SOLUTION OF THE HEAT TRANSFER OF LAMINAR FORCED-CONVECTION IN NON-CIRCULAR PIPES

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Summary

The heat transfer problems of forced-convection in non-circular pipes have many engineering applications. In this a paper a formal solution is given when the mapping function $z = w(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n$, which maps conformally the cross-section of the channel onto the unit circle γ in the ζ-plane is known. The expressions for the average velocity, average temperature, mixed mean temperature, heat transfer rate and the Nusselt number have been expressed in terms of the constants a_n.

§ 1. *Introduction*. The heat transfer problems of forced-convection in channels have been studied by a number of investigators. L. N. Tao[1]) has employed complex variable methods to solve some forced-convection problems of fully developed laminar flow in channels with heat sources and linearly varying wall temperature. In this paper complex variable methods developed by N. I. Muskhelishvili[2]) have been used. The method is illustrated by applications to cardioid and ovaloid cross-sections.

§ 2. *Basic equations*. Consider a steady fully developed laminar flow with arbitrary heat generation in a pipe of cross-section D bounded by a closed curve C. Let the axis of the pipe be in the Z-direction. The basic momentum and energy equations of the constant property non-dissipative fluid are

$$\nabla^2 u = \frac{1}{\mu} \frac{\partial p}{\partial Z} = C_1, \tag{1}$$

$$\nabla^2 t = \frac{\rho}{k} c_p \frac{\partial t}{\partial Z} u - Q/k = (C_2 u - C_3), \tag{2}$$

where c_p = specific heat constant pressure, k = thermal conductivity, μ = viscosity, φ = density, Q = heat source intensity, $C_1 = 1/\mu(\partial p/\partial Z)$, $C_2 = [(\rho/k)\, c_p(\partial t/\partial Z)]$, $C_3 = Q/k$. ∇^2 is the Laplace operator in two dimensions.

The boundary conditions for the problem are

$$u = 0, \tag{3.1}$$

$$t = t_w, \tag{3.2}$$

where u = local velocity, t = local temperature, t_w = wall temperature. Writing $z = x + iy$, $\bar{z} = x - iy$, $T = (t - t_w)$, (new temperature variable), $T = T_1(u) + T_2(M) = (T_1 + T_2)$, equations (1) and (2) can be written in the form

$$4 \frac{\partial^2 u}{\partial z \partial \bar{z}} = C_1, \tag{4}$$

$$4 \frac{\partial^2 T}{\partial z \partial \bar{z}} = (C_2 u - C_3). \tag{5}$$

From (5) we obtain

$$4 \frac{\partial^2 T_1}{\partial z \partial \bar{z}} = c_2 u, \tag{6}$$

$$4 \frac{\partial^2 T_2}{\partial z \partial \bar{z}} = -C_3 = -M, \tag{7}$$

Eliminating u from (4) and (6) we obtain

$$16 \left(\frac{\partial^4 T_1}{\partial z^2 \partial \bar{z}^2} \right) = C_1 C_2 = C_4. \tag{8}$$

Now we put

$$\theta_1 = T_1 - \frac{C_4}{64} (z\bar{z})^2, \tag{9}$$

$$\theta_2 = T_2 + \tfrac{1}{4} \int\int M \, dz \, d\bar{z}, \tag{10}$$

and equations (7) and (8) become

$$\frac{\partial^2 \theta_2}{\partial z \partial \bar{z}} = 0, \tag{11}$$

$$\frac{\partial^4\theta_1}{\partial z^2\partial\bar{z}^2} = 0, \tag{12}$$

the boundary conditions are

$$\left.\begin{aligned} \theta_1 &= -\frac{C_4}{64}(z\bar{z})^2 \quad &(13.1)\\ \frac{\partial^2\theta_1}{\partial z\partial\bar{z}} &= -\frac{C_4}{16}(z\bar{z}) \quad &(13.2)\end{aligned}\right\} \text{on} \quad B,$$

and

$$\theta_2 = \tfrac{1}{4}\textstyle\int\int M\,\mathrm{d}z\,\mathrm{d}\bar{z}, \quad \text{on} \quad B. \tag{13.3}$$

