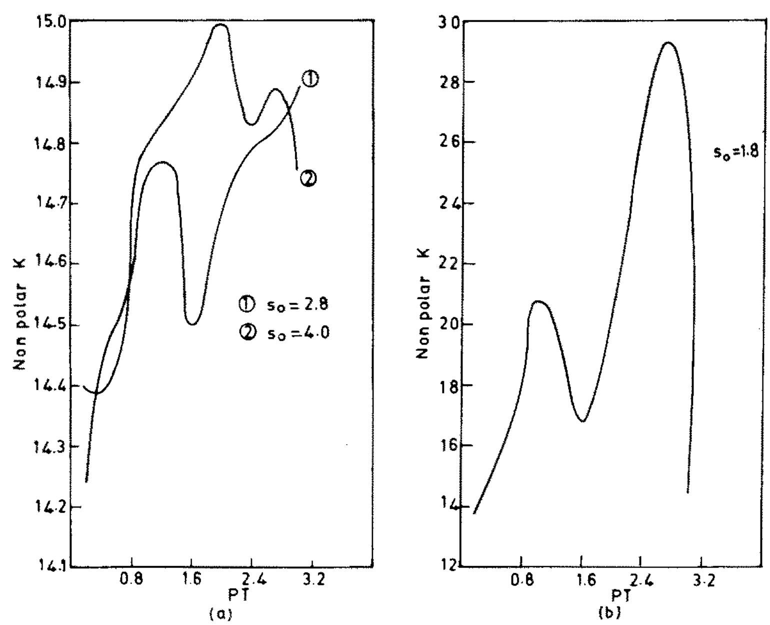

Fig. 9. Variation of nonpolar K with respect to PT.

from the spheroid. We can take

$$\Psi_0 = h_3 \sum_{n=1}^{\infty} A_n Q_n^{(1)}(i\tau) P_n^{(1)}(t) \tag{4.4}$$

where $\{A_n\}$ is an infinite set of constants and $Q_n^{(1)}$ and $P_n^{(1)}$ denote the Associated Legendre functions. The functions Ψ_1 and Ψ_2 can similarly be expressed in infinite series form involving

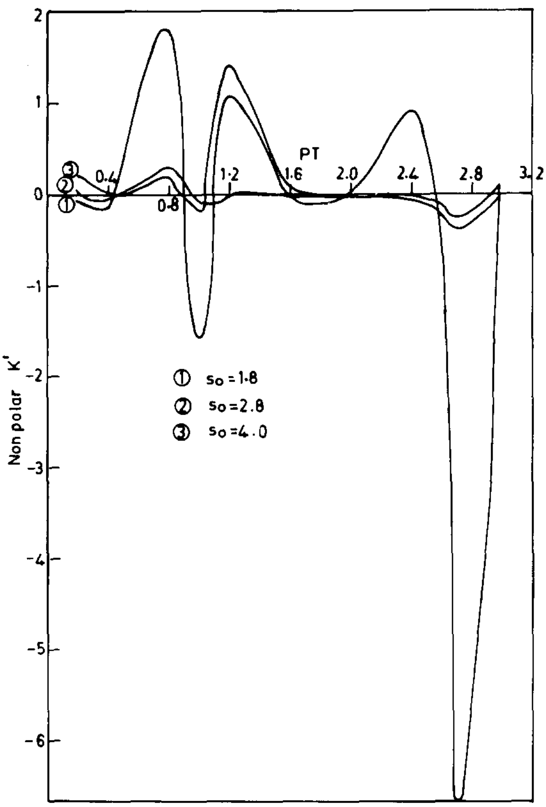

Fig. 10. Variation of nonpolar K' with respect to PT.

radial and angular spheroidal wave functions ([3], pp. 753–756) with the appropriate parameters. To ensure the regularity of these functions on the axis of symmetry, we have to restrict the angular wave functions to the first kind. To ensure further the regularity of the solution far away from the body we select the parameters α, β from the roots of the eqns (2.30) and (2.31) so as to have positive reàl parts and the radial wave functions to be of the third kind. Thus Ψ_1 is a linear combination of the functions

$$\{h_3 R_{1n}^{(3)}(i\alpha c, \tau) S_{1n}^{(1)}(i\alpha c, t)\} \tag{4.5}$$

and Ψ_2 is a combination of such functions with the parameter α in (4.5) replaced by β. In (4.5) above the functions $R_{1n}^{(3)}(i\alpha c, \tau)$ and $S_{1n}^{(1)}(i\alpha c, t)$ denote the oblate spheroidal radial and angular wave functions, respectively. These can be expressed as prolate spheroidal functions by changing $(i\alpha c, i\beta c)$ to $(\alpha c, \beta c)$ and τ to $i\tau$. Thus, we may write

$$\Psi_1 = h_3 \sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(\alpha c, i\tau) S_{1n}^{(1)}(\alpha c, t) \tag{4.6}$$

$$\Psi_2 = h_3 \sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(\beta c, i\tau) S_{1n}^{(1)}(\beta c, t) \tag{4.7}$$

where $\{B_n\}$ and $\{C_n\}$ are infinite sets of constants. The prolate spheroidal wave functions $R_{1n}^{(3)}(\lambda, i\tau)$ and $S_{1n}^{(1)}(\lambda, t)$ are given by ([3], pp. 751–756)

$$R_{1n}^{(3)}(\lambda, i\tau) = \left\{ i^{n+2} \sum_{r=0,1}^{\infty}{}' (r+1)(r+2) d_r^{1n}(\lambda) \right\}^{-1} \left(\frac{2(\tau^2+1)}{\pi \lambda \tau^3} \right)^{1/2}$$
$$\times \sum_{r=0,1}^{\infty}{}' (r+1)(r+2) d_r^{1n}(\lambda) K_{r+(3/2)}(\lambda\tau) \tag{4.8}$$

and

$$S_{1n}^{(1)}(\lambda, t) = \sum_{r=0,1}^{\infty}{}' d_r^{1n}(\lambda) P_{r+1}^{(1)}(t). \tag{4.9}$$

