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EXISTENCE OF PERIODIC SOLUTIONS OF THE EQUATIONS OF INCOMPRESSIBLE MICROSTRETCH FLUID FLOW

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Abstract—The flow of incompressible microstretch fluid is governed by a system of differential equations involving the velocity vector $\bar{q}$, the microprotation vector $\bar{\nu}$ and the scalar ν representing the microstretch of the fluid element. Let $R = R(t)$ be a bounded domain in space and let the field $(\bar{q}, \bar{\nu}, \nu)$ be prescribed at each point of the boundary $\partial R(t)$. If the domain $R(t)$ and the boundary data depend periodically on the time t, it is shown that under some assumptions on the initial distribution of the flow fields and the material constants of the fluid, there exists a unique, stable, periodic solution of the microstretch flow equations in $R(t)$, taking the prescribed values on the boundary $\partial R(t)$ (Theorem 2 of the paper). The proof rests on some relations describing the rate of decay of the energy functionals corresponding to the difference of two microstretch flows in the domain that have the same density and gyration parameters and are subject to the same boundary conditions.

1. INTRODUCTION

In this paper we employ the energy method to deduce from certain plausible hypotheses the existence of stable, periodic solutions of the equations of motion of incompressible, microstretch fluids. The theory of microstretch fluids initiated by Eringen[1] is a special case of the theory of simple microfluids (Eringen[2]) in which the gyration tensor ν_{ij} accounting for the intrinsic rotary motion and deformation of the fluid elements and the first stress moment tensor λ_{ijk} have the special structure (Ariman[3])

$$\nu_{ij} = \nu\delta_{ij} + \epsilon_{ijk}\nu_k, \tag{1}$$

$$\lambda_{ijk} = \lambda_i\delta_{jk} - \frac{1}{2}\epsilon_{jkr}m_{ir}. \tag{2}$$

The micromotion consists of a rotation about the centroid of the fluid element in an average sense and deformation consisting of a stretch due to the axial motions of cylindrical dumbbell elements. These are reckoned by the microrotation vector $\bar{\nu}$ and the microstretch ν which is a scalar field. When the fluid element has no micro-deformation the microstretch ν is zero, the gyration tensor ν_{ij} is antisymmetric and we have the theory of micropolar fluids (Eringen[4]).

The discussion concerning the existence of periodic and stable solutions of microstretch fluid flow equations hinges on some formulae describing the rate of decay of energy functionals for the difference of two microstretch flows in a spatial domain $R(t)$, both the flows having the same density ρ and gyration parameter j. Both the flow fields conform to the hyperstick or super-adherence condition at the boundary $\partial R(t)$ of the domain and there is no slip, no spin, no stretch of the fluid element relative to the boundary. Such relations concerning the time-rate of change of energy functionals have already been noticed by the authors[5]. However, some modification is necessary to facilitate further discussion on the existence of periodic solution. The modified version is reported below in Theorem 1. Criteria for the existence of stable, periodic solutions are given in Theorem 2. The present study provides an extension to the realm of micro-stretch fluids of the results seen earlier for micropolar fluids[6] and is inspired by the analysis of Serrin[7] on Navier–Stokes equations describing the flow of non-polar Newtonian viscous liquids.

2. ENERGY CRITERION FOR THE STABILITY OF MICROSTRETCH FLOW

The field equations of incompressible microstretch fluid flow are[1, 2]

$$\operatorname{div} \bar{q} = 0, \tag{3}$$

$$\frac{\partial j}{\partial t}+(\bar{q}\cdot \text{grad})j-2\nu j=0, \tag{4}$$

$$\rho\left[\frac{\partial \bar{q}}{\partial t}-\bar{q}\times \text{curl}\,\bar{q}+\text{grad}\left(\frac{1}{2}q^2\right)\right]$$
$$=-\text{grad}\,p+\lambda_0\,\text{grad}+k\,\text{curl}\,\bar{\nu}$$
$$-(\mu+k)\,\text{curl}\,\text{curl}\,\bar{q}+(\lambda_1+2\mu+k)\,\text{grad}\,(\text{div}\,\bar{q}), \tag{5}$$