The velocity field is related to θ_1 by

$$C_2 u = 4\frac{\partial^2\theta_1}{\partial z\partial\bar{z}} + \frac{C_4}{4}z\bar{z}. \tag{14}$$

The general solutions of (12) and (11) can be taken in the form

$$\theta_1 = \bar{z}\phi(z) + \overline{z\phi(z)} + \psi(z) + \overline{\psi(z)}, \tag{15}$$

$$\theta_2 = \phi_1(z) + \overline{\phi_1(z)}, \tag{16}$$

where $\phi(z)$, $\psi(z)$ are functions holomorphic in the region D and satisfying the boundary conditions (13.1), (13.2) and $\phi_1(z)$ is also a holomorphic function in D satisfying the condition (13.3). From the equation (15) we have

$$\frac{\partial^2\theta_1}{\partial z\partial\bar{z}} = \phi^1(z) + \overline{\phi^1(z)}. \tag{17}$$

Using the boundary conditions (13.1) and (13.2) in the equations (15) and (17) we obtain for the determination of the two analytic functions $\phi(z)$, $\psi(z)$

$$\bar{z}\phi(z) + \overline{z\phi(z)} + \psi(z) + \overline{\psi(z)} + \frac{C_4}{64}(z\bar{z})^2 = 0, \tag{18}$$

$$\phi'(z) + \overline{\phi'(z)} + \frac{C_4}{16}z\bar{z} = 0. \tag{19}$$

If the heat generation is uniform we obtain from (16) and (13.3)

$$\phi_1(z) + \overline{\phi_1(z)} = \frac{C_3}{4}z\bar{z}. \tag{20}$$

The velocity and temperature fields are given by

$$u = \frac{C_1}{4} z\bar{z} + (8/C_2) \operatorname{Re} \phi_1(z), \tag{21}$$

$$T = \frac{C_4}{64} (z\bar{z})^2 + 2 \operatorname{Re}[\bar{z}\phi(z) + \psi(z) + \phi_1(z)] - \frac{C_3}{4} z\bar{z}. \tag{22}$$

The average velocity u_m, the average temperature T_m the mixed mean temperature T_M, the heat transfer rate q, the heat transfer coefficient h and the Nusselt number Nu based on the mixed mean temperature are given by

$$Au_m = \int_D u \, \mathrm{d}A, \tag{23}$$

$$AT_m = \int_D T \, \mathrm{d}A, \tag{24}$$

$$u_m A T_M = \int_D uT \, \mathrm{d}A, \tag{25}$$

$$q = (C_2 u_m - C_3) \, kA, \tag{26}$$

$$h = -(q/ST_M) = (C_3 - C_2 u_m) \, kA/ST_M, \tag{27}$$

$$Nu = hD_e/k = 4(A/S)^2 \, (C_3 - C_2 u_m)/T_M, \tag{28}$$

where A is the area of the cross-section, D_e is the equivalent hydraulic diameter ($= 4A/S$), S is the circumferential length of the cross-section.

§ 3. *Conformal transformation.* Suppose that the cross-section of the channel be mapped onto the unit circle in the ζ-plane by the mapping function

$$z = w(\rho \, \mathrm{e}^{i\theta}) = w(\zeta). \tag{29}$$

Using (29) in (19), (20), (18) and multiplying by $(1/2\pi i)[\mathrm{d}\sigma/(\sigma - \zeta)]$ we easily find

$$X(\zeta) + \overline{X(0)} + \frac{C_4}{32\pi i} \int_\gamma \frac{w(\sigma) \, \overline{w(\sigma)}}{(\sigma - \zeta)} \, \mathrm{d}\sigma = 0, \tag{30}$$

$$Y(\zeta) + \overline{Y(0)} = \frac{C_3}{8\pi i} \int \frac{w(\sigma) \, \overline{w(\sigma)}}{(\sigma - \zeta)} \, \mathrm{d}\sigma, \tag{31}$$

$$\Psi(\zeta) + \overline{\Psi(0)} = -\frac{C_4}{128\pi i}\int\limits_\gamma \frac{[w(\sigma)\,\overline{w(\sigma)}]^2}{(\sigma-\zeta)}\,\mathrm{d}\sigma - $$

