STEADY SLOW ROTATION OF A SPHERE IN A VISCO-ELASTIC LIQUID

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Summary

In this paper, we obtain the flow due to slow steady rotation of a sphere in a visco-elastic liquid characterized by the constitutive relation given by Rivlin. The non-Newtonian effects are strongly dependent on a non-dimensional parameter K independent of the angular velocity of the sphere. If $1 < K \leqslant 3$, we notice four vortices symmetrically placed around the sphere. When K lies outside this range, the direction of the flow pattern is the same as that in the Newtonian case but displaced towards the sphere as K decreases. Also the expression for the couple on the sphere has been obtained which depends on K.

§ 1. *Introduction.* Rivlin[1]) considered a class of isotropic incompressible fluids, with rheological properties given by the equation of state, expressing S_{ij} the stress-tensor as a polynomial in the kinematic symmetric tensors

$$D_{ij} = U_{i,j} + U_{j,i} \tag{1a}$$

and

$$B_{ij} = A_{i,j} + A_{j,i} + 2U_{m,i}U_{m,j} \tag{1b}$$

where U_i and A_i denote the components of the velocity and acceleration in the i-th direction. In our present investigation, we consider a particular prototype of this class of fluids, characterized by the equation of state

$$S = -PI + \phi_1 D + \phi_2 B + \phi_3 D^2, \tag{2}$$

where P is the hydrostatic mean pressure, I is the unit tensor of rank 2 and the coefficients ϕ_1, ϕ_2, ϕ_3 are constants. These coefficients are, in general, functions of the invariants of the matrices

D and B. Visco-inelastic liquids characterized by Reiner[2]) can formally be obtained from (2) when $\phi_2 = 0$. Further, if ϕ_3 also vanishes, the liquid is Newtonian.

In the present paper we discuss the flow of a visco-elastic liquid given by (2) due to a sphere rotating steadily in it with a small angular velocity about one of its diameters. This problem was first studied for a classical (or Newtonian) viscous liquid by Stokes[3]). He conjectured that the sphere would act like a centrifugal fan receiving the fluid near the poles and throwing it away at the equator which was later confirmed by Khamrui[4]). Later, Datta[5]) extended this to the class of liquids characterized by Reiner[2]). Recently, the present author[6]) and Thomas and Walters[7]) considered the problem for visco-elastic liquids, whose constitutive relations involve relaxation and retardation times as suggested by Oldroyd.

As in [6]), we employ a method of successive approximation suggested by Collins[8]), in which it is assumed that the velocity components and the pressure can be expanded in ascending powers of a suitable parameter characteristic of the angular velocity of the sphere.

§ 2. *Basic equations.* Let a sphere of radius a rotate steadily with an angular velocity Ω about one of its diameters in a liquid, characterized by (2), extending to infinity in all directions. Let (U, V, W) denote the components of the velocity in the directions of the spherical polar coordinates R $(= ar)$, θ, ϕ respectively, with the origin at the centre of the sphere, $R =$ the distance measured from the centre, the axis $\theta = 0$ coinciding with the axis of rotation of the sphere and ϕ the azimuth.

We shall also introduce the non-dimensional variables defined by the following equations:

$$(U, V, W) = \frac{\phi_1}{\rho a}(u, v, w); \quad (A_R, A_\theta, A_\phi) = \frac{\phi_1^2}{\rho^2 a^3}(a_r, a_\theta, a_\phi);$$

$$P = \frac{\phi_1^2}{\rho a^2} p; \quad S_{ij} = \frac{\phi_1^2}{\rho a^2} s_{ij}; \quad D_{ij} = \frac{\phi_1}{\rho a^2} d_{ij}, \quad B_{ij} = \frac{\phi_1^2}{\rho^2 a^4} b_{ij}, \qquad (3)$$

$$\phi_2 = \rho a^2 \beta, \quad \phi_3 = \rho a^2 \nu_c$$

and

$$G = a\phi_1^2 g/\rho,$$

where ρ is the density of the liquid and G is the couple due to the liquid friction, on the sphere given by

$$G = -\int_0^\pi 2\pi[R^3 S_{R\phi}]_{R=a} \sin^2\theta \, d\theta. \tag{4}$$

The equations of steady motion and continuity for the liquid flow can now be written in the dimensionless form

