ON THE PULSATING FLOW SUPERPOSED ON THE STEADY LAMINAR MOTION OF A SECOND ORDER VISCOUS LIQUID BETWEEN TWO PARALLEL PLATES

by N. CH. PATTABHI RAMACHARYULU

Department of Mathematics, Regional Engineering College,
Warangal (A.P.), India

Summary

In this note, we study the pulsating flow superposed on the steady laminar motion of a second order viscous liquid between two parallel plates. The principal flow characters such as the mean velocity, skin friction, mean rate of work done by the internal friction, the coefficient of excess of work have been examined. The results have been obtained in terms of a non-dimensional non-Newtonian parameter ε. The flow for large frequencies has a boundary layer character. The results for the flow with a single Fourier component are illustrated and discussed in detail.

§ 1. *Introduction.* The theory of pulsating flow through tubes is mainly employed in examining the problems such as the supercharging system of reciprocating engines, the surging phenomena in power plants and the flow of blood through arteries. Further, in the stability consideration of the laminar motions and propogation of sound waves, the theory is treated by the principle of perturbations. Richardson [1], in an experiment on sound waves in resonators, and later Richardson and Tyler [2], on the reciprocating motion of a piston, noticed an annular effect: the mean square of the velocity attains maximum close to the pipe wall rather than at its axis. This problem is also of theoretical interest as it provides an exact solution to the equations of hydrodynamics of viscous incompressible fluids. The annular effect noticed in [1] and [2], has been explained by a theoretical analysis by Sexl [3] for the classical viscous liquid through a straight circular tube and by Khamrui [4] for the flow through an elliptic tube. Recently, Pipkin[5]) examined the same problem for visco-elastic liquids of the Rivlin-Ericksen type in a circular tube and the present author discussed such a flow through tubes of various cross-sections: equilateral triangle, ellipse and circular annulus (to be published). The main object in all these investigations is a pure periodic motion without any mean transmission of the fluid mass across the cross-section of the tube.

Allowing for the mean transmission of the fluid in the flow direction, Uchida[6]) discussed a general pulsating viscous fluid-flow through a straight circular tube. The principal characteristics of the flow such as the mean mass flow across the cross-section, the wall friction, the coefficient of excess of work have been determined and their variations with the frequency of excitation, illustrated. Recently, Fan and Chao[7]) examined a similar problem when the cross-section of the tube is a rectangle. The present investigation aims at the discussion of a general fluctuating flow of a *second order liquid* through two parallel plates. We obtain an exact solution and the effects on the principal flow characters mentioned above in terms of a non-dimensional non-Newtonian parameter ε.

§ 2. *Fundamental equations.* Assuming that the stress is more sensitive to recent deformation than to that at distant past, Coleman and Noll[8]) proved that up to the second order of a certain retardation parameter, the theory of simple fluids, yields the following constitutive relation for a class of second order incompressible fluids:

$$S = -PI + \phi_1 A^{(1)} + \phi_2 A^{(2)} + \phi_3 A^{(1)^2} \tag{1}$$

with

$$A_{ij}^{(1)} = U_{i,j} + U_{j,i} \quad \text{and} \quad A_{ij}^{(2)} = A_{i,j} + A_{j,i} + 2U_{m,i}U_{m,j}. \tag{2}$$

In these equations, S is the stress tensor, U_i and A_i are the velocity and acceleration components in the direction of the i-th coordinate X_i, P is the hydrostatic mean pressure, and ϕ_1, ϕ_2, ϕ_3 are material constants. These constants ϕ_1, ϕ_2, ϕ_3 may be named as the coefficients of viscosity, visco-elasticity and cross-viscosity respectively. It has been reported that high polymer solutions such as poly-isobutyline in cetane behave as second order fluids.

We choose a system of rectangular Cartesian coordinates (x, y, z) with the z-axis midway between the two plates separated by a distance $2h$ and the y-axis perpendicular to them. With this choice of the axes of reference, the two plates can be represented by

$$y = \pm h. \tag{3}$$

We further assume that the plates are infinite and all the physical quantities are independent of the x coordinate. The rectilinear flow between parallel plates is now characterized by the velocity $(0, 0, w)$. From the equation of continuity, we notice that w is independent of z and a function of y and t only. This indicates that the velocity is a constant in planes parallel to the plates which is acceptable for the fully developed laminar flows.

The components of the stress-tensor at the instant of time t can be obtained as

$$\begin{aligned} S_{xx} &= -P, \\ S_{yy} &= -P + \rho(2\beta + \nu_c)\left(\frac{\partial w}{\partial y}\right)^2, \\ S_{zz} &= -P + \rho\nu_c\left(\frac{\partial w}{\partial y}\right)^2, \\ S_{xy} &= 0, \\ S_{yz} &= \rho\left(\nu + \beta\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial y}, \end{aligned} \tag{4}$$

and

$$S_{xz} = 0$$

where

$$\phi_1 = \nu\rho, \quad \phi_2 = \beta\rho \quad \text{and} \quad \phi_3 = \nu_c\rho, \tag{5}$$

ρ being the density of the liquid.