The stream function of the flow is

$$\Psi = h_3 \left\{ \sum_{n=1}^{\infty} A_n Q_n^{(1)}(i\tau) P_n^{(1)}(t) + \sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(\alpha c, i\tau) S_{1n}^{(1)}(\alpha c, t) + \sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(\beta c, i\tau) S_{1n}^{(1)}(\beta c, t) \right\} \tag{4.10}$$

and the velocity and microrotation components are

$$\begin{aligned} u(\tau, t) = \frac{1}{c\sqrt{(\tau^2 + t^2)}} \bigg\{ & \sum_{n=1}^{\infty} A_n Q_n^{(1)}(i\tau) \frac{\mathrm{d}}{\mathrm{d}t} (\sqrt{(1 - t^2)} P_n^{(1)}(t)) \\ & + \sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(\alpha c, i\tau) \frac{\mathrm{d}}{\mathrm{d}t} (\sqrt{(1 - t^2)} S_{1n}^{(1)}(\alpha c, t)) \\ & + \sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(\beta c, i\tau) \frac{\mathrm{d}}{\mathrm{d}t} (\sqrt{(1 - t^2)} S_{1n}^{(1)}(\beta c, t)) \bigg\}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} V(\tau, t) = \frac{1}{c\sqrt{(\tau^2 + t^2)}} \bigg\{ & \sum_{n=1}^{\infty} A_n \frac{\mathrm{d}}{\mathrm{d}\tau} (\sqrt{(\tau^2 + 1)} Q_n^{(1)}(i\tau)) P_n^{(1)}(t) \\ & + \sum_{n=1}^{\infty} B_n \frac{\mathrm{d}}{\mathrm{d}\tau} (\sqrt{(\tau^2 + 1)} R_{1n}^{(3)}(\alpha c, i\tau)) S_{1n}^{(1)}(\alpha c, t) \\ & + \sum_{n=1}^{\infty} C_n \frac{\mathrm{d}}{\mathrm{d}\tau} (\sqrt{(\tau^2 + 1)} R_{1n}^{(3)}(\beta c, i\tau)) S_{1n}^{(1)}(\beta c, t) \bigg\}, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} C = & \frac{(\mu + k)\alpha^2 - i\rho\omega}{k} \sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(\alpha c, i\tau) S_{1n}^{(1)}(\alpha c, t) \\ & + \frac{(\mu + k)\beta^2 - i\rho\omega}{k} \sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(\beta c, i\tau) S_{1n}^{(1)}(\beta c, t). \end{aligned} \tag{4.13}$$

By the hyperstick boundary condition we have on $\tau = \tau_0$

$$u(\tau_0, t) = \frac{U\sqrt{(\tau_0^2 + 1)}t}{\sqrt{(\tau_0^2 + t^2)}} \tag{4.14}$$

$$v(\tau_0, t) = -\frac{U\sqrt{(1 - t^2)}\tau_0}{\sqrt{(\tau_0^2 + t^2)}} \tag{4.15}$$

$$C(\tau_0, t) = 0. \tag{4.16}$$

Determination of the constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$

From the boundary conditions (4.14)–(4.16), we can obtain the following equations involving the three sets of constants, on the same lines as in Section 3.

$$\begin{aligned} & A_n Q_n^{(1)}(i\tau_0) + \sum_{m=1}^{\infty} B_m R_{1m}^{(3)}(\alpha c, i\tau_0) d_{n-1}^{1m}(\alpha c) \\ & \quad + \sum_{m=1}^{\infty} C_m R_{1m}^{(3)}(\beta c, i\tau_0) d_{n-1}^{1m}(\beta c) \\ & = -\frac{1}{2} Uc\sqrt{(\tau_0^2 + 1)}\delta_{n1}. \end{aligned} \tag{4.17}$$

$$n(n+1)Q_n(i\tau_0)A_n + \sum_{m=1}^{\infty} B_m d_{n-1}^{1m}(\alpha c)\left[\frac{d}{d\tau}(\sqrt{(\tau^2+1)}R_{1m}^{(3)}(\alpha c, i\tau))\right]_{\tau_0}$$

$$+\sum_{m=1}^{\infty} C_m d_{n-1}^{1m}(\beta c)\left[\frac{d}{d\tau}(\sqrt{(\tau^2+1)}R_{1m}^{(3)}(\beta c, i\tau))\right]_{\tau_0} = -Uc\tau_0\delta_{n1}, \quad (4.18)$$

$$((\mu+k)\alpha^2 - i\rho\omega)B_n R_{1n}^{(3)}(\alpha c, i\tau_0)N_{nn}(\alpha c)$$

$$+((\mu+k)\beta^2 - i\rho\omega)\sum_{m=1}^{\infty} C_m R_{1m}^{(3)}(\beta c, i\tau_0)M_{nm}(\alpha c, \beta c) = 0. \quad (4.19)$$

$$n = 1, 2, 3, \ldots.$$

In the above the quantities $N_{nn}(\alpha c)$ and $M_{nm}(\alpha c, \beta c)$ are defined by

$$N_{nn}(\alpha c) = \int_{-1}^{1} S_{1n}^{(1)}(\alpha c, t)]^2\, dt = \sum_{r=0,1}^{\infty}{}' \frac{2(r+1)(r+2)}{2r+3}[d_r^{1n}(\alpha c)]^2, \quad (4.20)$$

$$M_{nm}(\alpha c, \beta c) = \int_{-1}^{1} S_{1n}^{(1)}(\alpha c, t)S_{1m}^{(1)}(\beta c, t)\, dt = \sum_{r=0,1}^{\infty}{}' \frac{2(r+1)(r+2)}{2r+3} d_r^{1n}(\alpha c)d_r^{1m}(\beta c). \quad (4.21)$$

From the above three sets of equations, the constants $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ can be determined (Appendix A2).