$$R_e \left[u \frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2 + w^2}{r} \right] = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 s_{rr}) + + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta s_{r\theta}) - \frac{s_{\theta\theta} + s_{\phi\phi}}{r}, \tag{5}$$

$$R_e \left[u \frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv - w^2\cot\theta}{r} \right] = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 s_{r\theta}) + + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta s_{\theta\theta}) + \frac{s_{r\theta} - s_{\phi\phi}\cot\theta}{r}, \tag{6}$$

$$R_e \left[u \frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{uw + vw\cot\theta}{r} \right] = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 s_{r\phi}) + + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta s_{\theta\phi}) + \frac{s_{\phi r} + s_{\theta\phi}\cot\theta}{r} \tag{7}$$

and

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) = 0, \tag{8}$$

where

$$R_e = \rho a^2 \Omega/\phi_1. \tag{9}$$

The equation of continuity (8) is identically satisfied by introducing the stream function ψ given by

$$u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}; \quad v = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{10}$$

The boundary conditions are

$$u = 0 = v, \quad w = R_e \sin\theta, \quad \text{when} \quad r = 1$$

and (11)

$$u = v = w = 0, \quad \text{when} \quad r = \infty.$$

To facilitate the investigation of the visco-elastic effects on the flow, we assume that the solution of the above equations can be expressed as a power series expansion in R_e:

$$X = R_e X^{(1)} + R_e^2 X^{(2)} + R_e^3 X^{(3)} + \dots, \tag{12}$$

where X may stand for any one of the physical quantities

$$u,\ v,\ w,\ \psi,\ d_{ij},\ a_i,\ b_{ij},\ s_{ij},\ p,\ g\ \dots .$$

The boundary conditions (11) in the successive approximations may be written as

$$\left.\begin{aligned} u^{(1)} &= v^{(1)} = 0, \quad w^{(1)} = \sin\theta \\ u^{(n+1)} &= v^{(n+1)} = w^{(n+1)} = 0 \ (n \geqslant 1) \end{aligned}\right\} \quad \text{on } r = 1, \tag{13}$$

$$u^{(n)} = v^{(n)} = w^{(n)} = 0, \quad \text{at} \quad r = \infty \quad (\text{for all } n).$$

and $p^{(n)}$ is finite in the region of the flow.

§ 3. *First approximation.* Substituting (12) in the muster of (5)–(7) and equating the coefficients of R_e we get the equations for the stresses

$$s_{ij}^{(1)} = -p^{(1)}\delta_{ij} + d_{ij}^{(1)} \tag{14}$$

and hence the equations of motion

$$-\frac{\partial p^{(1)}}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta} E^2\psi^{(1)} = 0, \tag{15a}$$

$$-\frac{\partial p^{(1)}}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial}{\partial r} E^2\psi^{(1)} = 0 \tag{15b}$$

and

$$E^2(w^{(1)} r \sin\theta) = 0, \tag{16}$$

where

$$E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

On eliminating $p^{(1)}$ from (15a) and (15b), we have the equation for $\psi^{(1)}$

$$E^4\psi^{(1)} = 0, \tag{17}$$

which yields the solution

$$\psi^{(1)} = \text{constant}, \tag{18}$$

and the pressure

$$p^{(1)} = \text{constant}. \tag{19}$$

(16) gives

$$w^{(1)} = \frac{\sin\theta}{r^2}, \tag{20}$$

which satisfies the appropriate boundary conditions.

From (20), we calculate $g^{(1)}$ the moment

$$g^{(1)} = -8\pi. \tag{21}$$

The results are the same as those noticed by Stokes for a Newtonian liquid[3]).