Using these stresses, the momentum equations for the fluid flow can be written as

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{6}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} + (2\beta + \nu_c)\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)^2, \tag{7}$$

and

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \left(\nu + \beta\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial y^2} \tag{8}$$

with the boundary conditions:

$$w(\pm h, t) = 0.$$

Since P is independent of x and is hydrostatically distributed in the y direction, the pressure gradient $-\rho^{-1}\,\partial P/\partial z$ must be spatially uniform but could be a function of time t. This can also be noticed as the integrability condition of the equations (6)–(8). Taking

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = f(t), \tag{10}$$

we obtain the pressure distribution:

$$P(x, y, z, t) = P_0(t) - \rho f(t)\, z + \rho(2\beta + \nu_c)\left(\frac{\partial w}{\partial y}\right)^2, \tag{11}$$

where $P_0(t)$ is a constant of integration. The velocity field can now be determined from the equation

$$\frac{\partial w}{\partial t} = f(t) + \left(\nu + \beta\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial y^2} \tag{12}$$

with condition of no slip on the boundaries:

$$w(\pm h, t) = 0. \tag{13}$$

§ 3. *Flow under the influence of a periodic pressure gradient.* If the period of excitation is $2\pi/\sigma$, we can express the pressure gradient in form of a Fourier series,

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = f(t) = a_0 + \sum_{n=1}^{\infty}(a_{cn}\cos n\sigma t + a_{sn}\sin n\sigma t) \tag{14a}$$

$$= a_0 + \mathrm{Re}\sum_{n=1}^{\infty} a_n\, e^{in\sigma t} \tag{14b}$$

where

$$a_n = a_{cn} - i a_{sn}. \tag{15}$$

The coefficients a_0, a_{cn}, a_{sn} are the Fourier coefficients of the function $f(t)$ when expanded in its fundamental period. Also, the coefficients a_{cn} and a_{sn} may be regarded as constants representing the amplitudes of the elemental vibrations of a pulsating pressure gradient superposed upon the constant pressure gradient a_0.

The solution of (12) can be obtained in the form:

$$w(y, t) = w_0(y) + \sum_{n=1}^{\infty}[w_{cn}(y)\cos n\sigma t + w_{sn}(y)\sin n\sigma t] \tag{16a}$$

$$= w_0(y) + \mathrm{Re}\sum_{n=1}^{\infty} w_n(y)\, e^{in\sigma t}, \tag{16b}$$

where

$$w_n(y) = w_{cn}(y) - iw_{sn}(y). \tag{17}$$

The differential equations for the functions $w_j(y)$ are

$$w_j''(y) - \frac{ij\sigma}{\nu + ij\beta\sigma} w_j(y) + \frac{a_j}{\nu + ij\beta\sigma} = 0 \tag{18}$$

where $i = \sqrt{-1}$ and $j = 0, 1, 2, 3, \ldots$, with the conditions

$$w_j(\pm h) = 0. \tag{19}$$

The solution of these equations can be obtained as

$$w_0(y) = \frac{a_0^2}{2\nu}(h^2 - y^2) \tag{20}$$

and

$$w_n(y) = \frac{ia_n}{n\sigma}\left[-1 + \frac{\cosh m_n y}{\cosh m_n h}\right], \tag{21}$$

where

$$m_n^2 = \frac{in\sigma}{(\nu + in\beta\sigma)} \tag{22}$$

and $n = 1, 2, 3, \ldots$.

For the further analysis of the problem, we introduce the following non-dimensional quantities:

$$k = h\sqrt{\sigma/\nu} \quad \text{and} \quad \xi = y/h \tag{23}$$

to denote the measures of the (non-dimensional) frequency of excitation and the distance from the axis respectively. Also, let

$$\varepsilon_n = \tan^{-1}(n\beta\sigma/\nu) \quad \text{and} \quad hm_n = k_n(r_n + is_n) \tag{24}$$

with

$$k_n = \sqrt{n}\, k. \tag{25}$$

Then

$$r_n = \sqrt{\cos\varepsilon_n}\cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\varepsilon_n) \quad \text{and} \quad s_n = \sqrt{\cos\varepsilon_n}\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\varepsilon_n). \tag{26}$$

The constant ε_n may be considered as the non-Newtonian parameter in the n-th mode of excitation. For Newtonian liquids and for the visco-inelastic liquids of the Reiner-Rivlin type, $\beta = 0$ i.e., $\varepsilon_n = 0$ for all n.

In terms of these non-dimensional constants, we rewrite the velocity field:

$$w(y, t) = \frac{a_0 h^2}{2\nu}(1 - \xi^2) + \\ + \frac{h^2}{\nu k^2} \sum_{n=1}^{\infty} \frac{a_{cn}}{n} \{h_n(\xi) \cos n\sigma t - [g_n(\xi) - 1] \sin n\sigma t\} + \\ + \frac{h^2}{\nu k^2} \sum_{n=1}^{\infty} \frac{a_{sn}}{n} \{[g_n(\xi) - 1] \cos n\sigma t + h_n(\xi) \sin n\sigma t\}, \tag{27}$$

where

$$g_n(\xi) = \frac{\cosh[k_n r_n(1 + \xi)] \cdot \cos[k_n s_n(1 - \xi)] + \cosh[k_n r_n(1 - \xi)] \cdot \cos[k_n s_n(1 + \xi)]}{\cosh(2k_n r_n) + \cos(2k_n s_n)} \tag{28}$$

and

$$h_n(\xi) = \frac{\sinh[k_n r_n(1 + \xi)] \cdot \sin[k_n s_n(1 - \xi)] + \sinh[k_n r_n(1 - \xi)] \cdot \sin[k_n s_n(1 + \xi)]}{\cosh(2k_n r_n) + \cos(2k_n s_n)}. \tag{29}$$

The mean velocity over one period across the cross-section of the channel is

$$U = \frac{\sigma}{2\pi} \int_0^{2\pi/\sigma} dt \frac{1}{2h} \int_{-h}^{h} w(y, t)\, dy = \frac{a_0 h^2}{3\nu} \tag{30}$$

and the mean pressure gradient over a period is

$$G = \frac{\sigma}{2\pi} \int_0^{2\pi/\sigma} \left(-\frac{1}{\rho}\right) \frac{\partial P}{\partial z}\, dt = a_0. \tag{31}$$

This shows that the mean velocity in the pulsating motion under the influence of a periodic pressure gradient (14) may thus be identified with the steady state flow subject to the same amount of the pressure gradient as that in the pulsating flow. Also, the mean velocity in the second order fluid-flow is the same as that for the Newtonian liquids.