Pressure distribution. From the eqns (2.22) and (2.23) we can see that the pressure $p(\tau, t)$ has the form

$$P(\tau, t) = i\rho\omega\sum_{n=1}^{\infty} n(n+1)A_n Q_n(i\tau)P_n(t). \quad (4.22)$$

Evaluation of the drag. The rate of deformation components are

$$E_{\xi\xi} = \frac{1}{c^3(\tau^2+t^2)}\left[\Psi_{\tau t} - \frac{(2\tau^2+t^2+1)\tau}{(\tau^2+t^2)(\tau^2+1)}\Psi_t - \frac{t\Psi_\tau}{\tau^2+t^2}\right],$$

$$E_{\eta\eta} = \frac{1}{c^3(\tau^2+t^2)}\left[-\Psi_{\tau t} + \frac{(1-\tau^2-2t^2)t}{(\tau^2+t^2)(1-t^2)}\Psi_\tau + \frac{\tau\Psi_t}{\tau^2+t^2}\right]$$

$$E_{\phi\phi} = \frac{1}{c^3(\tau^2+t^2)}\left(\frac{\tau\Psi_t}{\tau^2+1} + \frac{t\Psi_\tau}{1-t^2}\right),$$

$$E_{\xi\eta} = E_{\eta\xi} = \frac{(\tau^2+1)\Psi_{\tau\tau} - (1-t^2)\Psi_{tt}}{2c^3(\tau^2+t^2)\sqrt{((\tau^2+1)(1-t^2))}}$$

$$+\frac{t\sqrt{(1-t^2)}\Psi_t}{c^3(\tau^2+t^2)^2\sqrt{(\tau^2+1)}} - \frac{\tau\sqrt{(\tau^2+1)}\Psi_\tau}{c^3(\tau^2+t^2)^2\sqrt{(1-t^2)}} \quad (4.23)$$

$$E_{\eta\phi} = E_{\phi\eta} = E_{\xi\phi} = E_{\phi\xi} = 0.$$

and the non-zero spin component is

$$\omega_\phi = \frac{1}{2h_3}E^2\Psi. \quad (4.24)$$

The component of the stress vector $\bar{t}$ along the axis of the spheroid is

$$\frac{t\sqrt{(\tau^2+1)}t_{\xi\xi} - \tau\sqrt{(1-t^2)}t_{\xi\eta}}{\sqrt{(\tau^2+t^2)}} \quad (4.25)$$

As seen in Section 3, we have here again

$$T_{\xi\xi} = -P(\tau_0, t) \quad (4.26)$$

and

$$T_{\xi\eta} = \frac{\mu + k}{h_3} E^2\Psi, \tag{4.27}$$

$$(\mu + k)E^2\Psi = i\rho\omega(\Psi_1 + \Psi_2). \tag{4.28}$$

The body experiences only a drag in the direction of the axis of symmetry, given by

$$D = D_0 \, e^{i\omega t} \tag{4.29}$$

and

$$D_0 = -\frac{8\pi\rho\omega c^2}{3} A_1 i(\tau_0^2 + 1)Q_1(i\tau_0)$$
$$-\frac{8\pi\rho\omega c^2}{3} i\tau_0\sqrt{(\tau_0^2+1)} \left[\sum_{n=1}^{\infty} B_n R_{1n}^{(3)}(\alpha c, i\tau_0) d_0^{1n}(\alpha c) \right.$$
$$\left. + \sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(\beta c, i\tau_0) d_0^{1n}(\beta c) \right]. \tag{4.30}$$

This can be simplified into the form

$$D_0 = \frac{8\pi\rho\omega c^2}{3} i \left[\frac{Uc}{2} \tau_0(\tau_0^2 + 1) + A_1 \right] \tag{4.31}$$

and the drag on the spheroid is

$$MU\omega \, e^{i\omega t}(-K' - iK) \tag{4.32}$$

where M is the mass of the fluid displaced by the spheroid and

$$-K' - iK = i\left(1 + \frac{2A_1}{Uc^2\tau_0(\tau_0^2+1)}\right). \tag{4.33}$$

The couple vector on the spheroid $\tau = \tau_0$ is

$$\left(\frac{\gamma\sqrt{(\tau_0^2+1)}}{c\sqrt{(\tau_0^2+t^2)}} \left(\frac{\partial B}{\partial \tau}\right)_{\tau_0}\right) e^{i\omega t} \bar{e}_\phi \tag{4.34}$$

and its contribution to the resultant couple vector is zero. There is no contribution to the resultant couple vector from the stress vector $\bar{t}$ either. Thus there is no exertion of a couple on the body.

Table 3. Variation of K and K' (polar case)
$PL = 2.0$, $PJ = 0.5$, $PT = 0.4$, $\alpha^2 = 2.5$

τ_0	$-K$	$-K'$
0.5	0.12820135(2)	0.24863510(2)
1.0	0.15152625(3)	0.25839087(3)
1.5	0.12091938(4)	0.20167324(4)
1.8	0.33008845(4)	0.54283516(4)
2.0	0.36076282(4)	0.58620898(4)
2.5	0.33715552(3)	0.44424365(3)
3.0	0.42273145(3)	0.47940234(3)
3.5	0.44393018(2)	0.64816064(3)

Numerical results

The drag on the spheroid given in (4.31) involves only the single constant A_1 and the drag parameters K and K' are defined in (4.32). These are numerically evaluated for several parameter combinations involving the size of the spheroid, the imposed frequency ω and micropolarity constants. Tables 1 and 2 and the Figs. 11–16 show the variations of the drag parameters K, K' in the polar case. The Figs. 17–20 show the variations of K and K' in the nonpolar case.

