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STABILITY OF MICROSTRETCH FLUID MOTIONS

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Abstract—Criteria of stability of the unsteady motion of incompressible microstretch fluid in an arbitrary time-dependent domain are obtained using a general energy method introduced by Serrin. It is shown that the original motion is stable in the mean if either of the two sets of numbers $(\epsilon_1, \epsilon_2, \epsilon_3)$ or $(\sigma_1, \sigma_2, \sigma_3)$ consists of positive numbers only. These numbers are expressible in terms of the various Reynolds numbers of the original motion. The theorems giving the stability criteria are universal in the sense that they do not depend on the geometry of the domain or the actual distribution of the flow field quantities. The decay of energy of the flow in a rigid and fixed container as well as a theorem on the uniqueness of steady flows are deduced.

1. INTRODUCTION

THE THEORY of simple microfluids introduced by Eringen[1] deals with a class of fluids which respond to certain microscopic effects arising from the presence of microstructure and are influenced by spin inertia. Besides the usual translatory degrees of freedom reckoned by the velocity vector $\bar{q}$, the fluid element has degrees of freedom enabling it to possess intrinsic rotary motion and also deformation. The latter are governed by the three gyration vector fields ν_k. A simplified model of microfluids also introduced by Eringen[2] is the class of micropolar fluids with stretch. In this model (Ariman[3]) the gyration tensor ν_{kl} and the first stress moment tensor λ_{klm} have the form

$$\lambda_{kl} = \nu\delta_{kl} + \epsilon_{klr}\nu_r \tag{1}$$

$$\lambda_{klm} = \lambda_k\delta_{lm} - \frac{1}{2}\epsilon_{lmr}m_{kr}. \tag{2}$$

The micromotion here consists of a rotation about the centroid of the fluid element in an average sense described by the microrotation vector $\bar{\nu}$ and deformation consisting of a stretch due to the axial motions of cylindrical or dumb-bell elements, described by the scalar ν. When the fluid element has no deformation the gyration tensor is antisymmetric and we have the class of micropolar flows without stretch[4]. Both the theories of micropolar and microstretch fluids depart from the classical Navier–Stokes theory in two prominent features, viz the sustenance of couple stress and the non-symmetry of the stress tensor.

In this paper we consider the stability of microstretch fluid motions based on the energy method introduced earlier by Serrin[5] for investigation of the stability of viscous fluid motions governed by the Navier–Stokes equations. This powerful method was later extended by Joseph[6] for the discussion of stability of Boussinesq equations. One of the authors[7] has employed Serrin's method for the study of the stability of micropolar flows. The stability of Cosserat Fluid Motions has also been studied using Serrin's method by Shahimpoor and Ahmadi[8].

2. GOVERNING EQUATIONS OF INCOMPRESSIBLE MICROSTRETCH FLUID

We consider the motion of an incompressible microstretch fluid in an arbitrary time-dependent domain $R(t)$. The equations governing the flow are[2, 3]

$$\operatorname{div} \bar{q} = 0 \tag{3}$$

$$\frac{\partial j}{\partial t} + (\bar{q} \cdot \operatorname{grad})j - 2\nu j = 0 \tag{4}$$

$$\rho\left[\frac{\partial \bar{q}}{\partial t} - \bar{q} \times \operatorname{curl} \bar{q} + \operatorname{grad}\left(\frac{1}{2}q^2\right)\right] = \rho\bar{f} - \operatorname{grad} p + \lambda_0 \operatorname{grad} \nu + k \operatorname{curl} \bar{\nu}$$
$$- (\mu + k) \operatorname{curl} \operatorname{curl} \bar{q} + (\lambda_1 + 2\mu + k) \operatorname{grad} (\operatorname{div} \bar{q}) \tag{5}$$

$$\rho j\left[\frac{\partial \bar{\nu}}{\partial t}+(\bar{q}\cdot \text{grad})\bar{\nu}\right]=\rho \bar{l}-2k\bar{\nu}+k\,\text{curl}\,\bar{q}-\gamma\,\text{curl}\,\text{curl}\,\bar{\nu}+(\alpha+\beta+\gamma)\,\text{grad}\,(\text{div}\,\bar{\nu}) \quad (6)$$

$$\frac{1}{2}\rho j\left[\frac{\partial \nu}{\partial t}+(\bar{q}\cdot \text{grad})\nu\right]=\rho l+\alpha_0\nabla^2\nu-(\eta_0-\lambda_0)\nu. \quad (7)$$