Taking U as the standard velocity, the non-dimensional expression for the velocity can be written as

$$w^*(\xi, t) = w(y, t)/U = \tfrac{3}{2}(1 - \xi^2) +$$
$$+ \frac{3}{k^2} \sum_{n=1}^{\infty} \frac{a_{cn}}{na_0} \{h_n(\xi) \cos n\sigma t - [g_n(\xi) - 1] \sin n\sigma t\} +$$
$$+ \frac{3}{k^2} \sum_{n=1}^{\infty} \frac{a_{sn}}{na_0} \{[g_n(\xi) - 1] \cos n\sigma t + h_n(\xi) \sin n\sigma t\} \tag{32}$$

and the corresponding non-dimensional pressure gradient

$$\left(-\frac{1}{\rho}\,\frac{\partial P}{\partial z}\right)^* = \left(-\frac{1}{\rho}\,\frac{\partial P}{\partial z}\right)\left(\frac{2h}{\frac{1}{2}U^2}\right) =$$
$$= \frac{24}{Re}\left[1 + \sum_{n=1}^{\infty} \frac{a_{cn}}{a_0} \cos n\sigma t + \sum_{n=1}^{\infty} \frac{a_{sn}}{a_0} \sin n\sigma t\right], \tag{33}$$

where Re denotes the characteristic flow Reynolds' number:

$$Re = \frac{2hU}{\nu} = \frac{2a_0h^3}{3\nu^2} \tag{34}$$

and (a_{cn}/a_0), (a_{sn}/a_0) are the ratios of the amplitude of the periodic pressure gradient to that of the average pressure gradient.

Sectional mean velocity. The instantaneous mass flow across a section of the channel can be derived from the sectional mean velocity, which in the non-dimensional form is given by

$$w^*_{\mathrm{MV}}(t) = \frac{1}{2h} \int_{-h}^{h} \frac{w(y, t)}{U}\, \mathrm{d}y \tag{35}$$
$$= 1 + \frac{3}{k^2} \sum_{n=1}^{\infty} \frac{a_{cn}}{na_0} [C_n \cos n\sigma t + (1 - D_n) \sin n\sigma t] +$$
$$+ \frac{3}{k^2} \sum_{n=1}^{\infty} \frac{a_{sn}}{na_0} [C_n \sin n\sigma t - (1 - D_n) \cos n\sigma t], \tag{36}$$

where

$$C_n = \int_0^1 h_n(\xi)\, \mathrm{d}\xi = \frac{s_n \sinh(2k_nr_n) - r_n \sin(2k_ns_n)}{k_n \cos \varepsilon_n[\cosh(2k_nr_n) + \cos(2k_ns_n)]} \tag{37}$$

and

$$D_n = \int_0^1 g_n(\xi)\, \mathrm{d}\xi = \frac{r_n \sinh(2k_nr_n) + s_n \sin(2k_ns_n)}{k_n \cos \varepsilon_n[\cosh(2k_nr_n) + \cos(2k_ns_n)]}. \tag{38}$$

If

$$A_{\mathrm{MV}n} = \frac{3}{k_n^2}\,[C_n^2 + (1 - D_n)^2]^{\frac{1}{2}} \tag{39}$$

and

$$\theta_{\mathrm{MV}n} = \tan^{-1}[(1 - D_n)/C_n] \tag{40}$$

denote the amplitude coefficient and phase lag of the sectional mean velocity from the wave of the pressure gradient, we have

$$w^*_{\mathrm{MV}}(t) = 1 + \sum_{n=1}^{\infty} \frac{a_{cn}}{a_0} A_{\mathrm{MV}n} \cos(n\sigma t - \theta_{\mathrm{MV}n}) + \\ + \sum_{n=1}^{\infty} \frac{a_{sn}}{a_0} A_{\mathrm{MV}n} \sin(n\sigma t - \theta_{\mathrm{MV}n}). \tag{41}$$

Skin friction. The skin friction $= -S_{yz}$ on the wall is

$$S = \frac{3\rho U\nu}{h}\left[1 + \sum_{n=1}^{\infty} \frac{a_{cn}}{a_0} \{D_n \cos n\sigma t + C_n \sin n\sigma t\} + \right. \\ \left. + \sum_{n=1}^{\infty} \frac{a_{sn}}{a_0} \{D_n \sin n\sigma t - C_n \cos n\sigma t\}\right] \tag{42}$$

which in the non-dimensional form can be given as

$$S^* = S/(\tfrac{1}{2}\rho U^2) = \\ = \frac{12}{Re}\left[1 + \sum_{n=1}^{\infty} \frac{a_{cn}}{a_0} A_{\mathrm{SF}n} \cos(n\sigma t - \theta_{\mathrm{SF}n}) + \right. \\ \left. + \sum_{n=1}^{\infty} \frac{a_{sn}}{a_0} A_{\mathrm{SF}n} \sin(n\sigma t - \theta_{\mathrm{SF}n})\right], \tag{43}$$

where $A_{\mathrm{SF}n}$ and $\theta_{\mathrm{SF}n}$ denote the amplitude coefficient and the phase lag of the skin friction from the wave of the pressure gradient and are given by