$$\rho j\left[\frac{\partial \bar{\nu}}{\partial t}+(\bar{q}\,\text{grad})\bar{\nu}\right]=-2k\bar{\nu}+k\,\text{curl}\,\bar{q}$$
$$-\gamma\,\text{curl}\,\text{curl}\,\bar{\nu}+(\alpha+\beta+\gamma)\,\text{grad}\,(\text{div}\,\bar{\nu}), \tag{6}$$

$$\frac{1}{2}\rho j\left[\frac{\partial \nu}{\partial t}+(\bar{q}\cdot\text{grad})\nu\right]=\alpha_0\nabla^2\nu-(\eta_0-\lambda_0)\nu. \tag{7}$$

In the above muster of equations ρ is the density of the fluid, j denotes the gyration parameter and p is an undetermined pressure; the vectors $\bar{q}$, $\bar{\nu}$ are respectively the velocity and microrotation and the scalar field ν denotes the microstretch of the fluid elements. The terms representing the body force, body couple and trace of the first body moment are omitted. The constants (λ, μ, k, η_0, λ_0) are viscosity coefficients and (α, β, γ, α_0) are gyroviscosity coefficients. These are controlled by the restrictions[1]

$$3\lambda_1+2\mu+k\ge 0,\quad 2\mu+k\ge 0,\quad k\ge 0,$$
$$(\eta_0-\lambda_0)\ge 0,\quad (\eta_0-\lambda_0)(3\lambda_1+2\mu+k)\ge\frac{\lambda_0^2}{4},$$
$$3\alpha+\beta+\gamma\ge 0,\quad \gamma\ge 0,\quad |\beta|\le 0,\quad \alpha_0\ge 0. \tag{8}$$

The density ρ and the gyration parameter j are positive and the former is constant.

Consider the motion ($\bar{q}$, $\bar{\nu}$, ν) over the spatial domain $R(t)$ subject to the condition of super-adherence at the boundary $\partial R(t)$. If ($\bar{q}^*$, $\bar{\nu}^*$, ν^*) is another flow in the domain satisfying the super-adherence condition on $\partial R(t)$ and the two flows have the same density ρ and the gyration parameter j, the quantities

$$\bar{u}=\bar{q}^*-\bar{q},\quad \bar{\vartheta}=\bar{\nu}^*-\bar{\nu},\quad \theta=\nu^*-\nu \tag{9}$$

refer to the difference flow and vanish on the boundary $\partial R(t)$. Consider the energy functionals

$$T_1=\frac{1}{2}\int\rho(\bar{u})^2,\quad T_2=\frac{1}{2}\int\rho j(\bar{\vartheta})^2,$$
$$T_3=\frac{3}{4}\int\rho j\theta^2 \tag{10}$$

in which the integrals extend over the volume of the domain $R(t)$. (The conventional volume infinitesimal is omitted throughout the paper). The vector $\bar{u}$ is solenoidal in the domain $R(t)$ and on the boundary $\partial R(t)$ we have

$$\bar{u}=\bar{0},\quad \bar{\vartheta}=\bar{0},\quad \theta=0. \tag{11}$$

Following the procedure in[5] we can obtain the time—rate of change of the energy functionals in the form

$$\frac{dT_1}{dt}=\int\rho\bar{u}\cdot(\text{grad}\,\bar{u})\cdot\bar{q}+k\int\bar{\vartheta}\cdot\text{curl}\,\bar{u}-(\mu+k)\int(\text{curl}\,\bar{u})^2, \tag{12}$$

$$\frac{dT_2}{dt}=\int\rho j\bar{u}\cdot(\text{grad}\,\bar{\vartheta})\cdot\bar{\nu}+\int\rho j\nu^*(\bar{\vartheta})^2$$

$$+2\int \rho j\theta \bar{\nu}\cdot\bar{\vartheta} - 2k\int(\bar{\vartheta})^2 + k\int\bar{\vartheta}\cdot \operatorname{curl}\bar{u}$$

$$-\gamma\int(\operatorname{curl}\bar{\vartheta})^2 - (\alpha+\beta+\gamma)\int(\operatorname{div}\bar{\vartheta})^2, \tag{13}$$

$$\frac{\mathrm{d}T_3}{\mathrm{d}t} = \frac{3}{2}\int \rho j[\bar{u}\cdot(\operatorname{grad}\theta)]\nu + \frac{3}{2}\int \rho j\nu^*\theta^2$$

$$+3\int \rho j\nu\theta^2 - 3\alpha_0\int(\operatorname{grad}\theta)^2 - 3(\eta_0-\lambda_0)\int\theta^2. \tag{14}$$