$$-\frac{1}{2\pi i}\int\limits_\gamma \frac{\{\overline{w(\sigma)}\,\Phi(\sigma) + w(\sigma)\,\overline{\Phi(\sigma)}\}}{(\sigma-\zeta)}\,\mathrm{d}\sigma, \tag{32}$$

where

$$\phi(z) = \Phi(\zeta), \quad \psi(z) = \Psi(\zeta),$$
$$\phi'(z) = \Phi'(\zeta)/w'(\zeta) = X(\zeta), \quad \phi_1(z) = Y(\zeta),$$
$$\Phi(\zeta) = \int X(\zeta)\,w'(\zeta)\,\mathrm{d}\zeta. \tag{33}$$

Using Stokes' theorem we obtain from (29), (23), (24) and (25):

$$(C_2A)\,u_m = 4\,\mathrm{Im}\left[\frac{C_4}{64}\int\limits_\gamma w\bar{w}^2\,\mathrm{d}w + \int\limits_\gamma \Phi\,\mathrm{d}w\right], \tag{34}$$

$$AT_m = \mathrm{Im}\left[\frac{C_4}{384}\int\limits_\gamma w^2\bar{w}^3\,\mathrm{d}w - \frac{C_3}{16}\int\limits_\gamma w\bar{w}^2\,\mathrm{d}w + \right.$$
$$\left. + \frac{1}{2}\int\limits_\gamma \Phi\bar{w}^2\,\mathrm{d}w + \int\limits_\gamma (\Psi + Y)\,\bar{w}\,\mathrm{d}w\right], \tag{35}$$

$$(u_mA)\,T_M = \mathrm{Im}\left[\frac{C_1C_4}{2048}\int\limits_\gamma w^3\bar{w}^4\,\mathrm{d}w - \frac{C_1C_3}{96}\int\limits_\gamma w^2\bar{w}^3\,\mathrm{d}w - \right.$$
$$- \frac{C_3}{2C_2}\int\limits_\gamma Xw\bar{w}^2\,\mathrm{d}w + \frac{C_1}{48}\int\limits_\gamma Xw^2\bar{w}^3\,\mathrm{d}w + \frac{C_1}{12}\int\limits_\gamma \Phi w\bar{w}^3\,\mathrm{d}w +$$
$$+ \frac{C_1}{8}\int\limits_\gamma (\Psi + Y)\,w\bar{w}^2\,\mathrm{d}w + \frac{4}{C_2}\int\limits_\gamma X\bar{\theta}_0 w\,\mathrm{d}w +$$
$$\left. + \frac{2}{C_2}\int\limits_\gamma X\bar{w}^2\,\mathrm{d}w + \frac{4}{C_2}\int\limits_\gamma (\Psi + Y)(\Phi + \bar{w}X)\,\mathrm{d}w\right]. \tag{36}$$

Im denotes the imaginary part and $\sigma = \mathrm{e}^{i\theta}$ a point on the unit circle.

$$w = w(\sigma), \quad \Phi = \Phi(\sigma), \quad Y = Y(\sigma), \quad \Psi = \Psi(\sigma) \tag{36.1}$$
$$X = X(\sigma), \quad \theta_0 = \int \Phi(\zeta)\,w'(\zeta)\,\mathrm{d}\zeta,$$

$$A = \int_D \frac{\partial \bar{z}}{\partial \bar{z}}\, \mathrm{d}A = \frac{1}{2i} \int_C \bar{z}\, \mathrm{d}z. \tag{37}$$