§ 4. *Second approximation.* Substituting (12) in (5)–(7) and collecting the coefficients of R_e^2, we get the acceleration components

$$a_r^{(2)} = -\frac{\sin^2\theta}{r^5}\ ;\quad a_\theta^{(2)} = -\frac{\sin\theta\cos\theta}{r^5}\ ;\quad a_\phi^{(2)} = 0 \tag{22}$$

and the stresses,

$$\begin{aligned} s_{rr}^{(2)} &= -p^{(2)} + d_{rr}^{(2)} + 9(2\beta + \nu_c)\frac{\sin^2\theta}{r^6}, \\ s_{\theta\theta}^{(2)} &= -p^{(2)} + d_{\theta\theta}^{(2)}, \\ s_{\phi\phi}^{(2)} &= -p^{(2)} + d_{\phi\phi}^{(2)} + 9\nu_c\frac{\sin^2\theta}{r^6}, \\ s_{r\theta}^{(2)} &= d_{r\theta}^{(2)}, \\ s_{\theta\phi}^{(2)} &= d_{\theta\phi}^{(2)}, \\ s_{\phi r}^{(2)} &= d_{\phi r}^{(2)}. \end{aligned} \tag{23}$$

The equations of motion can now be reduced to the equations for $\psi^{(2)}$ and $w^{(2)}$

$$E^4\psi^{(2)} = \sin^2\theta\cos\theta\left[-\frac{6}{r^5} + \frac{12K}{r^7}\right] \tag{24}$$

and

$$E^2(w^{(2)}r\sin\theta) = 0, \tag{25}$$

where

$$K = 12(\beta + \nu_c). \tag{26}$$

(24), (25) yield the solution

$$\psi^{(2)} = -\tfrac{1}{8}\left(1 - \frac{1}{r}\right)^2\left[1 - \frac{K}{3}\left(1 + \frac{2}{r}\right)\right]\sin^2\theta\cos\theta \tag{27}$$

and

$$w^{(2)} = 0. \tag{28}$$

We thus get the velocity distributions

$$u^{(2)} = \frac{1}{8r^2}\left(1 - \frac{1}{r}\right)^2\left[1 - \frac{K}{3}\left(1 + \frac{2}{r}\right)\right](1 - 3\cos^2\theta), \tag{29a}$$

$$v^{(2)} = \frac{1}{4r^3}\left(1 - \frac{1}{r}\right)\left(1 - \frac{K}{r}\right)\sin\theta\cos\theta, \tag{29b}$$

$$w^{(2)} = 0, \tag{29c}$$

the pressure distribution

$$p^{(2)} = -\frac{1}{2r^3}\left[\left(1 - \frac{1}{r}\right)\left\{1 - \frac{K}{3}\left(1 + \frac{1}{r} + \frac{1}{r^2}\right)\right\} - \right.$$
$$\left. - \frac{3}{2}\left\{\left(1 - \frac{4}{3r}\right) - 4\beta\left(1 - \frac{5}{r^3}\right) - 4\nu_c\left(1 - \frac{7}{2r^2}\right)\right\}\sin^2\theta\right] + \text{constant} \tag{29d}$$

and the couple

$$g^{(2)} = 0. \tag{29e}$$

§ 5. *Third approximation.* Proceeding exactly in the same manner and collecting the coefficients of R_e^3, we get the nonvanishing component of the acceleration

$$a_\phi^{(3)} = \sin\theta\left[\left(\frac{1}{4r^5} - \frac{1}{4r^7}\right) + K\left(-\frac{1}{12r^5} - \frac{1}{4r^7} + \frac{1}{3r^8}\right)\right] +$$
$$+ \sin^3\theta\left[\left(-\frac{3}{8r^5} + \frac{1}{4r^6} + \frac{1}{8r^7}\right) + K\left(\frac{1}{8r^5} + \frac{1}{8r^7} - \frac{1}{4r^8}\right)\right], \tag{30}$$