In the above system of equations ρ is the density of the fluid, j denotes the gyration parameter, p is an undetermined pressure, $\bar{f}$ and $\bar{l}$ are, respectively, the body force and body couple per unit mass and l in eqn (7) is one third of the trace of the first body moment per unit mass. The vectors $\bar{q}$, $\bar{\nu}$ are, respectively, the velocity and microrotation vectors and the scalar ν denotes the microstretch of the fluid elements. The viscosity coefficients λ_1, μ, k, η_0, λ_0 and the gyroviscosity coefficients α, β, γ, α_0 are constant and are subject to the following restrictions[2].

$$3\lambda_1+2\mu+k\geq 0,\qquad 2\mu+k\geq 0,\qquad k\geq 0$$

$$\eta_0-\lambda_0\geq 0,\qquad (\eta_0-\lambda_0)(3\lambda_1+2\mu+k)\geq\frac{\lambda_0^2}{4} \quad (8)$$

$$3\alpha+\beta+\gamma\geq 0,\qquad \gamma\geq 0,\qquad |\beta|\leq\gamma,\qquad \alpha_0\geq 0.$$

The density ρ and the gyration parameter j are positive and the former is a constant.

Boundary conditions. We assume that on the boundary $\partial R(t)$ the field variables $\bar{q}$, $\bar{\nu}$ and ν are prescribed. If $\bar{x}$ is a point on the boundary, t is the time and $\bar{U}(\bar{x},t)$, $\bar{N}(\bar{x},t)$, $N_0(\bar{x},t)$ denote, respectively, the velocity, microrotation and microstretch of the element at $\bar{x}$ and at time t, we have

$$\bar{q}(\bar{x},t)=\bar{U}(\bar{x},t),$$

$$\bar{\nu}(\bar{x},t)=\bar{N}(\bar{x},t), \quad (9)$$

$$\nu(\bar{x},t)=N_0(\bar{x},t).$$

These conditions reflect a sort of super adherence of the fluid to the solid boundary.

3. SERRIN ENERGY EQUATION

We consider an incompressible microstretch fluid motion in the domain $R(t)$ specified by the field quantities $(\bar{q},\bar{\nu},\nu)$ and refer to this as the basic motion. The body force, body couple and the body moment trace are omitted. At some instant ($t=0$, say) the basic motion is altered to the starred motion $(\bar{q}^*,\bar{\nu}^*,\nu^*)$. The basic as well as the starred motions have the same density and gyration parameter j and are subject to the same conditions on the boundary $\partial R(t)$. In view of the hyperstic or super adherence condition, the field variables $\bar{q}$, $\bar{\nu}$, ν are zero at a rigid and fixed wall. To examine whether the starred flow approached the basic flow asymptotically and in the mean as $t\to\infty$ or differs radically from it, we employ the Liapunoff function representing the kinetic energy of the difference flow which is characterized by the velocity $\bar{u}=\bar{q}^*-\bar{q}$, the microrotation $\bar{\vartheta}=\bar{\nu}^*-\bar{\nu}$ and the microstretch $\theta=\nu^*-\nu$. The density ρ and gyration parameter j for the difference motion $(\bar{u},\bar{\vartheta},\theta)$ are the same as for the basic and starred motions. The kinetic energy of the difference motion is

$$T=T_1+T_2+T_3 \quad (10)$$

$$=\frac{1}{2}\int_R\rho(\bar{u})^2\,\mathrm{d}R+\frac{1}{2}\int_R\rho j(\bar{\vartheta})^2\,\mathrm{d}R+\frac{3}{4}\int\rho j\theta^2\,\mathrm{d}R. \quad (11)$$

The field quantities and the domain $R(t)$ are such that the divergence theorem is valid. In the sequel the volume infinitesimal $\mathrm{d}R$ in the volume integrals over the domain $R(t)$ will be omitted. The eqns (3)–(7) are valid for the basic flow $(\bar{q},\bar{\nu},\nu)$ as well as the starred flow $(\bar{q}^*,\bar{\nu}^*,\nu^*)$. We can easily obtain the following muster of differential equations for the difference flow $(\bar{u},\bar{\vartheta},\theta)$.