The integrals

$$\int \rho\bar{u}\cdot(\operatorname{grad}\bar{u})\cdot\bar{q},\qquad \int \rho j\bar{u}\cdot(\operatorname{grad}\bar{\vartheta})\cdot\bar{\nu},$$

$$\int \rho j\theta\nu\cdot\bar{\vartheta},\qquad \int \rho j[\bar{u}\cdot(\operatorname{grad}\theta)]\nu \tag{15}$$

that appear in the relations (12)–(14) are majorizable by use of Schwarz's inequality in the form

$$2\bar{u}\cdot(\operatorname{grad}\bar{u})\cdot\bar{q} \le \frac{2\mu+k}{2\rho}(\operatorname{grad}\bar{u})^2 + \frac{2\rho}{2\mu+k}(\bar{u})^2(\bar{q})^2, \tag{16}$$

$$2\bar{u}\cdot(\operatorname{grad}\bar{\vartheta})\cdot\bar{\nu} \le \frac{2\mu+k}{2\rho}(\operatorname{grad}\bar{\vartheta})^2 + \frac{2\rho}{2\mu+k}(\bar{u})^2(\bar{\nu})^2, \tag{17}$$

$$2\theta(\bar{\nu}\cdot\bar{\vartheta}) \le \frac{2\mu+k}{\rho d^2}\theta^2 + \frac{\rho d^2}{2\mu+k}(\bar{\nu}\cdot\bar{\vartheta})^2, \tag{18}$$

and

$$2[\bar{u}\cdot(\operatorname{grad}\theta)]\nu \le \frac{2\mu+k}{2\rho}(\operatorname{grad}\theta)^2 + \frac{2\rho}{2\mu+k}(\bar{u})^2\nu^2. \tag{19}$$

In (18) we may choose the constant d equal to the diameter of the ball that encloses the bounded domain $R(t)$. Using (16)–(19) in the relations (12)–(14) and noting that

$$\int(\operatorname{grad}\bar{u})^2 = \int(\operatorname{curl}\bar{u})^2 \tag{20}$$

we obtain the bounds for the time-rate of change of the energy functionals shown below:

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} \le -\frac{2\mu+3k}{4}\int(\operatorname{curl}\bar{u})^2 + \frac{\rho^2}{2\mu+k}\int(\bar{u})^2(\bar{q})^2 + k\int\bar{\vartheta}\cdot\operatorname{curl}\bar{u}, \tag{21}$$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} \le \frac{2\mu+k}{4}\int j(\operatorname{grad}\bar{\vartheta})^2 + \frac{\rho^2}{2\mu+k}\int j(\bar{u})^2(\bar{\nu})^2$$

$$+\int \rho j\nu^*(\bar{\vartheta})^2 + \frac{2\mu+k}{d^2}\int j\theta^2 + \frac{\rho^2 d^2}{2\mu+k}\int j(\bar{\nu}\cdot\bar{\vartheta})^2$$

$$-2k\int(\bar{\vartheta})^2 + k\int\bar{\vartheta}\cdot\operatorname{curl}\bar{u} - \gamma\int(\operatorname{curl}\bar{\vartheta})^2$$

$$-(\alpha+\beta+\gamma)\int(\operatorname{div}\bar{\vartheta})^2, \tag{22}$$

$$\frac{\mathrm{d}T_3}{\mathrm{d}t} \le \frac{3(2\mu+k)}{8}\int j(\operatorname{grad}\theta)^2 + \frac{3\rho^2}{2(2\mu+k)}\int j(\bar{u})^2\nu^2$$

$$+\frac{3}{2}\int \rho j\nu^*\theta^2 + 3\int \rho j\nu\theta^2 - 3\alpha_0\int(\operatorname{grad}\theta)^2$$