and the stresses

$$s_{ij}^{(3)} = -p^{(3)}\delta_{ij} + d_{ij}^{(3)},$$

excepting

$$s_{\phi r}^{(3)} = d_{\phi r}^{(3)} + \beta \left[\left\{\left(-\frac{3}{2r^6} + \frac{3}{2r^7}\right) + K\left(\frac{1}{2r^6} - \frac{1}{2r^9}\right)\right\} \sin\theta + \right.$$
$$\left. + \left\{\left(\frac{9}{4r^6} - \frac{15}{4r^7} + \frac{3}{2r^8}\right) + K\left(-\frac{3}{4r^6} + \frac{3}{2r^8} - \frac{3}{4r^9}\right)\right\} \sin^3\theta\right] +$$
$$+ \nu_c \left[\left\{\left(-\frac{3}{2r^6} + \frac{9}{2r^7} - \frac{3}{r^8}\right) + K\left(\frac{1}{2r^6} - \frac{3}{r^8} + \frac{5}{2r^9}\right)\right\} \sin\theta + \right.$$
$$\left. + \left\{\left(\frac{9}{4r^6} - \frac{15}{2r^7} + \frac{21}{4r^8}\right) + K\left(-\frac{3}{4r^6} + \frac{21}{4r^8} - \frac{9}{2r^9}\right)\right\} \sin^3\theta\right] \quad (31a)$$

and

$$s_{\theta\phi}^{(3)} = d_{\theta\phi}^{(3)} + \left[\beta\left\{\left(-\frac{9}{4r^6} + \frac{9}{2r^7} - \frac{9}{4r^8}\right) + \right.\right.$$
$$\left. + K\left(\frac{3}{4r^6} - \frac{9}{4r^8} + \frac{3}{2r^9}\right)\right\} +$$
$$\left. + \nu_c\left\{\left(-\frac{9}{4r^6} + \frac{15}{2r^7} - \frac{6}{r^8}\right) + K\left(\frac{3}{4r^6} - \frac{6}{r^8} + \frac{6}{r^9}\right)\right\}\right] \sin^2\theta\cos\theta. \quad (31b)$$

The equations for $\psi^{(3)}$ and $w^{(3)}$ in this case are

$$E^4\psi^{(3)} = 0 \quad (32)$$

and

$$E^2[w^{(3)}r\sin\theta] = \left[\left(\frac{1}{4r^4} - \frac{1}{4r^6}\right) + \right.$$
$$+ K\left(-\frac{1}{12r^4} + \frac{1}{8r^6} - \frac{2}{3r^7} + \frac{3}{4r^8}\right) +$$
$$\left. + K^2\left(-\frac{1}{8r^6} + \frac{3}{4r^8} - \frac{3}{4r^9}\right)\right] \sin^2\theta +$$
$$+ \left[\left(-\frac{3}{8r^4} + \frac{1}{4r^5} + \frac{1}{8r^6}\right) + K\left(\frac{1}{8r^4} - \frac{1}{4r^6} + \frac{3}{8r^7} - \frac{5}{16r^8}\right) + \right.$$
$$\left. + K^2\left(\frac{1}{8r^6} - \frac{5}{16r^8} + \frac{1}{4r^{10}}\right)\right] \sin^4\theta. \quad (33)$$

These equations yield the solution

$$\psi^{(3)} = 0 \quad (34)$$

and

$$w^{(3)} = F_1(r)\sin\theta + F_2(r)\sin^3\theta, \quad (35)$$

where

$$F_1(r) = \frac{\log r}{35r^4} + \frac{1}{1200r^2}\left(1 - \frac{1}{r}\right)\left(1 - \frac{74}{r} + \frac{25}{r^2}\right) +$$

$$+ \frac{K}{1680r^2}\left(1 - \frac{1}{r}\right)\left(1 + \frac{36}{r} - \frac{20}{r^2} + \frac{15}{r^3} - \frac{35}{r^4}\right) -$$

$$- \frac{K^2}{1080r^2}\left(1 - \frac{1}{r}\right)\left(1 + \frac{1}{r} - \frac{185}{22r^2} + \frac{145}{22r^3} + \frac{145}{22r^4} - \frac{175}{11r^5}\right)$$

and

$$F_2(r) = - \frac{\log r}{28r^4} + \frac{1}{16r^3}\left(1 - \frac{1}{r}\right)\left(1 - \frac{1}{4r}\right) -$$

$$- \frac{K}{48r^3}\left(1 - \frac{1}{r}\right)^2\left(1 + \frac{1}{2r^2}\right) -$$

$$- \frac{23}{2112r^4} K^2\left(1 - \frac{1}{r}\right)\left(1 - \frac{10}{23r} - \frac{10}{23r^2} + \frac{12}{23r^3}\right).$$