$$\operatorname{div} \bar{u} = 0 \tag{12}$$

$$(\bar{u} \cdot \operatorname{grad}) j - 2\theta j = 0 \tag{13}$$

$$\rho\left[\frac{\partial \bar{u}}{\partial t} + (\bar{q}^* \cdot \operatorname{grad})\bar{u} + (\bar{u} \cdot \operatorname{grad})\bar{q}\right] = -\operatorname{grad}(p^* - p) + \lambda_0 \operatorname{grad} \theta + k \operatorname{curl} \bar{\vartheta} - (\mu + k) \operatorname{curl} \operatorname{curl} \bar{u} \tag{14}$$

$$\rho j\left[\frac{\partial \bar{\vartheta}}{\partial t} + (\bar{q}^* \cdot \operatorname{grad})\bar{\vartheta} + (\bar{u} \cdot \operatorname{grad})\bar{\nu}\right] = -2k\bar{\vartheta} + k \operatorname{curl} \bar{u} - \gamma \operatorname{curl} \operatorname{curl} \bar{\vartheta} + (\alpha + \beta + \gamma) \operatorname{grad}(\operatorname{div} \bar{\vartheta}) \tag{15}$$

$$\frac{1}{2}\rho j\left[\frac{\partial \theta}{\partial t} + (\bar{q}^* \cdot \operatorname{grad})\theta + (\bar{u} \cdot \operatorname{grad})\nu\right] = \alpha_0 \nabla^2 \theta - (\eta_0 - \lambda_0)\theta. \tag{16}$$

In view of the hyperstick boundary conditions (9) valid for both the basic and starred flows, the difference flow $(\bar{u}, \bar{\vartheta}, \theta)$ is such that

$$\bar{u} = \bar{0}, \qquad \bar{\vartheta} = \bar{0}, \qquad \theta = 0 \qquad \text{on } \partial R(t). \tag{17}$$

From (10), (11) and the Leibnitz rule, we have

$$\frac{dT_1}{dt} = \int \rho \bar{u} \cdot \frac{\partial \bar{u}}{\partial t}. \tag{18}$$

Using (14) we see that (18) can be written also in the form

$$\frac{dT_1}{dt} = \int \{-\rho \bar{u} \cdot [(\bar{q}^* \cdot \operatorname{grad})\bar{u}] - \rho \bar{u} \cdot [(\bar{u} \cdot \operatorname{grad})\bar{q}] - \bar{u} \cdot \operatorname{grad}(p^* - p) + \lambda_0 \bar{u} \cdot (\operatorname{grad} \theta) + k\bar{u} \cdot \operatorname{curl} \bar{\vartheta} - (\mu + k)\bar{u} \cdot \operatorname{curl} \operatorname{curl} \bar{u}\}. \tag{19}$$

The above equation can be simplified by the use of divergence theorem and noting that $\bar{u}$ is a solenoidal vector and that it vanishes on the boundary $\partial R(t)$. We see that time-rate of change of the energy functional T_1 can be expressed in either of the following two ways

$$\frac{dT_1}{dt} = -\int \rho \bar{u} \cdot D \cdot \bar{u} + k \int \bar{\vartheta} \cdot \operatorname{curl} \bar{u} - (\mu + k) \int (\operatorname{curl} \bar{u})^2, \tag{20}$$

$$\frac{dT_1}{dt} = \int \rho \bar{u} \cdot (\operatorname{grad} \bar{u}) \cdot \bar{q} + k \int \bar{\vartheta} \cdot \operatorname{curl} \bar{u} - (\mu + k) \int (\operatorname{curl} \bar{u})^2. \tag{21}$$

In (20) D denotes the rate of deformation matrix of the basic flow velocity $\bar{q}$ and grad $\bar{u}$ in (21) denotes matrix with components $(\operatorname{grad} \bar{u})_{ij} = u_{j,i}$.