§ 4. *General solution.* Let the cross-section of the channel be mapped conformally onto the unit circle γ in the ζ-plane by the mapping formula

$$z = w(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n. \tag{38}$$

Using (38) in (37), (30), (31), (33), (34), (35), (36) and (32) we easily find

$$A = \pi \sum_{n=1}^{\infty} n a_n \bar{a}_n, \tag{39}$$

$$X(\zeta) = -\frac{C_4}{16} \left(\sum_{1}^{\infty} b_n \zeta^n + \tfrac{1}{2} b_0 \right), \tag{40}$$

$$Y(\zeta) = \frac{C_3}{4} \left(\sum_{1}^{\infty} b_n \zeta^n + \tfrac{1}{2} b_0 \right), \tag{41}$$

$$\Phi(\zeta) = -\frac{C_4}{16} \left[\sum_{0}^{\infty} \frac{c_n \zeta^{n+1}}{(n+1)} - \tfrac{1}{2} b_0 \sum_{0}^{\infty} a_n \zeta^n \right] = -\frac{C_4}{16} \sum_{0}^{\infty} e_n \zeta^n. \tag{42}$$

$$8 A u_m = -\pi C_1 [4 (\sum_{n=1}^{\infty} a_{n+1} \bar{c}_n + \tfrac{1}{2} b_0 \sum_{n=1}^{\infty} n a_n \bar{a}_n) - \\ - (b_0 d_0 + \sum_{n=1}^{\infty} b_n d_{-n} + \bar{b}_n d_n)], \tag{43}$$

$$A T_m = 2\pi \left[\frac{C_4}{384} \sum_{-\infty}^{\infty} l_{\gamma-1} b_{-r} - \frac{C_4}{32} \sum_{-\infty}^{\infty} d_r \bar{t}_{-r} + \right. \\ + \frac{C_4}{64} \{ 4 \sum_{1}^{\infty} (t_r + \bar{t}_r)\, d_{-r} - \sum_{1}^{\infty} h_r d_{-r} + 4 t_0 d_0 - \tfrac{1}{2} h_0 d_0 \} + \\ \left. + \frac{C_3}{16} \{ 4 (\sum_{1}^{\infty} b_r d_{-r} + \tfrac{1}{2} b_0 d_0) - \sum_{-\infty}^{\infty} b_{-r} d_r \} \right], \tag{44}$$

$$\Psi(\zeta) = \frac{C_4}{64} [4 \sum_{1}^{\infty} (t_n + \bar{t}_n)\, \zeta^n - \sum_{1}^{\infty} h_n \zeta^n + 4 t_0 - \tfrac{1}{2} h_0], \tag{45}$$

$$(u_m A)\ T_M = \frac{\pi C_1 C_3}{48}\ [5 \sum_1^\infty b_r l_{-(\gamma+1)} + 2b_0 l_{-1} - \sum_1^\infty b_{-r} l_{\gamma-1} +$$

$$+ 6\{(\tfrac{3}{4} b_0^2 d_0 + b_0 \sum_1^\infty b_r d_{-r} - \sum_1^\infty S_r d_{-r}) - (\sum_1^\infty b_r f_{-(r+1)} + \tfrac{1}{2} b_0 f_{-1})\}] +$$

$$+ \frac{\pi C_1 C_4}{3072} \Big[3 \sum_{-\infty}^\infty b_r L_{-(r+1)} - 8(\sum_1^\infty b_r L_{-(r+1)} + \tfrac{1}{2} b_0 L_{-1}) -$$

$$- 32 \sum_{-\infty}^\infty \bar{t}_r l_{-(\gamma+1)} + 12\{4 \sum_1^\infty (t_r + \bar{t}_r)\, l_{-(\gamma+1)} - \sum_1^\infty h_r l_{-(\gamma+1)} +$$

$$+ l_{-1}(4t_0 - \tfrac{1}{2} h_0)\} + 48\{\sum_0^\infty (B_r - b_0 e_r)\, \bar{H}_{-(r+1)}\} -$$

$$- 24\{4 \sum_1^\infty (t_r + \bar{t}_r)\, f_{-(r+1)} - \sum_1^\infty h_r f_{-(r+1)} + f_{-1}(4t_0 - \tfrac{1}{2} h_0)\} -$$

$$- 24\{4 \sum_0^\infty U_r d_{-r} - \sum_0^\infty V_r d_{-r} + (\tfrac{1}{2} h_0 - 4t_0) \sum_0^\infty b_r d_{-r} -$$

$$- \tfrac{1}{2} b_0 (4 \sum_1^\infty (t_r + \bar{t}_r)\, d_{-r} - \sum_1^\infty h_r d_{-r} + 4t_0 d_0 - \tfrac{1}{2} h_0 d_0)\} +$$

$$+ 96 \sum_0^\infty (A_r' - b_0 \bar{A}_r) \frac{W_r}{(r+1)} \Big]. \tag{46}$$

In deducing the above formula the following results have been used.