Table 4
$\tau_0 = 1.0,\quad PL = 2.5,\quad PJ = 1.0,\quad \alpha^2 = 4.0$

PT	$-K$	$-K'$
0.4	0.61928784(3)	0.71018237(3)
0.8	0.16669482(3)	0.57144836(2)
1.0	0.12248933(3)	0.11553275(3)
1.2	0.12143762(3)	0.86516220(3)
1.6	0.55913055(2)	0.87156296(2)

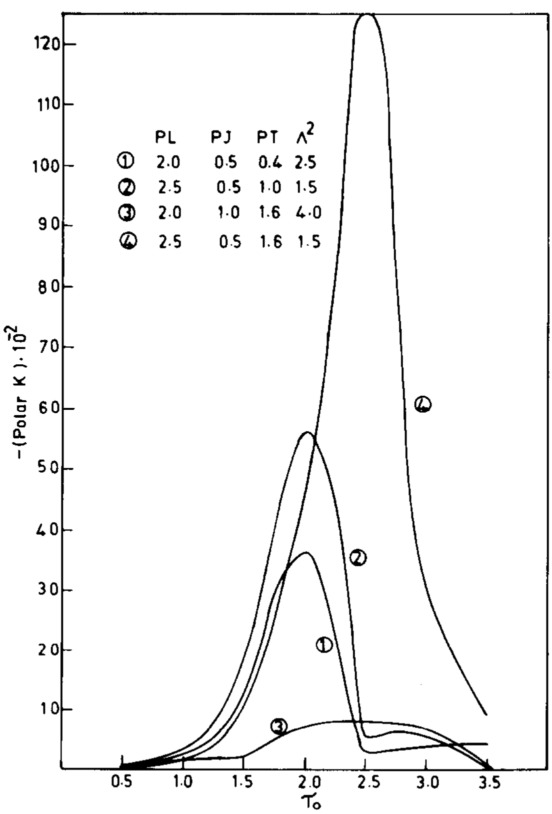

Fig. 11. Variation of polar K with respect to τ_0.

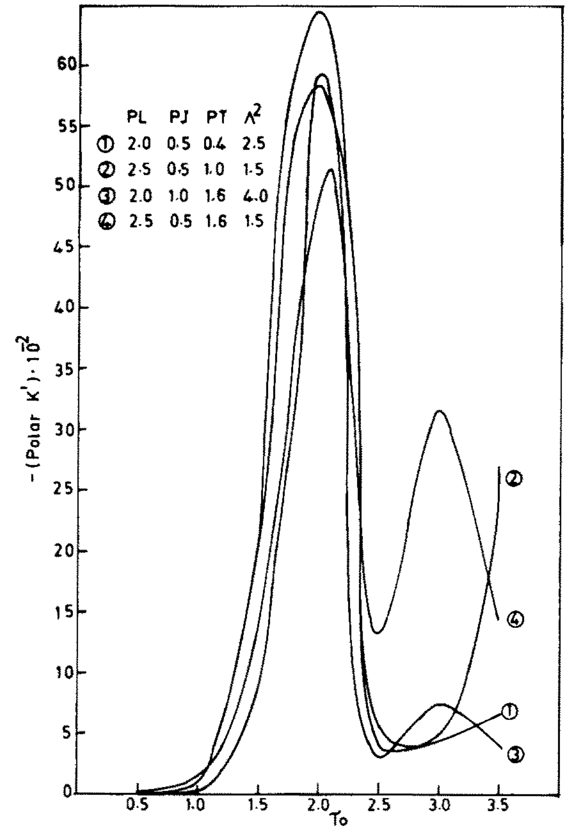

Fig. 12. Variation of polar K' with respect to τ_0.

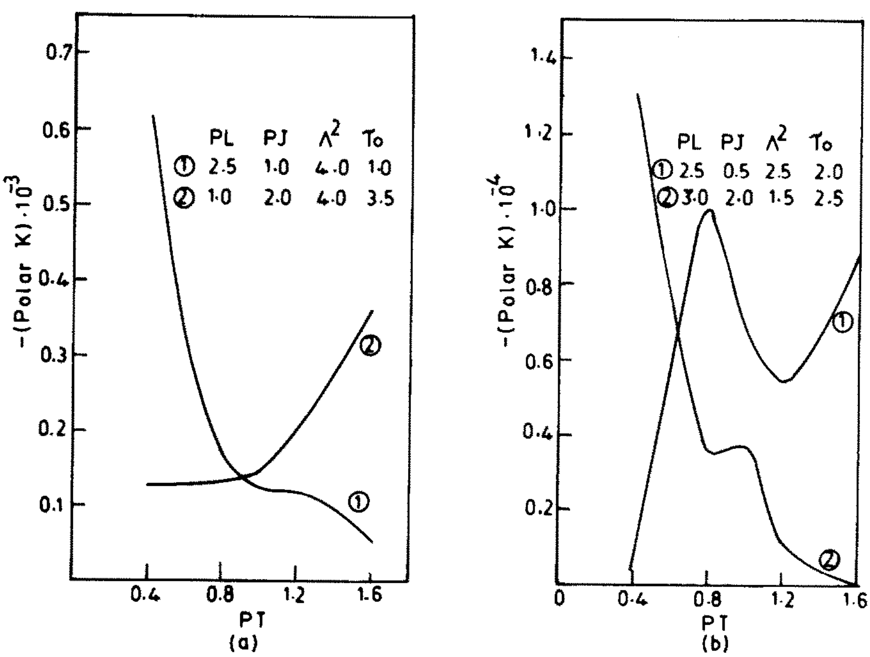

Fig. 13. Variation of polar K with respect to PT.