From (10) and (11) we have

$$\frac{dT_2}{dt} = \frac{1}{2}\int \rho \frac{\partial}{\partial t}[j(\bar{\vartheta})^2] \tag{22}$$

and with the use of (15) we see that

$$\frac{dT_2}{dt} = \int \frac{1}{2}\rho \frac{\partial j}{\partial t}(\bar{\vartheta})^2 - \rho j\bar{\vartheta} \cdot [(\bar{q}^* \cdot \operatorname{grad})\bar{\vartheta}] - \rho j\bar{\vartheta} \cdot [(\bar{u} \cdot \operatorname{grad})\bar{\nu}] - 2k(\bar{\vartheta})^2 + k\bar{\vartheta} \cdot \operatorname{curl} \bar{u} - \gamma\bar{\vartheta} \cdot \operatorname{curl} \operatorname{curl} \bar{\vartheta} + (\alpha + \beta + \gamma)\bar{\vartheta} \cdot \operatorname{grad}(\operatorname{div} \bar{\vartheta}). \tag{23}$$

A selective use of the divergence theorem leads to the following simplified version of (23)

$$\frac{dT_2}{dt} = \int \{\rho j\nu^*(\bar{\vartheta})^2 - \rho j\bar{u} \cdot E \cdot \bar{\vartheta} - 2k(\bar{\vartheta})^2 + k\bar{\vartheta} \cdot \operatorname{curl} \bar{u} - \gamma(\operatorname{curl} \bar{\vartheta})^2 - (\alpha + \beta + \gamma)(\operatorname{div} \bar{\vartheta})^2\}. \tag{24}$$

In the above, E denotes the matrix gradient of the microrotation vector $\bar{\nu}$ of the basic flow.

From (10) and (11) we get

$$\frac{\mathrm{d}T_3}{\mathrm{d}t}=\frac{3}{4}\int\rho\left(\frac{\partial j}{\partial t}\theta^2+2j\theta\frac{\partial\theta}{\partial t}\right) \tag{25}$$

and this can be brought into the form

$$\frac{\mathrm{d}T_3}{\mathrm{d}t}=3\int\left\{\frac{1}{2}\rho j\nu^*\theta^2-\frac{1}{2}\rho j\theta[(\bar{u}\cdot\mathrm{grad})\nu]-\alpha_0(\mathrm{grad}\,\theta)^2-(\eta_0-\lambda_0)\theta^2\right\}. \tag{26}$$

From (20), (24) and (26) we see that the time-rate of change of the energy functional T is given by

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t}=&-\int\rho\bar{u}\cdot D\cdot\bar{u}-\int\rho j\,\bar{u}\cdot E\cdot\bar{\vartheta}+\int\rho j\nu^*(\bar{\vartheta})^2+\frac{3}{2}\int\rho j\nu^*\theta^2\\&-\frac{3}{2}\int\rho j\theta[(\bar{u}\cdot\mathrm{grad})\nu]-\frac{k}{2}\int(\mathrm{curl}\,\bar{u}-2\bar{\vartheta})^2\\&-\left(\mu+\frac{1}{2}k\right)\int(\mathrm{curl}\,\bar{u})^2-\gamma\int(\mathrm{curl}\,\bar{\vartheta})^2\\&-(\alpha+\beta+\gamma)\int(\mathrm{div}\,\bar{\vartheta})^2-3\alpha_0\int(\mathrm{grad}\,\theta)^2-3(\eta_0-\lambda_0)\int\theta^2.\end{aligned} \tag{27}$$

Comparison of (20) and (21) shows that the time-rate of change of the energy functional T can also be expressed in the alternative form

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t}=&\int\rho\bar{u}\cdot(\mathrm{grad}\,\bar{u})\cdot\bar{q}-\int\rho j\bar{u}\cdot E\cdot\bar{\vartheta}+\int\rho j\nu^*(\bar{\vartheta})^2+\frac{3}{2}\int\rho j\nu^*\theta^2\\&-\frac{1}{2}\int\rho j\theta[(\bar{u}\cdot\mathrm{grad})\nu]-\frac{k}{2}\int(\mathrm{curl}\,\bar{u}-2\bar{\vartheta})^2-\left(\mu+\frac{1}{2}k\right)\int(\mathrm{curl}\,\bar{u})^2\\&-\gamma\int(\mathrm{curl}\,\bar{\vartheta})^2-(\alpha+\beta+\gamma)\int(\mathrm{div}\,\bar{\vartheta})^2-3\alpha_0\int(\mathrm{grad}\,\theta)^2-3(\eta_0-\lambda_0)\int\theta^2.\end{aligned} \tag{28}$$