$$A_{\mathrm{SF}n} = [C_n^2 + D_n^2]^{\frac{1}{2}} \tag{44}$$

and

$$\theta_{\mathrm{SF}n} = \tan^{-1} C_n/D_n. \tag{45}$$

Mean rate of the total work done. The energy dissipation function due to internal friction of the fluid is given by

$$\Phi = S_{yz}A_{yz}^{(1)} = \\ = \frac{9\rho\nu U^2}{h^2}\,[\xi^2 - A(\xi, t)\,\xi + B(\xi, t)], \tag{46}$$

where

$$A(\xi, t) = \sum_{n=1}^{\infty} \frac{a_{cn}}{a_0 k_n^2} \{\cos n\sigma t[2h_n'(\xi) - g_n'(\xi)\tan \varepsilon_n] - \sin n\sigma t[2g_n'(\xi) + h_n'(\xi)\tan \varepsilon_n]\} + \sum_{n=1}^{\infty} \frac{a_{sn}}{a_0 k_n^2} \{\cos n\sigma t[2g_n'(\xi) + h_n'(\xi)\tan \varepsilon_n] + \sin n\sigma t[2h_n'(\xi) - g_n'(\xi)\tan \varepsilon_n]\} \tag{47}$$

and

$$B(\xi, t) = \sum_{l,n=1}^{\infty} \frac{a_{cn}a_{cl}}{a_0^2 k_n^2 k_l^2} [\{\cos n\sigma t[h_n'(\xi) - g_n'(\xi)\tan \varepsilon_n] - \sin n\sigma t[g_n'(\xi) + h_n'(\xi)]\tan \varepsilon_n\} \times \{h_l'(\xi) \cos l\sigma t - g_l'(\xi) \sin l\sigma t\} + \sum_{l,n=1}^{\infty} \frac{a_{sn}a_{sl}}{a_0^2 k_n^2 k_l^2} [\{\cos n\sigma t[g_n'(\xi) + h_n'(\xi)\tan \varepsilon_n] + \sin n\sigma t[h_n'(\xi) - g_n'(\xi)\tan \varepsilon_n]\} \times \{g_l'(\xi) \cos l\sigma t + h_l'(\xi) \sin l\sigma t\} + \sum_{l,n=1}^{\infty} \frac{a_{cn}a_{sl}}{a_0^2 k_n^2 k_l^2} [\{\cos n\sigma t[h_n'(\xi) - {}_n g_n'(\xi)\tan \varepsilon] - \sin n\sigma t[g_n'(\xi) + h_n'(\xi)\tan \varepsilon_n]\} \times \{g_l'(\xi) \cos l\sigma t + h_l'(\xi) \sin l\sigma t\} + \{\cos l\sigma t[g_l'(\xi) + h_l'(\xi)\tan \varepsilon_l] + \sin l\sigma t[h_l'(\xi) - g_l'(\xi)]\tan \varepsilon_l\} \times \{h_n'(\xi) \cos n\sigma t - g_n'(\xi) \sin n\sigma t\}. \tag{48}$$

The rate of dissipation across the section is

$$W_i = \int_{-h}^{h} \Phi \, \mathrm{d}y = 2h \int_0^1 \Phi \, \mathrm{d}\xi,$$

from which we obtain the total mean rate of change of dissipation of energy due to internal friction across the cross-section:

$$\overline{W}_i = \frac{\sigma}{2\pi} \int_0^{2\pi/\sigma} W_i \, \mathrm{d}t = \frac{6\rho U^2 \nu}{h} \left[1 + \frac{3}{2k^2} \sum_1^{\infty} \frac{C_n}{n} \left\{\frac{a_{cn}^2}{a_0^2} + \frac{a_{sn}^2}{a_0^2}\right\}\right]. \tag{49}$$

The rate of increase of the total kinetic energy of the fluid in a unit length of the channel is

$$W_{\mathrm{k}} = \tfrac{1}{2}\rho \int_{-h}^{h} \frac{\partial}{\partial t} w^2 \, \mathrm{d}y.$$

From this, we notice that the total mean rate of change of the kinetic energy across the cross section is

$$\overline{W}_{\mathrm{k}} = \frac{\sigma}{2\pi} \int_{0}^{2\pi/\sigma} W_{\mathrm{k}} \, \mathrm{d}t = 0. \tag{50}$$

Further, the rate of the total work done by the external forces (i.e., the exciting pressure gradient) is

$$W_{\mathrm{e}} = 2hUw^{*}_{\mathrm{MV}} \left(-\frac{\partial P}{\partial z}\right).$$

With this, we get the total mean rate of the external force

$$\overline{W}_{\mathrm{e}} = \frac{\sigma}{2\pi} \int_{0}^{2\pi/\sigma} W_{\mathrm{e}} \, \mathrm{d}t =$$

$$= \frac{6\rho\nu U^2}{h^2} \left[1 + \frac{3}{2k^2} \sum_{n=1}^{\infty} \frac{C_n}{n} \left\{\left(\frac{a_{cn}}{a_0}\right)^2 + \left(\frac{a_{sn}}{a_0}\right)^2\right\}\right]. \tag{51}$$

The results (49)–(51) show that the pressure gradient does work equal to the energy loss due the dissipation of energy after full cycle of motions. Also the kinetic energy changes instantaneously but as a total, there is no loss in that energy after a complete cycle. Energy loss is thus caused by dissipation as shown in (49) and is increased by the existence of the components of the fluctuating motion. In this respect, it will not be advantageous to send mass of fluid by pulsating motions.