$$w = w(\sigma) = \sum_0^\infty a_n \sigma^n, \tag{47}$$

$$w\bar{w} = \sum_{-\infty}^\infty b_n \sigma^n, \quad b_n = \sum_{\gamma=0}^\infty a_{n+r} \bar{a}_r, \quad \bar{b}_n = b_{-n}, \tag{47.1}$$

$$w'\bar{w} = \sum_{-\infty}^\infty d_n \sigma^{n-1}, \quad d_n = \sum_{\gamma=0}^\infty (n + r)\, a_{n+r} \bar{a}_r,$$
$$d_{-n} = \sum_{\gamma=0}^\infty r a_r \bar{a}_{n+r}, \tag{47.2}$$

$$w\bar{w}^2 w' = \sum_{-\infty}^\infty l_n \sigma^n, \quad l_n = \sum_{-\infty}^\infty d_{n+r+1} b_{-r}, \tag{47.3}$$

$$w^2 \bar{w}^3 w' = \sum_{-\infty}^\infty L_n \sigma^n, \quad L_n = \sum_{-\infty}^\infty l_{n+\gamma} b_{-r}, \tag{47.4}$$

$$w^3 \bar{w}^4 w' = \sum_{-\infty}^\infty P_n \sigma^n, \quad P_n = \sum_{-\infty}^\infty L_{n-r} b_r, \tag{47.5}$$

$$ww' = \sum_{0}^{\infty} \bar{A}_n \sigma^n, \quad \bar{A}_n = \sum_{\gamma=0}^{\infty} (r+1)\, a_{r+1} a_{n-r}, \tag{47.6}$$

$$(w\bar{w})^2 = \sum_{-\infty}^{\infty} h_n \sigma^n, \quad h_n = \sum_{-\infty}^{\infty} b_{n+r} \bar{b}_{-r}, \tag{47.7}$$

$$\bar{w}^2 w' = \sum_{-\infty}^{\infty} \bar{H}_n \sigma^n, \quad \bar{H}_n = \sum_{\gamma=0}^{\infty} d_{n+r+1} \bar{a}_r, \tag{47.8}$$

$$\bar{\Phi} w = \sum_{-\infty}^{\infty} t_n \sigma^n, \quad t_n = \sum_{\gamma=0}^{\infty} a_{n+r} \bar{e}_r, \tag{48.1}$$

$$\Phi' \bar{w} = \sum_{-\infty}^{\infty} g_n \sigma^n, \quad g_n = \sum_{\gamma=0}^{\infty} (n+r+1)\, e_{n+r+1} \bar{a}_r, \tag{48.2}$$

$$\bar{\Phi} w' = \sum_{-\infty}^{\infty} f_n \sigma^n, \quad f_n = \sum_{\gamma=1}^{\infty} (n+r+1)\, a_{n+r+1} \bar{e}_\gamma, \tag{48.3}$$

$$\Phi \bar{w} = \sum_{-\infty}^{\infty} \bar{t}_n \sigma^n, \quad \bar{t}_n = \sum_{\gamma=0}^{\infty} \bar{a}_r e_{n+r}, \tag{48.4}$$

$$\Phi \bar{w}^2 w' = \sum_{-\infty}^{\infty} H_n \sigma^n, \quad H_n = \sum_{-\infty}^{\infty} d_{n+r+1} \bar{t}_{-r}, \tag{48.5}$$

$$X w \bar{w}^2 w' = -\frac{C_4}{16} \Big[\sum_{-\infty}^{\infty} M_n \sigma^n - \tfrac{1}{2} b_0 \sum_{-\infty}^{\infty} l_n \sigma^n\Big], \quad M_n = \sum_{\gamma=0}^{\infty} l_{n-\gamma} b_r, \tag{48.6}$$