The eqns (27) and (28) are basic to the discussion of mean stability and either of them may be called the Serrin energy equation. Since the viscosity coefficients k, $2\mu+k$, $\eta_0-\lambda_0$ and the gyroviscosity coefficients γ, $\alpha+\beta+\gamma$ and α_0 are all positive, we see that the terms in (20), (24) and (26) involving the functionals

$$\int(\mathrm{curl}\,\bar{u})^2,\quad\int(\bar{\vartheta})^2,\quad\int(\mathrm{curl}\,\bar{\vartheta})^2,\quad\int(\mathrm{div}\,\bar{\vartheta})^2,\quad\int(\mathrm{grad}\,\theta)^2\quad\text{and}\quad\int\theta^2 \tag{29}$$

show tendency to stabilize the basic flow. The functional $\int\bar{\vartheta}\cdot\mathrm{curl}\,\bar{u}$ in (20) or (21) and (24) is an interaction effect and may inhibit stability. However, this interaction effect gets founded on combining the eqns (20) and (24). The rest of the terms in (20), (24) and (26) can be individually destabilizing.

4. CRITERIA FOR UNIVERSAL STABILITY

The muster of symbols employed in the sequel is listed below:

(i) d = diameter of the ball which encloses the bounded volume $R(t)$

(ii) $-m$ = lower bound for the eigen values of the strain-rate matrix D of the basic flow velocity $\bar{q}$

(iii) n = upper bound for the magnitude of the matrix E; i.e. n = u.b. . $\{tr(EE^T)\}^{1/2}$

(iv) V_0 = maximum speed of the basic flow over $R(t)$

(v) J = upper bound of j

(vi) M = upper bound of the microstretches $|\nu|$, $|\nu^*|$
(vii) p = upper bound of $|\text{grad } \nu|$.

In each of the above, the bound is over the domain $R(t)$ and over the time interval $(0, t)$. Further we define the following symbols.

(viii) $a = \min(\alpha + \beta + \gamma, \gamma)$
(ix) $m_1 = 2a/(2\mu + k)d^2$
(x) $m_2 = 2\rho V_0 d/(2\mu + k) = R2$
(xi) $m_3 = M/pd$
(xii) $m_4 = a/\alpha_0$
(xiii) $m_5 = (\eta_0 - \lambda_0)d^2/\alpha_0$
(xiv) $m_6 = 2\rho m d^2/(2\mu + k) = R1$
(xv) $m_7 = 2\rho n J d/(2\mu + k) = Rm$
(xvi) $m_8 = 2\rho p J d/(2\mu + k) = Rs.$ (30)

From the definitions of m, n and p it is readily seen that

$$\bar{u} \cdot D \cdot \bar{u} \geq -m(\bar{u})^2 \tag{31}$$

$$\bar{u} \cdot E \cdot \bar{\vartheta} \geq -n|\bar{u}|\,|\bar{\vartheta}| \geq -\frac{1}{2}n\left(d(\bar{\vartheta})^2 + \frac{(\bar{u})^2}{d}\right) \tag{32}$$

$$\theta(\bar{u} \cdot \text{grad})\nu \geq -p|\theta|\,|\bar{u}| \geq -\frac{1}{2}p\left(d\theta^2 + \frac{(\bar{u})^2}{d}\right). \tag{33}$$

Employing the bound M defined in (30(vi)) and the inequalities (31), (32), (33) in (27) we see that

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} = \leq & \int \rho m(u)^2 + \frac{1}{2}\int \rho j n\left(d(\bar{\vartheta})^2 + \frac{(\bar{u})^2}{d}\right) + M\int \rho j(\bar{\vartheta})^2 + \frac{3}{2}M\int \rho j\theta^2 \\ & + \frac{3}{4}\int \rho j p\left(d\theta^2 + \frac{(\bar{u})^2}{d}\right) - \frac{1}{2}k\int (\text{curl } \bar{u} - 2\bar{\vartheta})^2 \\ & - \left(\mu + \frac{1}{2}k\right)\int (\text{curl } \bar{u})^2 - \gamma\int (\text{curl } \bar{\vartheta})^2 - (\alpha + \beta + \gamma)\int (\text{div } \bar{\vartheta})^2 \\ & - 3\alpha_0\int (\text{grad } \theta)^2 - 3(\eta_0 - \lambda_0)\int \theta^2.\end{aligned} \tag{34}$$