If the coefficient of excess of work is defined as the extra energy dissipated due to the pulsation of amplitude equal to the constant term in (14), i.e., when $\sqrt{a_{cn}^2 + a_{sn}^2} = a_0$, we have in the n-th mode, the coefficient of excess of work given by

$$(\mathrm{C.E.W.})_n = 3C_n/2k_n^2. \tag{52}$$

§ 4. *Flow for large frequencies.* When k $(= h\sqrt{\sigma/\nu})$ is sufficiently large, we have

$$\begin{matrix}\cosh \\ \sinh\end{matrix} [k_n r_n(1 \pm \xi)] \simeq \tfrac{1}{2}\, e^{k_n r_n (1 \pm \xi)}. \tag{53}$$

If $\eta_1 = 1 - \xi$ and $\eta_2 = 1 + \xi$ are the (non-dimensional) distances measured from the two walls, we now obtain the velocity:

$$\begin{aligned} w^*(\xi, t) &= \tfrac{3}{2}(1 - \xi^2) + \\ &+ \frac{3}{k^2} \sum_{n=1}^{\infty} \left\{ \frac{a_{cn}}{n a_0} \sin n\sigma t - \frac{a_{sn}}{n a_0} \cos n\sigma t \right\} + \\ &+ \frac{3}{k^2} \sum_{n=1}^{\infty} \left[\frac{a_{cn}}{n a_0} \{ e^{-k_n r_n \eta_1} \sin(k_n s_n \eta_1 - n\sigma t) + \right. \\ &\left. + e^{-k_n r_n \eta_2} \sin(k_n s_n \eta_2 - n\sigma t) \} \right] + \\ &+ \frac{3}{k^2} \sum_{n=1}^{\infty} \left[\frac{a_{sn}}{n a_0} \{ e^{-k_n r_n \eta_1} \cos(k_n s_n \eta_1 - n\sigma t) + \right. \\ &\left. + e^{-k_n r_n \eta_2} \cos(k_n s_n \eta_2 - n\sigma t) \} \right]. \end{aligned} \tag{54}$$

This, in the neighbourhood of the wall, becomes

$$\begin{aligned} w^*(\xi, t) &= \tfrac{3}{2}(1 - \xi^2) + \\ &+ \frac{3}{k^2} \sum_{n=1}^{\infty} \left\{ \frac{a_{cn}}{n a_0} \sin n\sigma t - \frac{a_{sn}}{n a_0} \cos n\sigma t \right\} + \\ &+ \frac{3}{k^2} \sum_{n=1}^{\infty} \left[\frac{a_{cn}}{n a_0} e^{-k_n r_n \eta} \sin(k_n s_n \eta - n\sigma t) + \right. \\ &\left. + \frac{a_{sn}}{n a_0} e^{-k_n r_n \eta} \cos(k_n s_n \eta - n\sigma t) \right], \end{aligned} \tag{55}$$

where η is the distance from the fluid element to the nearer wall. Also, at large distances from the walls, i.e., in a core round the axis for which

$$\eta >> \delta = \frac{1}{\mathrm{inf.}(k_n r_n)}, \tag{56}$$

the exponential terms in (54) damp out and the velocity approaches

$$w^*(\xi, t) = \tfrac{3}{2}(1 - \xi^2) + \frac{3}{k^2} \sum_{n=1}^{\infty} \left[\frac{a_{cn}}{n a_0} \sin n\sigma t - \frac{a_{sn}}{n a_0} \cos n\sigma t \right]. \tag{57}$$

This shows that the flow has a boundary layer character: the unsteady effects of the viscosity and visco-elasticity coefficients on the steady laminar motion are predominent in a certain neighbourhood of the walls. The expression for δ given in (56) may be taken as the thickness of this boundary layer beyond which we have a pulsating inviscid flow superposed on a steady classical viscous flow.

Also, for large k,

$$C_n \simeq \frac{\sqrt{\cos \varepsilon_n}}{2k_n \cos(\frac{1}{4}\pi - \frac{1}{2}\varepsilon_n)} \tag{58a}$$

and

$$D_n \simeq \frac{\sqrt{\cos \varepsilon_n}}{2k_n \sin(\frac{1}{4}\pi - \frac{1}{2}\varepsilon_n)}. \tag{58b}$$

We then have

$$A_{\mathrm{MV}n} \simeq 3/k_n^2 \to 0, \tag{59a}$$

$$\theta_{\mathrm{MV}n} \simeq \tfrac{1}{2}\pi, \tag{59b}$$

$$A_{\mathrm{SF}n} \simeq 1/(k_n\sqrt{\cos \varepsilon_n}) \to 0, \tag{59c}$$

$$\theta_{\mathrm{SF}n} \simeq \tfrac{1}{4}\pi - \tfrac{1}{2}\varepsilon_n, \tag{59d}$$

and

$$(\mathrm{C.E.W.})_n \to 0. \tag{59e}$$

§ 5. *Flow for small frequencies.* When k is small, we obtain

$$g_n(\xi) \doteqdot 1 - \tfrac{1}{4}nk^2(1 - \xi^2) \sin 2\varepsilon_n, \tag{60}$$

and

$$h_n(\xi) \doteqdot \tfrac{1}{2}nk^2(1 - \xi^2) \cos^2 \varepsilon_n.$$

We then have the velocity distribution:

$$w^*(\xi, t) \doteqdot \tfrac{3}{2}(1 - \xi^2)\left[1 + \sum_{n=1}^{\infty} \frac{a_{cn}}{na_0} \cos \varepsilon_n \cos(n\sigma t - \varepsilon_n) + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{a_{sn}}{na_0} \cos \varepsilon_n \sin(n\sigma t - \varepsilon_n)\right], \tag{61}$$

which is parabolic as in the case of a steady viscous flow while the magnitude varies periodically with the pressure gradient with a phase lag of ε_n and the amplitude reduced in the ratio $\cos \varepsilon_n$ in the n-th mode of the pulsation.