$$X w^2 \bar{w}^3 w' = -\frac{C_4}{16} \Big[\sum_{-\infty}^{\infty} N_n \sigma^n - \tfrac{1}{2} b_0 \sum_{-\infty}^{\infty} L_n \sigma^n\Big], \quad N_n = \sum_{\gamma=0}^{\infty} L_{n-r} b_r, \tag{48.7}$$

$$\Phi w \bar{w}^3 w' = -\frac{C_4}{16} \sum_{-\infty}^{\infty} Q_n \sigma^n, \quad Q_n = \sum_{-\infty}^{\infty} l_{n-\gamma} \bar{t}_r, \tag{48.8}$$

$$Y w \bar{w}^2 w' = \frac{C_3}{4} \Big[\sum_{-\infty}^{\infty} M_n \sigma^n - \tfrac{1}{2} b_0 \sum_{-\infty}^{\infty} l_n \sigma^n\Big], \tag{48.9}$$

$$\Phi X = \frac{C_4^2}{256} \Big[\sum_{0}^{\infty} B_n \sigma^n - \tfrac{1}{2} b_0 \sum_{0}^{\infty} e_n \sigma^n\Big], \quad B_n = \sum_{\gamma=0}^{\infty} e_{n-r} b_r, \tag{48.10}$$

$$\Phi X \bar{w}^2 w' = \frac{C_4^2}{256} \Big[\sum_{-\infty}^{\infty} \Big(\sum_{\gamma=0}^{\infty} B_r \bar{H}_{n-r}\Big) \sigma^n - \tfrac{1}{2} b_0 \sum_{-\infty}^{\infty} \Big(\sum_{\gamma=0}^{\infty} e_r \bar{H}_{n-r}\Big) \sigma^n\Big], \tag{48.11}$$

$$\theta_0(\sigma) = -\frac{C_4}{16} \sum_{0}^{\infty} \frac{W_n \sigma^{n+1}}{(n+1)}, \quad W_n = \sum_{\gamma=0}^{\infty} (r+1)\, a_{r+1} e_{n-r}, \tag{48.12}$$

$$\frac{1}{2\pi i}\int_{\gamma} X\bar{\theta}_0 w w' \,\mathrm{d}\sigma = \frac{C_4^2}{256}\left[\sum_0^\infty \frac{A_r' W_r}{(r+1)} - \tfrac{1}{2}b_0 \sum_0^\infty \frac{\bar{A}_r W_r}{(r+1)}\right], \quad (48.13)$$

$$\bar{\Phi} X = -\frac{C_4^2}{1024}\,[4\sum_0^\infty U_n\sigma^n - \sum_0^\infty V_n\sigma^n + (\tfrac{1}{2}h_0 - 4t_0)\sum_0^\infty b_n\sigma^n -$$

$$-\tfrac{1}{2}b_0\{4\sum_0^\infty (t_r + \bar{t}_r)\,\sigma^\gamma - \sum_0^\infty b_n\sigma^n + (\tfrac{1}{2}h_0 - 4t_0)\}], \quad (48.14)$$

where

$$U_n = \sum_0^\infty (t_r\, X\, \bar{t}_r)\, b_{n-r}, \quad V_n = \sum_0^\infty h_r b_{n-r}. \quad (49)$$

§ 5. *Applications.*

Cross-section a cardioid. The mapping function

$$z = w(\zeta) = R(1+\zeta)^2, \qquad R > 0, \quad (47)$$

maps the cross-section of the channel onto the unit circle in the ζ-plane.

Using (50) in (47) to (48.1) we find the non-vanishing constants

$$a_0 = R,\quad a_1 = 2R,\quad a_2 = R,\quad b_0 = 6R^2,\quad b_1 = 4R^2,$$
$$b_2 = R^2,\quad c_1 = 8R^3,\quad c_2 = 10R^3,\quad c_3 = 2R^3,$$
$$d_0 = d_1 = 6R^2,\quad d_2 = d_{-1} = 2R^2, \quad (50.1)$$