Using the functional inequalities [5, 9, 10] in (34)

$$\int (\text{curl } \bar{u})^2 = \int (\text{grad } \bar{u})^2 \geq \frac{80}{d^2}\int (\bar{u})^2 \tag{35}$$

$$\int \{(\text{div } \bar{\vartheta})^2 + (\text{curl } \bar{\vartheta})^2\} \geq \frac{3\pi^2}{d^2}\int (\bar{\vartheta})^2 \tag{36}$$

$$\int (\text{grad } \theta)^2 \geq \frac{3\pi^2}{d^2}\int \theta^2 \tag{37}$$

we find that

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} \leq & \left(2m + \frac{nJ}{d} + \frac{3pj}{2d} - \frac{80(2\mu + k)}{\rho d^2}\right)T_1 + \left(nd + 2M - \frac{6a\pi^2}{\rho J d^2}\right)T_2 \\ & + \left(pd + 2M - \frac{12\alpha_0\pi^2}{\rho J d^2} - \frac{4(\eta_0 - \lambda_0)}{\rho J}\right)T_3.\end{aligned} \tag{38}$$

The above inequality can be written in the form

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq -\frac{2m}{m_6}\epsilon_1 T_1 - \frac{2m_1}{m_7}nd\epsilon_2 T_2 - \frac{4m_1}{m_4 m_8}pd\epsilon_3 T_3 \tag{39}$$

and the numbers ϵ_1, ϵ_2, ϵ_3 are given by

$$-\epsilon_1 = m_6 + \frac{1}{2} m_7 + \frac{3}{4} m_8 - 80,$$

$$-\epsilon_2 = \frac{m_7}{2m_1} + \frac{m_3 m_8}{m_1} - 3\pi^2, \tag{40}$$

$$-\epsilon_3 = \frac{(1+2m_3)m_4 m_8}{4m_1} - m_5 - 3\pi^2.$$

From (39) we can deduce the following criterion for universal stability of the microstretch flow.

Theorem 1. If the numbers ϵ_1, ϵ_2, ϵ_3 are positive the functional T of the incompressible microstretch flow in the bounded domain $R(t)$ tends to zero as $t \to \infty$ and the basic flow $(\bar{q}, \bar{\nu}, \nu)$ is stable in the mean.

When the three numbers ϵ_1, ϵ_2, ϵ_3 are all positive and b denotes the minimum of the quantities

$$b = \min\left(\frac{2m\epsilon_1}{m_6}, \frac{2m_1 n d\epsilon_2}{m_7}, \frac{4m_1 p d\epsilon_3}{m_4 m_8}\right) \tag{41}$$

we see from (39) and (41) that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq -bT. \tag{42}$$

If the flow is valid over the time interval $0 \leq t < \tau$, the energy functional $T(t)$ satisfies the inequality

$$T(t) \leq T(t_0)\exp[-b(t-t_0)] \tag{43}$$

for any pair of values (t_0, t) such that $0 \leq t_0 \leq t < \tau$. This shows that the energy functional for the difference motion decays faster than the exponential. From (40) and (41) it is clear that disturbances in microstretch fluid flows are damped more rapidly than in the corresponding micropolar flows without stretch. This is to be expected as there is an additional dissipative mechanism here.

The dimensionless numbers m_6, m_7, m_8 defined in (30) may be recognized as the Reynolds numbers of the basic flow involving the velocity $\bar{q}$, microrotation $\bar{\nu}$ and microstretch ν, respectively. Denoting these by the suggestive symbols Rl, Rm and Rs we can express the criterion for the universal stability of the flow $(\bar{q}, \bar{\nu}, \nu)$ also in the following form.

$$Rl + \frac{1}{2} Rm + \frac{3}{4} Rs < 80$$

$$Rm + 2m_3 Rs < 6m_1\pi^2 \tag{44}$$

$$Rs < \frac{4m_1(m_5 + 3\pi^2)}{(1+2m_3)m_4}.$$

Since each of the numbers Rl, Rm and Rs is non-negative, it is possible to determine the optimal values of Rl, Rm and Rs in a universally stable, incompressible microstretch fluid motion whenever the quantities m_1, m_3, m_4 and m_5 are numerically assigned.