Also, we have

$$C_n \to \tfrac{1}{3}nk^2 \cos^2 \varepsilon_n \tag{62a}$$

and

$$D_n \to 1 - \tfrac{1}{6}nk^2 \sin 2\varepsilon_n. \tag{62b}$$

We then get

$$A_{\mathrm{MV}n} \to \cos \varepsilon_n, \tag{63a}$$

$$\theta_{\mathrm{MV}n} \to \varepsilon_n, \tag{63b}$$

$$A_{\mathrm{SF}n} \to 1 - \tfrac{1}{6}nk^2 \sin 2\varepsilon_n \doteqdot 1, \tag{63c}$$

$$\theta_{\mathrm{SF}n} \to \tan^{-1}(\tfrac{1}{3}nk^2 \cos^2 \varepsilon_n) \doteqdot 0 \tag{63d}$$

and

$$(\mathrm{C.E.W.})_n \to \tfrac{1}{2} \cos^2 \varepsilon_n. \tag{63e}$$

§ 6. *Flow under harmonic pressure gradient.* When the flow is influenced by a pressure gradient with a single harmonic component, we take

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} = a_0 + a_{c1} \cos \sigma t =$$

$$= a_0(1 + a \cos \sigma t). \tag{64}$$

In this case, $a_{cn}/a_0 = a$, $a_{sn} = 0$, $n = 1$ in (14). For simplicity, we drop the suffix n and adopt the same notation as that used above, i.e.,

$$m = r + is, \quad \varepsilon = \tan^{-1}\beta\sigma/\nu,$$

$$r = \sqrt{\cos \varepsilon} \cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\varepsilon), \quad s = \sqrt{\cos \varepsilon} \sin(\tfrac{1}{4}\pi - \tfrac{1}{2}\varepsilon), \tag{65}$$

$$g(\xi) = \frac{\cosh kr(1+\xi) \cos ks(1-\xi) + \cosh kr(1-\xi) \cos ks(1+\xi)}{\cosh 2kr + \cos 2ks}, \tag{66}$$

$$h(\xi) = \frac{\sinh kr(1+\xi) \sin ks(1-\xi) + \sinh kr(1-\xi) \sin ks(1+\xi)}{\cosh 2kr + \cos 2ks}, \tag{67}$$

$$C = \frac{s \sinh 2kr - r \sin 2ks}{k \cos \varepsilon(\cosh 2kr + \cos 2ks)} \tag{68}$$

and

$$D = \frac{r \sinh 2kr + s \sin 2ks}{k \cos \varepsilon(\cosh 2kr + \cos 2ks)}. \tag{69}$$

The flow is characterized by the velocity field:

$$w^*(\xi, t) = \tfrac{3}{2}(1 - \xi^2) + \frac{3a}{k^2} \{h(\xi) \cos \sigma t + [1 - g(\xi)] \sin \sigma t\} \tag{70}$$

and by the pressure field:

$$\frac{1}{\rho}\{P(\xi, z, t) - P_0(t)\} = \frac{-3\nu U^2}{h}(1 + a\cos\sigma t)\,z +$$

$$+ \frac{2\beta + \nu_c}{h^2}\left[-3\xi + \frac{3a}{k}\{h'(\xi)\cos\sigma t - g'(\xi)\sin\sigma t\}\right]^2. \quad (71)$$

The sectional mean velocity is now given by

$$w^*_{MV}(t) = 1 + aA_{MV}\cos(\sigma t - \theta_{MV}), \quad (72)$$

and the skin friction by

$$S^*(t) = \frac{12}{Re}[1 + aA_{SF}\cos(\sigma t - \theta_{SF})]. \quad (73)$$

Also the coefficient of excess of work is given by

$$\text{C.E.W.} = 3C/2k^2 \quad (74)$$

and the boundary layer thickness δ is

$$\delta = 1/[k\cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\varepsilon)\cdot\sqrt{\cos\varepsilon}], \quad (75)$$

the value of which in the Newtonian case ($\varepsilon = 0$) is

$$\delta_N = \sqrt{2}/k. \quad (76)$$

The variations of the principal coefficients of the flow:

$$A_{MV}, \quad \theta_{MV}, \quad A_{SF}, \quad Q_{SF}, \quad \text{C.E.W.} \quad \text{and} \quad \delta/\delta_N$$

versus k have been illustrated in figs. 1 to 6 respectively, with $\varepsilon = -60°, -30°, 0°, +30°, +60°$. The numerical data for these are obtained with the help of an I.B.M. 1620 computer.

§ 7. *Discussion of the results.* (i) When the pulsation frequency is extremely low, the coefficient of the amplitude of the mean velocity (A_{MV} is $\cos\varepsilon$ and for extremely rapid pulsations, it is $3/k^2 \simeq 0$ (fig. 1).