$$-3(\eta_0-\lambda_0)\int\theta^2. \tag{23}$$

Let the positive constants V_0, $|\bar{\nu}_0|$, M and J be defined in the form

$$\begin{aligned} V_0 &= \max |\bar{q}|, \quad |\bar{\nu}_0| = \max |\bar{\nu}| \\ M &= \max (|\nu|, \quad |\nu^*|) \\ J &= \max (j) \end{aligned} \tag{24}$$

the maxima being over the domain $R(t) \times (0, \tau]$ for any fixed positive constant τ. The time-rate of change of the total energy functionals

$$T = T_1 + T_2 + T_3 \tag{25}$$

is then bounded in the form

$$\begin{aligned} \frac{dT}{dt} \leq & -\frac{2\mu + k}{4} \int (\text{curl } \bar{u})^2 + \frac{\rho^2 V_0^2}{2\mu + k} \int (\bar{u})^2 \\ & + \frac{2\mu + k}{4} \int (\text{grad } \bar{\vartheta})^2 + \frac{\rho J |\bar{\nu}_0|^2}{2\mu + k} \int (\bar{u})^2 \\ & + M \int \rho j (\bar{\vartheta})^2 + \frac{2\mu + k}{d^2} \int j\theta^2 \\ & + \frac{\rho d^2 |\bar{\nu}_0|^2}{2\mu + k} \int j(\bar{\vartheta})^2 - \gamma \int (\text{curl } \bar{\vartheta})^2 \\ & - (\alpha + \beta + \gamma) \int (\text{div } \bar{\vartheta})^2 - \frac{k}{2} \int (\text{curl } \bar{u} - 2\bar{\vartheta})^2 \\ & + \frac{3(2\mu + k)}{8} \int j(\text{grad } \theta)^2 + \frac{3\rho^2 |\bar{\nu}_0|^2 J}{2(2\mu + k)} \int (\bar{u})^2 \\ & + \frac{9M}{2} \int \rho j \theta^2 - 3\alpha_0 \int (\text{grad } \theta)^2 - 3(\eta_0 - \lambda_0) \int \theta^2. \end{aligned} \tag{26}$$

We may recall the inequalities [8, 9]

$$\int (\text{curl } \bar{u})^2 \geq \frac{80}{d^2} \int (\bar{u})^2, \tag{28}$$

$$\int (\text{grad } \bar{\vartheta})^2 \geq \frac{3\pi^2}{d^2} \int (\bar{\vartheta})^2, \tag{28}$$

$$\int (\text{grad } \theta)^2 \geq \frac{3\pi^2}{d^2} \int \theta^2 \tag{29}$$

in (26) to recast the bound for dT/dt. Further, on introducing the positive constant

$$a = \min (\gamma, \alpha + \beta + \gamma) \tag{30}$$

we see that

$$\begin{aligned} \gamma \int (\text{curl } \bar{\vartheta})^2 + (\alpha + \beta + \gamma) \int (\text{div } \bar{\vartheta})^2 \\ \geq a \int \{(\text{curl } \bar{\vartheta})^2 + (\text{div } \bar{\vartheta})^2\} \\ = a \int (\text{grad } \bar{\vartheta})^2. \end{aligned} \tag{31}$$

From (26)–(31) we can deduce that

$$\frac{dT}{dt} \leq c_1 \int (\bar{u})^2 + c_2 \int j(\bar{\vartheta})^2 + c_3 \int j\theta^2 \tag{32}$$

and this can be written in the equivalent form

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq d_1T_1 + d_2T_2 + d_3T_3, \tag{33}$$

the constants d_1, d_2, d_3 being given by

$$d_1 = \frac{2\rho V_0^2 + 2\rho J|\bar{\nu}_0|^2 + 3\rho JM^2}{2\mu + k} - \frac{40(2\mu + k)}{\rho d^2}, \tag{34}$$

$$d_2 = 2M + \frac{3\pi^2(2\mu + k)}{2\rho d^2} + \frac{2\rho d^2|\bar{\nu}_0|^2}{2\mu + k} - \frac{6\pi^2 a}{\rho J d^2}, \tag{35}$$

$$d_3 = 6M + \frac{4(2\mu + k)}{3\rho d^2} + \frac{3\pi^2(2\mu + k)}{2\rho d^2} - \frac{12\pi^2\alpha_0}{\rho J d^2} - \frac{4(\eta_0 - \lambda_0)}{\rho J}. \tag{36}$$