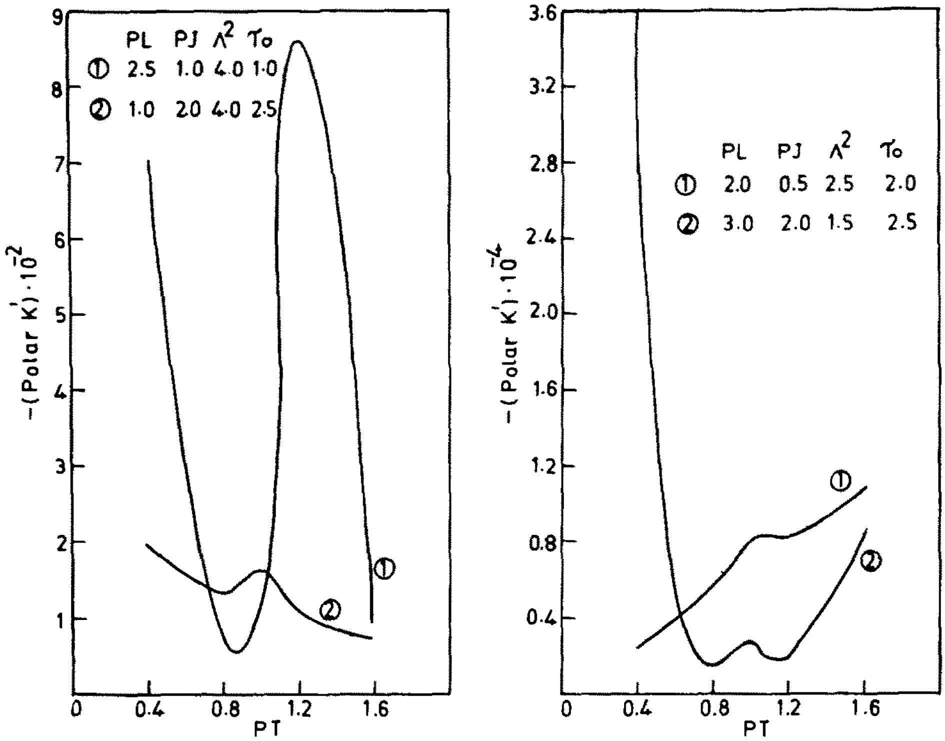

Fig. 14. Variation of polar K' with respect to PT.

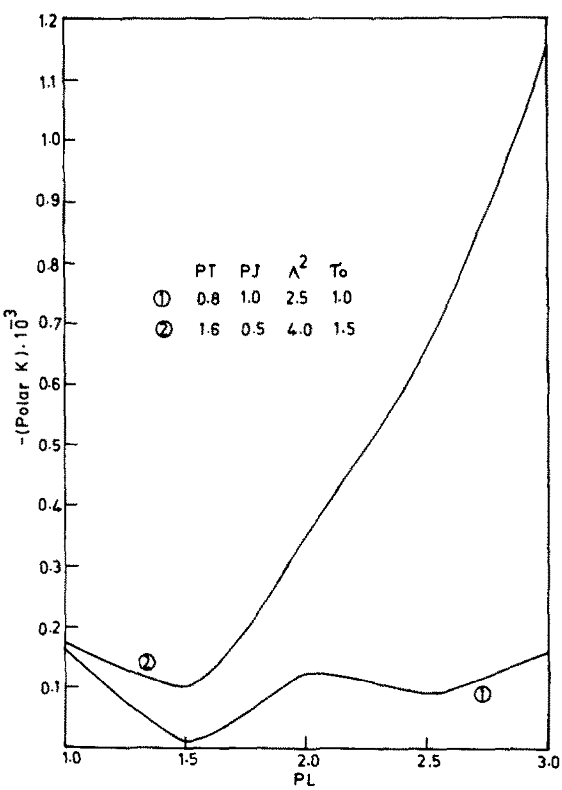

Fig. 15(a).

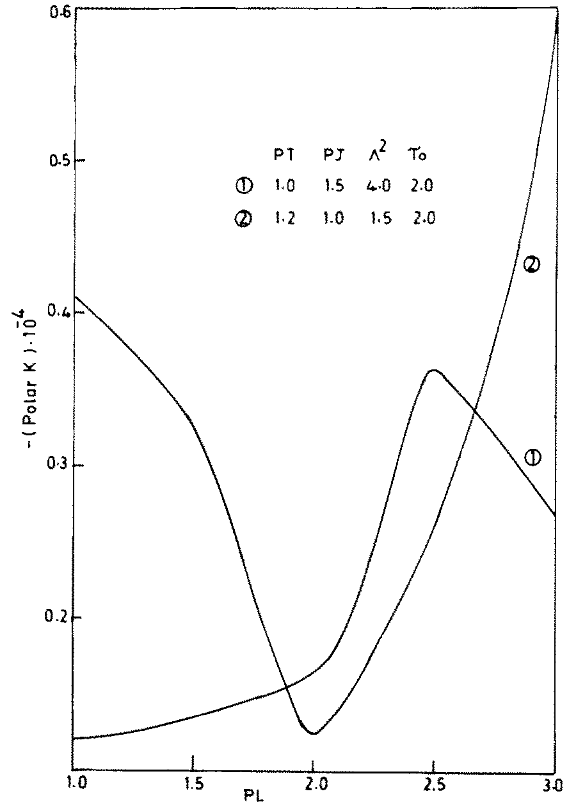

Fig. 15(b).

Fig. 15. Variation of polar *K* with respect to *PL*.