An alternative criterion for the universal stability of the flow $(\bar{q}, \bar{\nu}, \nu)$ is possible if we start with the identity (28). From Schwarz's inequality and (30, (iv)) we have

$$\rho\bar{u} \cdot (\text{grad } \bar{u}) \cdot \bar{q} \leq \frac{2\mu + k}{4}(\text{grad } \bar{u})^2 + \frac{\rho^2 V_0^2}{2\mu + k}(\bar{u})^2. \tag{45}$$

Using this inequality to replace the first term in (28) and proceeding as we did earlier to reach

(34), we see that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq -\frac{1}{4}(2\mu + k)\int (\operatorname{grad}\bar{u})^2 + \frac{\rho^2 V_0^2}{2\mu + k}\int (\bar{u})^2 + \frac{1}{2}\int \rho jn\left(d(\bar{\vartheta})^2 + \frac{(\bar{u})^2}{d}\right)$$
$$+ M\int \rho j(\bar{\vartheta})^2 + \frac{3}{2}M\int \rho j\theta^2 + \frac{3}{4}\int \rho jp\left(d\theta^2 + \frac{(\bar{u})^2}{d}\right)$$
$$-\frac{1}{2}k\int (\operatorname{curl}\bar{u} - 2\bar{\vartheta})^2 - \gamma\int (\operatorname{curl}\bar{\vartheta})^2 - (\alpha + \beta + \gamma)\int (\operatorname{div}\bar{\vartheta})^2$$
$$\times 3\alpha_0 \int (\operatorname{grad}\theta)^2 - 3(\eta_0 - \lambda_0)\int \theta^2. \tag{46}$$

and it is then clear that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq \left(\frac{2\rho V_0^2}{2\mu + k} + \frac{nJ}{d} + \frac{3pJ}{2d} - \frac{40(2\mu + k)}{\rho d^2}\right)T_1 + \left(nd + 2M - \frac{6a\pi^2}{Jd^2}\right)T_2$$
$$+ \left(pd + 2M - \frac{12\alpha_0\pi^2}{\rho Jd^2} - \frac{4(\eta_0 - \lambda_0)}{\rho J}\right)T_3. \tag{47}$$

Let σ_1, σ_2, σ_3, c be the numbers such that

$$\sigma_1 = -(m_2)^2 - m_7 - \frac{3}{2}m_8 + 80, \qquad \sigma_2 = \epsilon_2, \qquad \sigma_3 = \epsilon_3$$
$$c = \min\left\{\frac{2\mu + k}{2\rho d^2}\sigma_1, \frac{2m_1 nd}{m_7}\sigma_2, \frac{4m_1 pd\sigma_3}{m_4 m_8}\right\}. \tag{48}$$

If σ_1, σ_2, σ_3 are all positive, we see that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq -cT \tag{49}$$

and hence, if the flow is valid over the time interval $0 \leq t < \tau$,

$$T(t) \leq T(t_0)\exp[-c(t - t_0)] \tag{50}$$

for any pair of values (t_0, t) such that $0 \leq t_0 \leq t < \tau$; this leads to an alternative criterion for the universal stability as seen in Theorem 2. If the numbers σ_1, σ_2, σ_3 are positive the Liapunoff function T for the incompressible microstretch flow in the bounded domain $R(t)$ tends to zero as $t \to \infty$ and the basic flow $(\bar{q}, \bar{\nu}, \nu)$ is stable in the mean.

The above criterion for universal stability is expressed by the inequalities

$$(R2)^2 + Rm + \frac{3}{2}Rs < 80$$
$$Rm + 2m_3 Rs < 6m_1\pi^2$$
$$Rs < \frac{4m_1(m_5 + 3\pi^2)}{(1 + 2m_3)m_4} \tag{51}$$

and comparison of (44) and (51) shows that the two criteria differ only in the first of the three inequalities. The measures $R1$ and $R2$ employed to signify the Reynolds number of the flow are different and the above change is thus understandable.