Let us define the dimensionless numbers

$$Re = \frac{\rho V_0 d}{\mu + \frac{k}{2}}, \quad Rm = \frac{\rho\sqrt{(J)}d|\bar{\nu}_0|}{\mu + \frac{k}{2}}, \quad Rs = \sqrt{\left(\frac{3}{2}\right)}\frac{\rho\sqrt{(J)}dM}{\mu + \frac{k}{2}}. \tag{37}$$

From (33)–(36) we find that $\mathrm{d}T/\mathrm{d}t$ is negative whenever we have the conditions

$$Re^2 + Rm^2 + Rs^2 < 80, \tag{38}$$

$$\frac{\rho Md^2}{\mu + \frac{k}{2}} + \frac{1}{2}\left(\frac{\rho d^2|\bar{\nu}_0|}{\mu + \frac{k}{2}}\right)^2 - 6\pi^2\left(\frac{a}{(2\mu + k)J} - \frac{1}{4}\right) < 0, \tag{39}$$

and

$$\frac{\rho Md^2}{\mu + \frac{k}{2}} + \frac{4}{9} + \frac{\pi^2}{2} - \frac{2\pi^2\alpha_0}{\left(\mu + \frac{k}{2}\right)J} - \frac{2(\eta_0 - \lambda_0)d^2}{3\left(\mu + \frac{k}{2}\right)J} < 0. \tag{40}$$

For the requirements in (39), (40) it is necessary to have

$$(2\mu + k)J - 4a < 0, \tag{41}$$

and

$$6\pi^2\alpha_0 + 2(\eta_0 - \lambda_0)d^2 < \left(\frac{4}{3} + \frac{3\pi^2}{2}\right)\left(\mu + \frac{k}{2}\right)J \tag{42}$$

involving the material constants $\mu + k/2$, $\eta_0 - \lambda_0, a$ and α_0 besides the constants J and d. The conditions (41), (42) are in response to the demand that the microstretch flow $(\bar{q}, \bar{\nu}, \nu)$ is asymptotically stable in the sense that the Liapunoff measure of stability T tends to zero as the time t tends to infinity.

Let the numbers ϵ_1, ϵ_2, ϵ_3 be defined in the form

$$\epsilon_1 = (80 - Re^2 - Rm^2 - Rs^2)\frac{\mu + \frac{k}{2}}{\rho d^2}, \tag{43}$$

$$\epsilon_2 = \left\{6\pi^2\left(\frac{a}{(2\mu + k)J} - \frac{1}{4}\right) - \frac{\rho Md^2}{\mu + \frac{k}{2}} - \frac{1}{2}\left(\frac{\rho d^2|\bar{\nu}_0|}{\mu + (k/2)}\right)^2\right\}\frac{2\left(\mu + \frac{k}{2}\right)}{\rho d^2}, \tag{44}$$

$$\epsilon_3 = \left\{ \frac{2\pi^2\alpha_0}{\left(\mu + \frac{k}{2}\right)J} + \frac{2(\eta_0 - \lambda_0)d^2}{3\left(\mu + \frac{k}{2}\right)J} - \frac{\rho M d^2}{\mu + \frac{k}{2}} - \frac{4}{9} - \frac{\pi^2}{2} \right\} \frac{6\left(\mu + \frac{k}{2}\right)}{\rho d^2} \tag{45}$$

and let

$$\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3). \tag{46}$$

We see then

Theorem 1. If the spatial domain $R(t)$ is enclosable in a ball of diameter d and $(\bar{q}, \bar{\nu}, \nu)$ are the velocity, microrotation and microstretch of an incompressible microstretch fluid in $R(t)$ which satisfies the hyperstick condition on the boundary $\partial R(t)$, the energy functional $T(\bar{u}, \bar{\vartheta}, \theta)$ of the arbitrary difference $(\bar{u} = \bar{q}^* - \bar{q}, \bar{\vartheta} = \bar{\nu}^* - \bar{\nu}, \theta = \nu^* - \nu)$ is such that