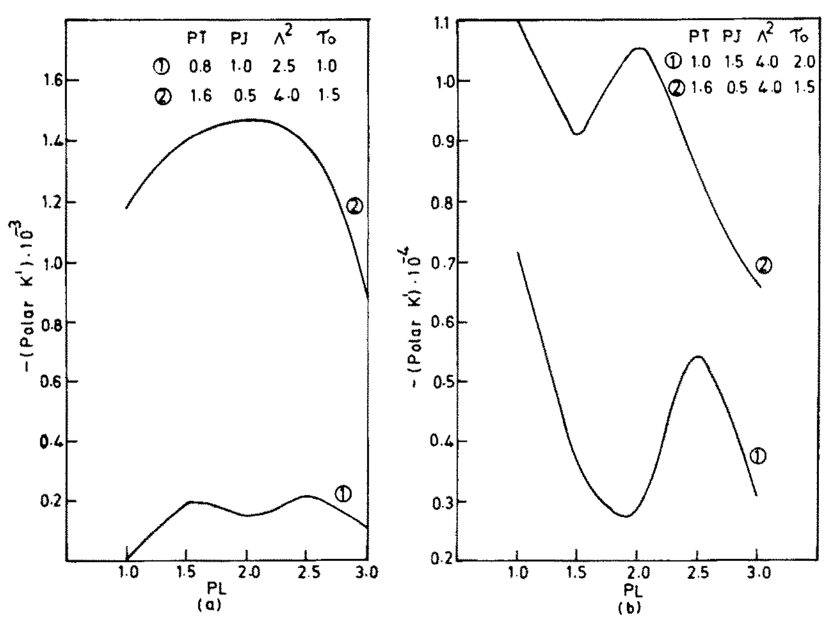

Fig. 16. Variation of polar *K′* with respect to *PL*.

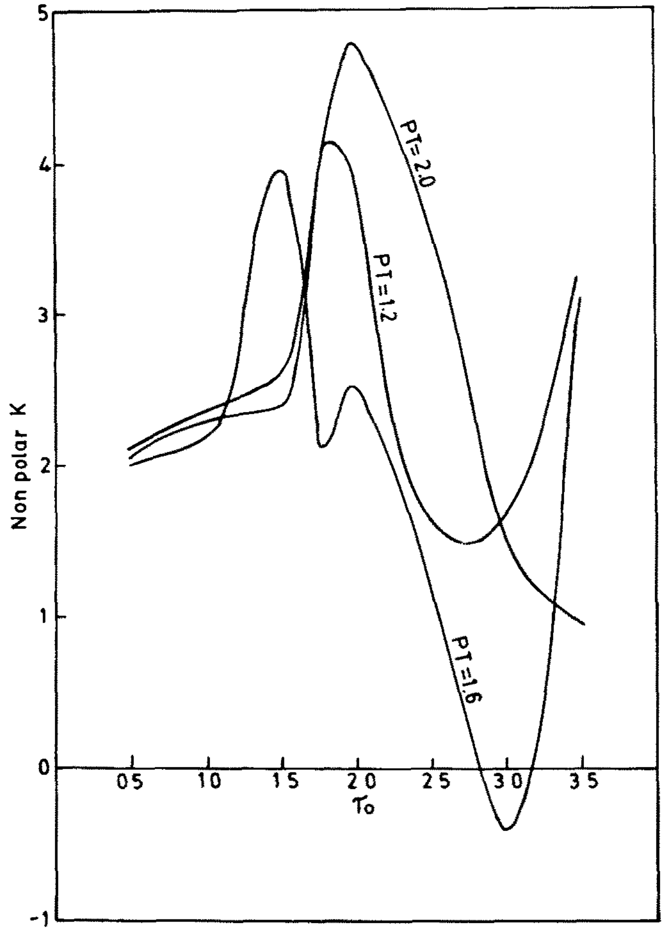

Fig. 17. Variation of nonpolar K with respect to τ_0.

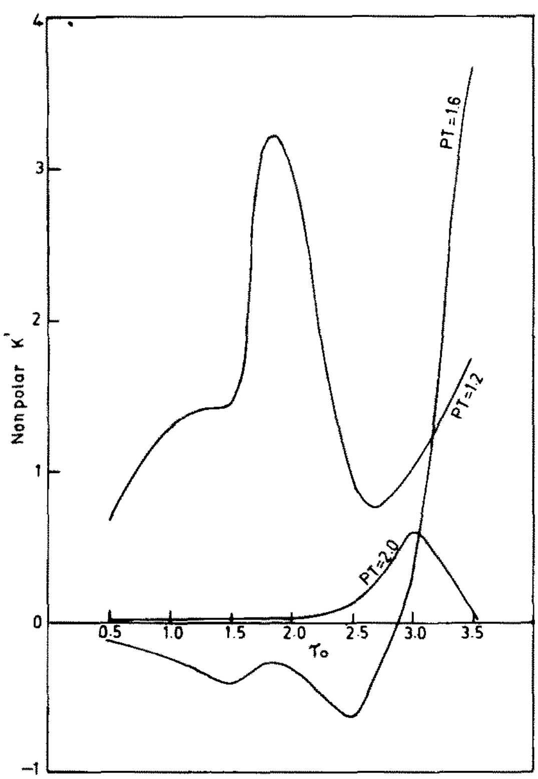

Fig. 18. Variation of nonpolar K' with respect to τ_0.