When $\varepsilon < 0$, A_{MV} increases first for small frequencies before dying out to zero at large frequencies. The frequency at which A_{MV} attains its maximum increases, while the maximum value of A_{MV} itself reduces but rather slowly with decrease of ε. The rate

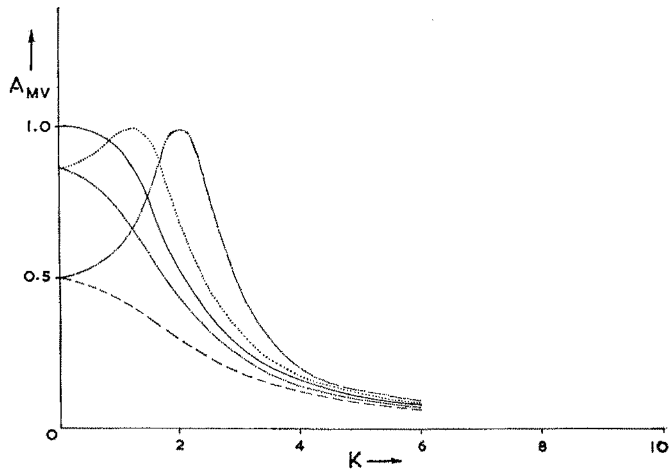

Fig. 1. Variation of the amplitude of mean velocity (A_{MV}) vrs the frequency (k).

$\varepsilon = -60°$ ——·····——·····——
$\varepsilon = -30°$ ····················
$\varepsilon = 0°$ ————————————
$\varepsilon = +30°$ —··—··—··—··—··
$\varepsilon = +60°$ - - - - - - - - - - - - - - -

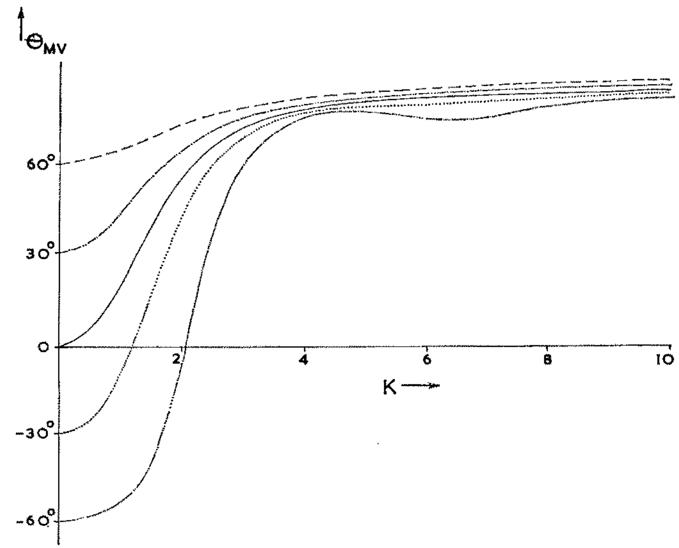

Fig. 2. Variation the phase lag of mean velocity (θ_{MV}) vrs the frequency (k).

$\varepsilon = -60°$ ——·····——·····——
$\varepsilon = -30°$ ····················
$\varepsilon = 0°$ ————————————
$\varepsilon = +30°$ —··—··—··—··—··
$\varepsilon = +60°$ - - - - - - - - - - - - - - -

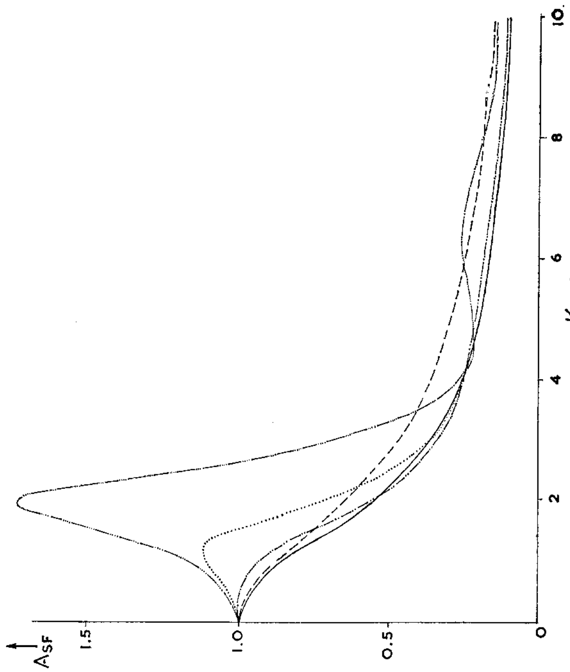

Fig. 3. Variation of the amplitude of skin friction (A_{SF}) vrs frequency (k).

$\varepsilon = -60°$ ——····——····——
$\varepsilon = -30°$ ··················
$\varepsilon = \ \ \ 0°$ ————————
$\varepsilon = +30°$ —·—·—·—·—·—
$\varepsilon = +60°$ ------------------

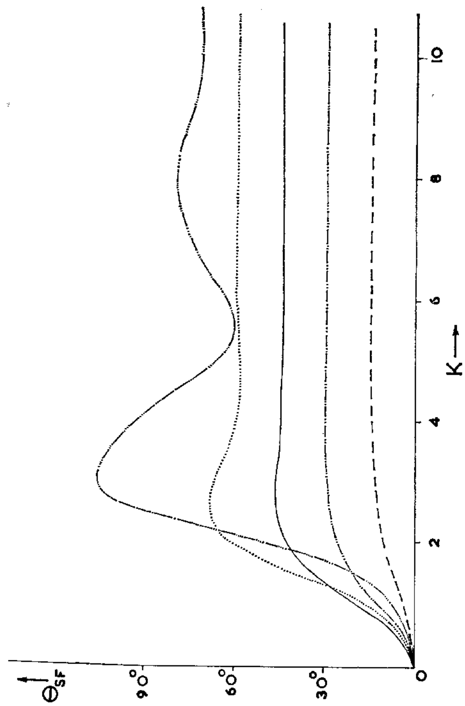

Fig. 4. Variation of the phase lag of skin friction (θ_{SF}) vrs the frequency (k).