$$T(\bar{u}, \bar{\vartheta}, \theta) \leq T_0 \exp(-\epsilon t) \tag{47}$$

whenever the conditions (38)–(40) are satisfied by the primary flow. The constant T_0 in (47) denotes the initial value at $t = 0$ of the energy functional T corresponding to the disturbance $(\bar{u}, \bar{\vartheta}, \theta)$ and ϵ has the meaning given in (46).

3. EXISTENCE OF PERIODIC SOLUTIONS

To prove the theorem on the existence of periodic solutions of the microstretch flow equations we start with the following assumptions.

C1. The field $\{\bar{q}(\bar{x}, t), \bar{\nu}(\bar{x}, t), \nu(\bar{x}, t)\}$ is prescribed at each point of the boundary $\partial R(t)$ of a bounded, spatial domain $R(t)$.

C2. The domain $R(t)$ and the assigned boundary values of the field $\{\bar{q}(\bar{x}, t), \bar{\nu}(\bar{x}, t), \nu(\bar{x}, t)\}$ depend periodically on t.

C3. To every continuous initial distribution of the field $\{\bar{q}, \bar{\nu}, \nu\}$ and a suitable initial distribution of the gyration parameter $j(\bar{x}, t)$ there corresponds a solution of the flow equations, satisfying the prescribed conditions on the boundary $\partial R(t)$ and the flow is valid for all time $t \geq 0$.

C4. The conditions in (41) and (42) are satisfied.

C5. There is one solution of the flow equations for which the numbers $(\epsilon_1, \epsilon_2, \epsilon_3)$ introduced in (43)–(45) are positive. This solution is equicontinuous in $\bar{x} = (x, y, z)$ for all t.

C6. The gyration parameter $j(\bar{x}, t)$ is periodic in t with period the same as the boundary values of the field.

Theorem 2. Under the assumptions C1 to C6 there exists a unique, stable periodic solution $\{\bar{q}(\bar{x}, t), \bar{\nu}(\bar{x}, t), \nu(\bar{x}, t)\}$ of the microstretch flow equations in $R(t)$ with its values on $\partial R(t)$ coinciding with the assigned values.

It is convenient to write $\{\bar{q}(\bar{x}, t), \sqrt{(j)}\bar{\nu}(\bar{x}, t), \sqrt{[(3/2)j]}\ \nu(\bar{x}, t)\}$ taking the fields $\bar{q}(\bar{x}, t)$, $\sqrt{(j)}\bar{\nu}(\bar{x}, t)$, $\sqrt{(3j/2)}\nu(\bar{x}, t)$ in conjunction as a single seven—component vector $\bar{A}(\bar{x}, t)$ and refer to the quantity $(Re^2 + Rm^2 + Rs^2)^{1/2} = R$ as the Reynolds number of the flow $\bar{A}(\bar{x}, t)$.

We may take the period of the assigned boundary values and also of $R(t)$ equal to one. Let $\bar{A}(\bar{x}, t)$ be the solution guaranteed by condition C5. The sequence of functions $\Phi(\bar{x}) = \bar{A}(\bar{x}, n)$ $(n = 1, 2, 3, ..)$ is bounded and equicontinuous in $\bar{x}$ and hence by Arzela's theorem ([10], p. 59) this sequence contains a subsequence which converges uniformly to a continuous vector function $\bar{A}(\bar{x})$ in the domain $R(t)$. We shall see presently that the entire sequence $\bar{A}(\bar{x}, n)$ converges to $\bar{A}(\bar{x})$. If this is not true, there would be another subsequence converging uniformly to the continuous vector $\bar{B}(\bar{x})$. Put

$$\bar{A}'(\bar{x}, t) = \bar{A}(\bar{x}, +t + m - n), \quad t \geq 0 \tag{48}$$

and let $m > n$. This vector field is a solution of the microstretch flow equations and satisfies the assigned conditions on the boundary $\partial R(t)$. Let