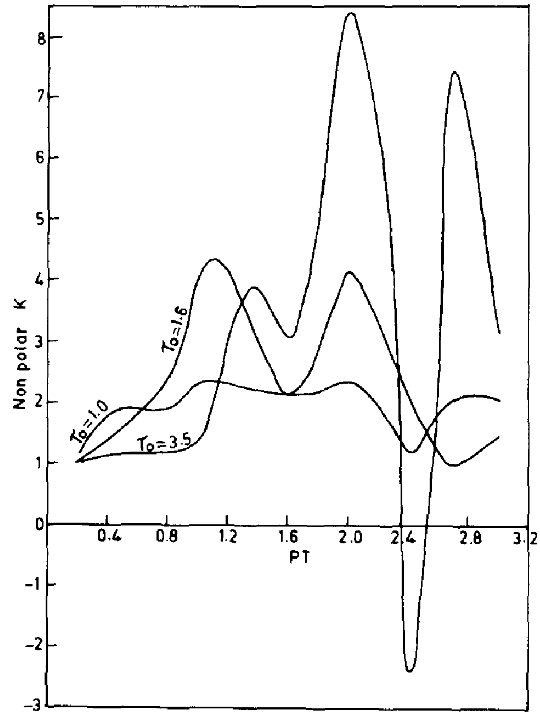

Fig. 19. Variation of nonpolar K with respect to PT.

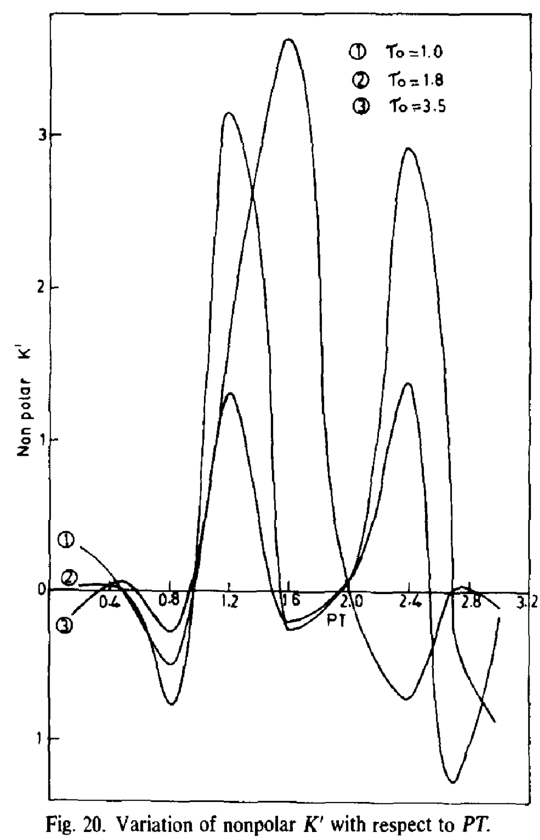

Fig. 20. Variation of nonpolar K' with respect to PT.

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APPENDIX A1

Determination of the constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$

The infinite sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ that occur in the expression (3.11) for the stream function and in the expression (3.14) for the microrotation component are determined from the three eqns (3.22), (3.23) and (3.27). From the eqns (3.22) and (3.23) we can eliminate the quantity A_n and the result is

$$\sum_{m=1}^{\infty} B_m d_{n-1}^{1m}(i\alpha c)\left\{n(n+1)Q_n(s_0)R_{1m}^{(3)}(i\alpha c, s_0) - Q_n^{(1)\prime}(s_0)\left[\frac{d}{ds}(\sqrt{(s^2-1)}R_{1m}^{(3)}(i\alpha c, s))\right]_{s_0}\right\} + \sum_{m=1}^{\infty} C_m d_{n-1}^{1m}(i\beta c)\left\{n(n+1)Q_n(s_0)R_{1m}^{(3)}(i\beta c, s_0) - Q_n^{(1)\prime}(s_0)\left[\frac{d}{ds}(\sqrt{(s^2-1)}R_{1m}^{(3)}(i\beta c, s))\right]_{s_0}\right\} = -\frac{Uc}{\sqrt{(s_0^2-1)}}\delta_{n1}. \tag{A1.1}$$

From the eqn (3.27) we see that

$$B_m = -\left(\frac{(\mu+k)\beta^2 - i\rho\omega}{(\mu+k)\alpha^2 - i\rho\omega}\right)\frac{\sum_{p=1}^{\infty} C_p R_{1p}^{(3)}(i\beta c, s_0)M_{mp}(i\alpha c, i\beta c)}{N_{mm}(i\alpha c)R_{1m}^{(3)}(i\alpha c, s_0)}. \tag{A1.2}$$

The quantities B_m can be eliminated between (A1.1) and (A1.2) and the result is an infinite system of linear algebraic equations for the unknown C_m. If we define

$$\theta^2 = \frac{\rho\omega c^2}{\mu + k} \tag{A1.3}$$

we see that the system can be written in the form

$$\sum_{p=1}^{\infty} \gamma_{np}C_p = \frac{Uc\delta_{n1}}{(s_0^2-1)^{3/2}Q_1'(s_0)} \tag{A1.4}$$

and

$$\gamma_{np} = R_{1p}^{(3)}(i\beta c, s_0)\left\{d_{n-1}^{1p}(i\beta c)\left(\frac{\frac{d}{ds}R_{1p}^{(3)}(i\beta c, s)}{R_{1p}^{(3)}(i\beta c, s)} + \frac{s}{s^2-1} - \frac{n(n+1)Q_n(s)}{(s^2-1)Q_n'(s)}\right)_{s_0} - \frac{c^2\beta^2 - i\theta^2}{c^2\alpha^2 - i\theta^2}\sum_{m=1}^{\infty}\left(\frac{d_{n-1}^{1m}(i\alpha c)M_{mp}(i\alpha c, i\beta c)}{N_{mm}(i\alpha c)}\right) \times \left(\frac{\frac{d}{ds}R_{1m}^{(3)}(i\alpha c, s)}{R_{1m}^{(3)}(i\alpha c, s)} + \frac{s}{s^2-1} - \frac{n(n+1)Q_n(s)}{(s^2-1)Q_n'(s)}\right)_{s=s_0}\right\} \tag{A1.5}$$

for $n, p = 1, 2, 3, \ldots$.