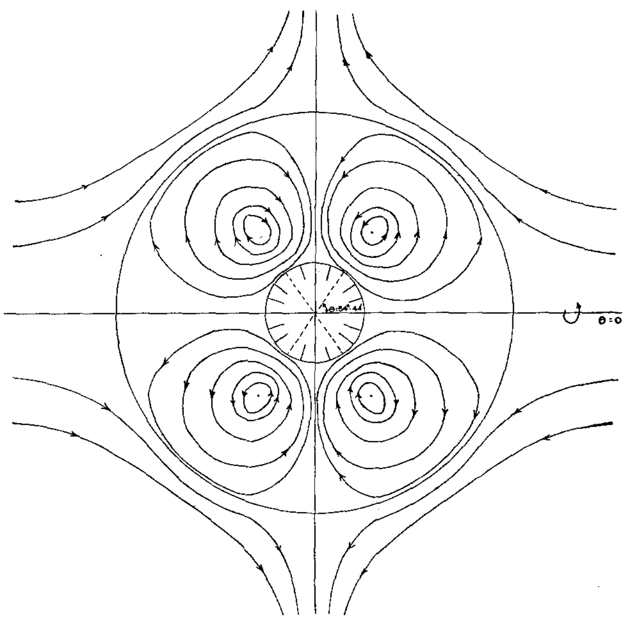

Fig. 1. Flow pattern for $K = 2$.

From this, we obtain the couple $g^{(3)}$

$$g^{(3)} = \frac{\pi}{15}\left(-\frac{1}{10} - \frac{K}{14} + \frac{K^2}{9}\right). \tag{36}$$

We notice that these results are in agreement with those obtained by Collins when $K = 0$[8].

§ 6. *Discussion of the results.* (i) It may be noticed that the flow pattern obtained during the motion due to a rotating sphere is strongly dependent on K. The general motion is obtained by superposing, on the rotational velocity, the motion given by $\psi^{(2)}$ in the meridional plane which indicate the formation of four secondary vortices as discussed in (iv).

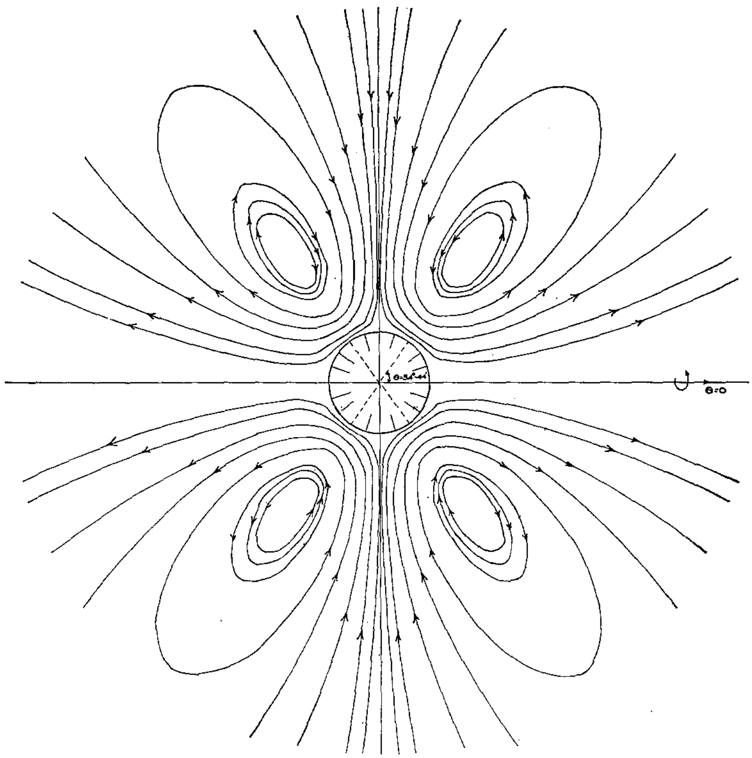

Fig. 2. Flow pattern for $K = 3$.

(ii) The radial velocity (u) changes sign on the cone C: $\theta = \cos^{-1} 1/\sqrt{3}$ and also on the sphere $r = r^*$ $(= 2K/(3 - K))$.