$$T\{\bar{A}(\bar{x},t)\}=\frac{1}{2}\int\rho|\bar{A}(\bar{x},t)|^2=\frac{1}{2}\int\rho(\bar{q})^2+\frac{1}{2}\int\rho j(\bar{\nu})^2+\frac{3}{4}\int\rho j\nu^2 \tag{49}$$

in which the domain of integration is $R(t)$. The flow defined by each of the two vector functions $\bar{A}(\bar{x},t)$ and $\bar{A}'(\bar{x},t)$ satisfies the conditions (38)–(40) since the numbers $(\epsilon_1,\epsilon_2,\epsilon_3)$ are positive for both the flows. Hence by Theorem 1, eqn (47)

$$T\{\bar{A}'(\bar{x},t)-\bar{A}(\bar{x},t)\}\le T_0\exp(-\epsilon t) \tag{50}$$

where ϵ is defined in (46) and

$$T_0=(T\{\bar{A}'(\bar{x},t)-\bar{A}(\bar{x},t)\})_{t=0}. \tag{51}$$

Since both the flows $\bar{A}(\bar{x},t)$ and $\bar{A}'(\bar{x},t)$ satisfy the conditions (38)–(40) we can see that

$$T_0\le\frac{1}{2}\rho\int\left\{(\bar{q}'-\bar{q})^2+j(\bar{\nu}'-\bar{\nu})^2+\frac{3}{2}j(\nu'-\nu)^2\right\}_{t=0} \tag{52}$$

where the domain of integration is the spatial region $R(0)$. We have therefore the bound

$$T_0\le(\rho)(2V_0^2+2J|\bar{\nu}_0|^2+3JM^2)(\text{volume } R(0))=C \tag{53}$$

and from (50), (53) we see that

$$T\{\bar{A}'(\bar{x},t)-A(\bar{x},t)\}\le C\,\mathrm{e}^{-\epsilon t}. \tag{54}$$

Choosing $t=n$ in (54) we have

$$T\{\Phi_m(\bar{x})-\Phi_n(\bar{x})\}\le C\,\mathrm{e}^{-\epsilon n} \tag{55}$$

and hence we see that

$$\lim_{m,n\to\infty}T\{\Phi_m(\bar{x})-\Phi_n(\bar{x})\}=0. \tag{56}$$

The domain of integration in (55), (56) is $R(n)=R(0)$. Allowing m and n to infinity through sequences of integers such that $\Phi_n(x)\to\bar{A}(\bar{x})$ and $\Phi_m(\bar{x})\to\bar{B}(\bar{x})$ we see that the result (56) poses a contradiction to the earlier provision that $\bar{A}(\bar{x})$ and $\bar{B}(\bar{x})$ are different. We have, therefore, to conclude that the entire sequence $\bar{A}(\bar{x},n)$ converges uniformly to the continuous vector function $\bar{A}(\bar{x})$.

From condition C(3) there exists a flow $\bar{A}^*(\bar{x},t)$ such that

$$\bar{A}^*(\bar{x},0)=\bar{A}(\bar{x}). \tag{57}$$

We shall see that the solution $\bar{A}^*(\bar{x},t)$ is periodic and stable. To this end, put

$$\bar{A}''(\bar{x},t)=\bar{A}(\bar{x},t+n). \tag{58}$$

From (54) we have

$$T\{\bar{A}^*(\bar{x},t)-\bar{A}''(\bar{x},t)\}\le T_1\exp(-\epsilon t) \tag{59}$$

in which ϵ has the same meaning as before and

$$T_1 = (T\{\bar{A}^*(\bar{x}, t) - \bar{A}^n(\bar{x}, t)\})_{t=0} = T\{\bar{A}(\bar{x}) - \Phi_n(\bar{x})\}. \tag{60}$$

The choice of $t = 1$ in (59) leads to

$$T\{\bar{A}^*(\bar{x}, 1) - \Phi_{n+1}(\bar{x})\} \le T_1 e^{-\epsilon} < T_1 \tag{61}$$

and allowing n to infinity, we get

$$T\{\bar{A}^*(\bar{x}, 1) - \bar{A}(\bar{x})\} = 0, \tag{62}$$

since $\lim_{n\to\infty} T_1 = 0$. From (62) we can conclude that

$$\bar{A}^*(\bar{x}, 1) = \bar{A}(\bar{x}) = \bar{A}(\bar{x}, 0) \tag{63}$$

and $\bar{A}^*(\bar{x}, t)$ is, therefore, periodic with period equal to one.