The constants $\{A_n\}$ are also directly expressible in terms of the constants $\{C_m\}$ apart from a term involving the parameter U. From eqns (3.19) and (3.21) it is possible to eliminate the entire block of terms involving B_n's and we see that

$$A_n Q_n^{(1)\prime}(s_0) = -\frac{1}{2}U_c(s_0^2-1)^{1/2}\delta_{n1} - \frac{(\mu+k)(\alpha^2-\beta^2)}{(\mu+k)\alpha^2 - i\rho\omega} \times \sum_m [C_m d_{n-1}^{1m}(i\beta c)R_{1m}^{(3)}(i\beta c, s_0)] \tag{A1.6}$$

$(n = 1, 2, 3, \ldots)$.

The parameters $d_{n-1}^{1m}(i\alpha c)$, $d_{n-1}^{1p}(i\beta c)$ which appear in the coefficients γ_{np} of the above linear system may be deemed to be zero when $((m-n)/(p-n))$ take odd integral values. The system (A1.4) can be split into two subsystems corresponding to even and odd values of n. The subsystem with $n = 2, 4, 6$, etc. is a homogeneous system and we may take the solution $C_2, C_4, C_6, \ldots$ of this infinite system to be trivial. The quantities $M_{nm}(i\alpha c, i\beta c)$ defined in (3.26) can also be deemed to be zero whenever $n + m$ is an odd positive integer. From the eqn (3.27) and the above observation concerning $M_{nm}(i\alpha c, i\beta c)$, it follows that the coefficients B_n are zero for all even integral values of n in view of the similar result concerning the coefficients C_n. From (3.22) or (3.23) we can now see that the coefficients A_n are also zero for even positive integral values of n.

Thus the stream function involves only the three infinite sets of constants $\{A_{2n+1}\}$, $\{B_{2n+1}\}$, $\{C_{2n+1}\}$. The constants $\{C_{2n+1}\}$ are determined from the system

$$\sum_{p=0}^{\infty} \gamma_{2n+1,2p+1} C_{2p+1} = \frac{UC\delta_{2n+1,1}}{(s_0^2-1)^{3/2} Q_1'(s_0)} \tag{A1.7}$$

$n = 0, 1, 2, 3, \ldots$

The constants $\{B_{2n+1}\}$ are then determined from (3.27) and $\{A_{2n+1}\}$ are determined thereafter from (3.22).

APPENDIX A2

Determination of the constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$

The infinite sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ that occur in the expression (4.10) for the stream function and in the expression (4.13) for the microrotation component C are determined from the three eqns (4.17)–(4.19). As in the Appendix A1 we can derive the following system of equations for the constants $\{C_n\}$.

$$\sum_{p=1}^{\infty} \gamma_{np} C_p = \frac{UC\delta_{n1}}{(\tau_0^2+1)^{3/2}\left[\frac{d}{d\tau} Q_1(i\tau)\right]_{\tau_0}} \tag{A2.1}$$

where

$$\begin{aligned}\gamma_{np} = R_{1p}^{(3)}(\beta c, i\tau_0)\Bigg\{ & d_{n-1}^{1p}(\beta c)\left(\frac{n(n+1)Q_n(i\tau_0)}{(\tau_0^2+1)\left[\frac{d}{d\tau} Q_n(i\tau_0)\right]_{\tau_0}} - \frac{\tau_0}{\tau_0^2+1} - \frac{\left[\frac{d}{d\tau} R_{1p}^{(3)}(\beta c, i\tau)\right]_{\tau_0}}{R_{1p}^{(3)}(\beta c, i\tau_0)}\right) \\ & - \frac{c^2\beta^2 - i\theta^2}{c^2\alpha^2 - i\theta^2} \sum_{m=1}^{\infty} \frac{d_{n-1}^{1m}(\alpha c) M_{mp}(\alpha c, \beta c)}{N_{mm}(\alpha c)} \left(\frac{n(n+1)Q_n(i\tau_0)}{(\tau_0^2+1)\left[\frac{d}{d\tau} Q_n(i\tau)\right]_{\tau_0}} \right. \\ & \left. - \frac{\tau_0}{\tau_0^2+1} - \frac{\left[\frac{d}{d\tau} R_{1m}^{(3)}(\alpha c, i\tau)\right]_{\tau_0}}{R_{1m}^{(3)}(\alpha c, i\tau_0)}\right)\end{aligned} \tag{A2.2}$$

$n, p = 1, 2, 3, \ldots$

We can also see that

$$\begin{aligned} A_n Q_n^{(1)}(i\tau_0) = & -\frac{1}{2} Uc\sqrt{(\tau_0^2+1)}\delta_{n1} - \frac{(\mu+k)(\alpha^2-\beta^2)}{(\mu+k)(\alpha^2) - i\rho\omega} \\ & \times \sum_m C_m d_{n-1}^m(\beta c) R_{1m}^{(3)}(\beta c, i\tau_0). \end{aligned} \tag{A2.3}$$

The constants $\{A_{2n}\}, \{B_{2n}\}, \{C_{2n}\}$ are zero as in the case of the prolate spheroid and the stream function involves the three sets of constants $\{A_{2n+1}\}$, $\{B_{2n+1}\}$, $\{C_{2n+1}\}$. The constants $\{C_{2n+1}\}$ are determined from the system

$$\sum_{p=0}^{\infty} \gamma_{2n+1,2p+1} C_{2p+1} = \frac{UC\delta_{2n+1,1}}{(\tau_0^2+1)^{3/2}\left[\frac{d}{d\tau} Q_1(i\tau)\right]_{\tau_0}} \tag{A2.4}$$

$n = 0, 1, 2, \ldots$

The constants $\{B_{2n+1}\}$ are then determined from (4.19) and $\{A_{2n+1}\}$ from (4.17).