5. AN APPLICATION

If the boundary $\partial R(t)$ consists of rigid fixed walls, any motion initially present in the domain $R(t)$ will presumably die out due to lack of supply of energy. By choosing the basic flow to be trivial so that $\bar{q} = \bar{0}$, $\bar{\nu} = \bar{0}$, $\nu = 0$ we see from either of the eqns (27), (28) that

$$\frac{dT}{dt} = \int \rho j \cdot \nu^*(\bar{\vartheta})^2 + \frac{3}{2}\int \rho j \nu^* \theta^2 - \frac{k}{2}\int (\text{curl }\bar{u} - 2\bar{\vartheta})^2$$

$$- \left(\mu + \frac{k}{2}\right)\int (\text{curl }\bar{u})^2 - \gamma \int (\text{curl }\bar{\vartheta})^2 - (\alpha + \beta + \gamma)\int (\text{div }\bar{\vartheta})^2$$

$$- 3\alpha_0 \int (\text{grad }\theta)^2 - 3(\eta_0 - \lambda_0)\int \theta^2. \tag{52}$$

In the above $\bar{u} = \bar{q}^*$, $\bar{\vartheta} = \bar{\nu}^*$, $\theta = \nu^*$ specify the flow under consideration. The quantities J and M defined in ((30), (v) and (vi)) are such that

$$J = \max j, \qquad M = \max |\nu^*|. \tag{53}$$

Proceeding as before we have

$$\frac{dT}{dt} \le -\frac{80(2\mu + k)}{\rho d^2} T_1 + 2\left(M - \frac{3a\pi^2}{\rho J d^2}\right)T_2 + \left(2M - \frac{4(\eta_0 - \lambda_0)d^2 + 12\alpha_0\pi^2}{\rho J d^2}\right)T_3. \tag{54}$$

If we define

$$m_0 = \min\left(\frac{3a\pi^2}{\rho J d^2}, \frac{2(\eta_0 - \lambda_0)d^2 + 6\alpha_0\pi^2}{\rho J d^2}\right) \tag{55}$$

and employ the symbol sgn to specify the sign of the expression in the parentheses, it follows that

$$\text{sgn}\left(\frac{dT}{dt}\right) = -1 \tag{56}$$

for the class of incompressible microstretch flows in the domain $R(t)$ for which

$$M = \max |\nu| < m. \tag{57}$$

We can then deduce the result[10]

$$T(t) \le T(t_0)\exp[-s(t - t_0)] \tag{58}$$

where

$$s = \min\left\{\frac{80(2\mu + k)}{\rho d^2}, 2(m - M)\right\} \tag{59}$$

and t_0 has the same significance as in (43) and (50).

6. UNIQUENESS THEOREMS

(i) Let $(\bar{q}, \bar{\nu}, \nu)$ and $(\bar{q}^*, \bar{\nu}^*, \nu^*)$ be two possible steady flows over the domain $R(t)$ and subject to the hyperstick boundary condition (9). Let m and V_0 be defined for the flow $(\bar{q}, \bar{\nu}, \nu)$ as in ((30), (ii) and (iv)). The kinetic energy T of the difference motion $(\bar{u}, \bar{\vartheta}, \theta)$ is then constant. If either of the sets of inequalities (44), (51) is satisfied, the law of decay in (43) or (50) is to be valid and it follows that $T(t)$ is zero. This implies that the two flows are identical.

(ii) Let $(\bar{q}, \bar{\nu}, \nu)$ and $(\bar{q}^*, \bar{\nu}^*, \nu^*)$ be two possible unsteady flows in the bounded domain $R(t)$ such that they have the same field distribution at $t = 0$ and are subject to the same boundary condition on $\partial R(t)$. Then the difference flow $(\bar{u}, \bar{\vartheta}, \theta)$ is controlled by the law of decay in (43) or (50) if the conditions (44) or (51) are satisfied. At $t = 0$ the flow $(\bar{u}, \bar{\vartheta}, \theta)$ is trivial and hence it follows that $T(t) = 0$ for all $t > 0$. Thus the two flows are identical.

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