$\varepsilon = -60°$ ——····——····——
$\varepsilon = -30°$ ··················
$\varepsilon = \ \ \ 0°$ ————————
$\varepsilon = +30°$ —·—·—·—·—·—
$\varepsilon = +60°$ ------------------

of decay of A_{MV} at large frequencies is greater than that for a Newtonian liquid.

When $\varepsilon \geqslant 0$, A_{MV} decreases steadily as the frequency of pulsation increases. Also the rate of this decay decreases as ε increases and is much less than that in Newtonian case.

(ii) The coefficient of the phase lag of the mean velocity ($= \theta_{\mathrm{MV}}$) increases from ε at small frequencies to $\pi/2$ at large frequencies (fig. 2). The rate of change of θ_{MV} decreases as ε increases. Further, the phase lag is less or greater than that for a Newtonian liquid according to whether ε is negative or positive.

(iii) The coefficient of amplitude of skin friction (A_{SF}) decreases from 1 to 0 as the frequency of the pulsation increases from extremely small values to large values (fig. 3).

For $\varepsilon < 0$, A_{SF} increases first (as in the case of A_{MV}) for small frequencies before reducing to zero at large frequencies. The maximum of A_{SF} increases very rapidly as ε decreases. The frequency at which this is maximum also increases but very slowly with the decrease of ε. The rate of decay of A_{SF} for large frequencies is much greater than that in the Newtonian case.

For $\varepsilon \geqslant 0$, A_{SF} decreases steadily with the increasing frequency.

(iv) The phase lag of the skin friction (θ_{SF}) changes from 0 to $(\frac{1}{4}\pi - \frac{1}{2}\varepsilon)$ as the frequency of the pulsation increases (fig. 4).

At small frequencies, the rate of variation of θ_{SF} for $\varepsilon < 0$ is much less and for $\varepsilon > 0$, this variation is greater than that for a Newtonian liquid. Further, the phase of the sectional mean velocity is much delayed from the pulsating pressure gradient while that of the shearing stress is less delayed from it. This delay becomes more as ε increases.

(v) The coefficient of excess of work (C.E.W.) decreases from $\frac{1}{2}\cos^2\varepsilon$ at extremely slow pulsations to the value zero for rapid pulsations (fig. 5).

When $\varepsilon < 0$, C.E.W. increases first at small frequencies and later decreases rapidly to zero. The frequency at which the maximum of C.E.W. is noticed, increases as ε decreases, whereas the maximum of C.E.W. itself decreases at a lower rate.

For $\varepsilon > 0$, this coefficient dies out to zero with a rate decreasing as ε becomes large.

(iv) The boundary layer thickness coefficient ($= \delta/\delta_{\mathrm{N}}$) attains maximum for $\varepsilon = 30°$. Also $\delta < \delta_{\mathrm{N}}$ in the range $0 < \varepsilon < 57°4'$.

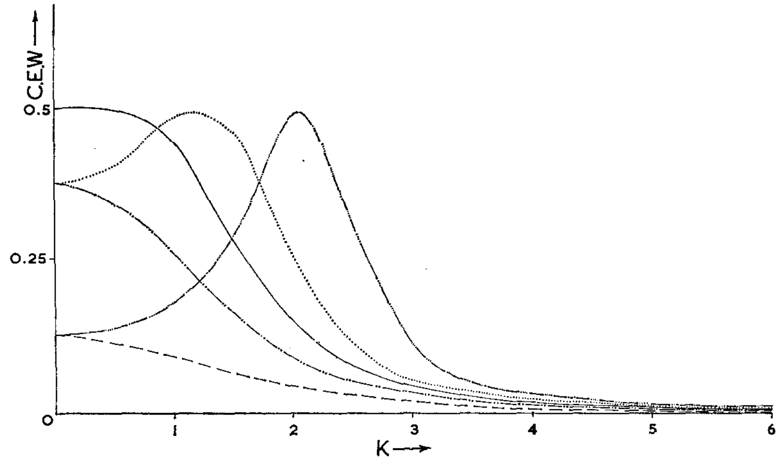

Fig. 5. Variation of the coefficient of excess of work (C.E.W.) vrs. the frequency (k).

$\varepsilon = -60°$ ——— · · · ——— · · · ———
$\varepsilon = -30°$ · · · · · · · · · · · · · · · · · · ·
$\varepsilon = 0°$ ————————————
$\varepsilon = +30°$ — · · — · · — · · — · · — · ·
$\varepsilon = +60°$ -

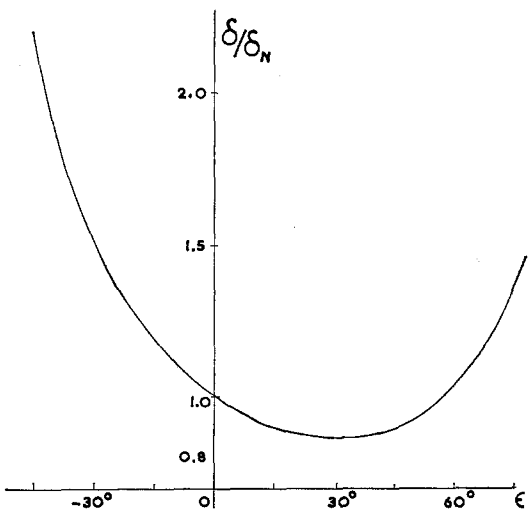

Fig. 6. Variation of the coefficient of the boundary layer thickness (δ/δ_N) vrs. the non-Newtonian parameter (ε).

Outside this range, the effect of ε is to increase the boundary layer thickness (fig. 6).

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