By Theorem 1 we know that

$$T\{\bar{A}^*(\bar{x}, t) - \bar{A}(\bar{x}, t)\} \to 0 \text{ as } t \to \infty. \tag{64}$$

Since both $\bar{A}^*(\bar{x}, t)$ and $\bar{A}(\bar{x}, t)$ are equicontinuous, we see that

$$\bar{A}^*(\bar{x}, t) - \bar{A}(\bar{x}, t) \to 0 \text{ as } t \to \infty \tag{65}$$

and so

$$\max_{(\bar{x})} |\bar{A}^*(\bar{x}, t)| - \max_{(\bar{x})} |\bar{A}(\bar{x}, t)| \le 0(t). \tag{66}$$

Since $\bar{A}^*(\bar{x}, t)$ is periodic, we see that

$$\max_{(\bar{x})} |\bar{A}^*(\bar{x}, t)| \le \max_{(\bar{x})} |\bar{A}(\bar{x}, t)|. \tag{67}$$

Hence the Reynolds numbers R for the two flows are such that

$$R^2(\bar{A}^*) \le R^2(\bar{A}) \tag{68}$$

and the numbers $(\epsilon_1, \epsilon_2, \epsilon_3)$ defined in (43)–(45) are such that

$$\epsilon_i(\bar{A}^*) \ge \epsilon_i(\bar{A}). \qquad (i = 1, 2, 3). \tag{69}$$

Since the set of numbers $(\epsilon_1, \epsilon_2, \epsilon_3)$ is positive for the flow $\bar{A}(\bar{x}, t)$, it follows from (69) that the set is positive also for the flow $\bar{A}^*(\bar{x}, t)$ and hence it is stable. This completes the proof of Theorem 2.

4. ADDITIONAL REMARKS

(a) The boundary conditions prescribed must be compatible with a flow for which the restrictions (38)–(40) are satisfied. Thus, for sufficiently low valued periodic boundary prescription of $\bar{A}(\bar{x}, t)$ there exists a periodic flow to which every other flow tends in the mean eventually.

(b) When the assigned boundary conditions are *steady*, Theorem 2 assures the existence of a unique, stable, time-independent solution of the microstretch flow equations, taking the prescribed values on the boundary.

(c) The condition C(3) is mathematically stringent as the flow is to be valid for all $t \geq 0$. However it is enough if this condition holds for those initial data for which the conditions (38)–(40) hold.

(d) Theorem 2 is not to be deemed as the standard type of mathematical existence theorem. Conditions C3 and C5 may not hold for certain types of boundary data. The extent of applicability of the Theorem is not well-defined and this aspect of the preblem needs investigation.

REFERENCES

[1] A. C. ERINGEN, *Int. J. Engng Sci.* **7**, 115 (1969).
[2] A. C. ERINGEN, *Int. J. Engng Sci.* **2**, 205 (1964).
[3] T. ARIMAN, *Rheologica Acta.* **9**, 542 (1970).
[4] A. C. ERINGEN, *J. Math. Mech.* **16**, 1 (1966).
[5] S. K. LAKSHMANA RAO and K. VENKATAPATHI RAJU, *Int. J. Engng. Sci.* To be published.
[6] S. K. LAKSHMANA RAO, *Int. J. Engng. Sci.* **9**, 1143 (1971).
[7] J. SERRIN, *Arch. Ration. Mech. Analysis* **3**, 120 (1959).
[8] L. E. PAYNE and H. F. WEINBERGER, *Nonlinear Problems.* The University of Wisconsin Press, (1963).
[9] S. K. LAKSHMANA RAO, *Int. J. Engng Sci.* **9**, 1151 (1971).
[10] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, Vol. 1. p. 59. Intersciences, New York (1953).

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