We thus notice that the sphere acts like a centrifugal fan as conjectured by Stokes[3]): "the motion at a distance from the rotating sphere consists of a flow inwards the poles and outwards the equator". We notice the non-Newtonian effect: the reversal of the sign of u when $K = 3$ and also on the sphere $r = r^*$ when $1 < K < 3$. For the range $K \leqslant 1$ or $K > 3$ and also when $r > r^*$ for $1 < K < 3$, the flow direction would be in accordance with the Stokes conjecture.

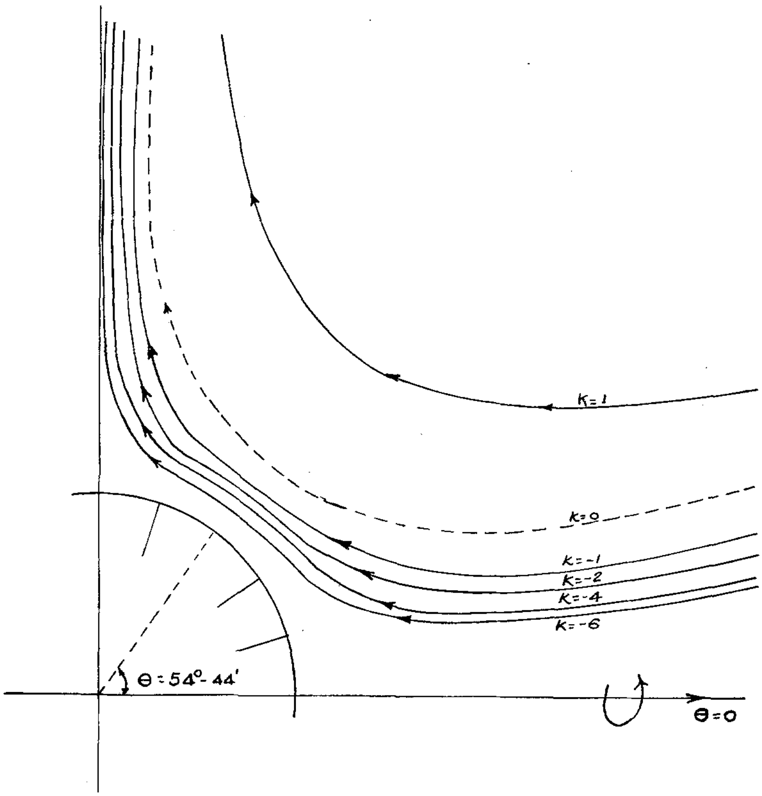

Fig. 3. The stream lines $\psi^{(2)} = 0.002$ for $K = 1, 0, -1, -2, -4, -6$, ($K = 0$ corresponds to the Newtonian case).

(iii) The poloidal component v of the velocity vanishes along the axis of rotation, equatorial plane of the sphere and also on the surface $r = K$ (> 1). The non-Newtonian effect is noticed in the region $1 < r < K$, is negative for $0 < \theta < 90°$ and positive for $90° < \theta < 180°$ which is opposite to that in the Newtonian case.

(iv) When $1 < K \leqslant 3$:

The stream line pattern in the meridional plane consists of circulatory flow around four vortices placed symmetrically round the sphere at the points given by $r = K$, $\theta = \cos^{-1}(1/\sqrt{3})$ the directions of the vortices in the adjacent quandrants being opposite and same in the opposite quadrants. The strength of each vortex is $\sqrt{2}\,(K^2 - 2K + 2)(1 - K)/12K^5$. The flow patterns for the two cases $K = 2$, $K = 3$ have been shown in the figures 1 and 2.

(v) The flow pattern for $K \leq 1$ has been shown in fig. 3. It may be mentioned that the stream lines $\psi^{(2)} =$ constant are displaced towards the sphere as K decreases. A similar effect may be noticed for the range $K > 3$.

(vi) The couple on the sphere due to the liquid is given by

$$g = R_e\left[-8\pi + \frac{R_e^2\zeta\pi}{15}\left(-\frac{1}{10} - \frac{K}{14} + \frac{K^2}{9}\right)\right] \tag{37}$$

which depends on K whose values could be determined from the observations of the couple on the sphere for small values of R_e.

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