$$e_1 = 6R^3,\quad e_2 = 7R^3,\quad e_3 = (10/3)\,R^3,\quad e_4 = (1/2)\,R^3,$$
$$f_{-1} = 26R^4,\quad f_0 = 12R^4,\quad f_{-2} = (62/3)\,R^3,$$
$$f_{-3} = (23/3)\,R^4,\quad f_{-4} = R^4, \quad (50.2)$$

$$g_0 = 44R^4,\quad g_1 = 36R^4,\quad g_2 = 14R^4,\quad g_3 = 2R^4,$$
$$l_0 = 70R^4,\quad l_1 = 42R^4,\quad l_2 = 14R^4,\quad l_3 = 2R^3,$$
$$l_{-2} = 42R^4,\ l_{-3} = 14R^4,\quad l_{-4} = 2R^4,\quad l_{-1} = 70R^4, \quad (50.3)$$

$$h_0 = 70R^4,\quad h_1 = 56R^4,\quad h_2 = 28R^4,\quad h_3 = 8R^4,$$
$$h_4 = R^4,\quad t_0 = 19R^4,\quad t_1 = 6R^4,\quad \bar{t}_1 = (70/3)\,R^4,$$
$$\bar{t}_2 = (85/6)\,R^4,\ \bar{t}_3 = (13/3)\,R^4,\ \bar{t}_4 = (1/2)\,R^4,\ \bar{t}_{-1} = 6R^4,$$
$$S_0 = 36R^4,\quad S_1 = 48R^4,\quad U_0 = 228R^6,$$
$$U_1 = 328R^6,\quad v_0 = 420R^6,\quad V_1 = 616R^6, \quad (50.4)$$

$$L_0 = 924R^6,\quad L_{-1} = 924R^6,\quad L_{-2} = 660R^6,\quad L_{-3} = 330R^6,$$
$$L_1 = 660R^6,\quad B_1 = 36R^5,\quad B_2 = 66R^5,\quad B_3 = 54R^5,$$
$$B_4 = (70/3)\,R^5,\quad B_5 = (16/3)\,R^5,\quad B_0 = (1/2)\,R^5,$$
$$\bar{H}_{-1} = 20R^3,\quad \bar{H}_{-2} = 20R^3,\quad \bar{H}_{-3} = 10R^3,\quad \bar{H}_{-4} = 2R^3. \tag{50.5}$$

Using (50.1) to (50.5) in the general formulae (39) to (46) we obtain after simplification

$$A = 6\pi R^2, \tag{51}$$

$$X(\zeta) = -(C_1/16)\,R^2(3 + 4\zeta + \zeta^2), \tag{52}$$

$$Y(\zeta) = (C_3/4)\,R^2(3 + 4\zeta + \zeta^2), \tag{53}$$

$$\Phi(\zeta) = -(C_4/96)\,R^3(36\zeta + 42\zeta^2 + 20\zeta^3 + 3\zeta^4), \tag{54}$$

$$\Psi(\zeta) = (C_4/192)\,R^4(123 + 184\zeta + 86\zeta^2 + 28\zeta^3 + 3\zeta^4), \tag{55}$$

$$u_m = (17/24)\,C_1R^2, \tag{56}$$

$$T_m = (C_4/144)\,R^4(97 + 102\delta). \tag{57}$$

The velocity and temperature fields are given by

$$u = (C_1/4)\,R^2[\rho^4 + 4\rho^2 + 4\rho(\rho^2 - 1)\cos\theta - 5], \tag{58}$$

$$\begin{aligned} T = {} & (C_4/64)\,R^4D_0(D_0 - 16\delta) - \\ & - (C_4/96)\,R^4[(84\rho^4 + 144\rho^2 - 123) + \\ & + 8\rho(5\rho^4 + 30\rho^2 - 14)\cos\theta + \\ & + 2\rho^2(3\rho^4 + 40\rho^2 - 1)\cos 2\theta + \\ & + 12\rho^3(\rho^2 + 1)\cos 3\theta + 3\rho^4\cos 4\theta - \\ & - 4\delta(3 + 4\rho\cos\theta + \rho^2\cos 2\theta)]. \end{aligned} \tag{59}$$

The mixed mean temperature is given by

$$T_M = (C_4/32640)\,R^4(30503 + 31040\delta), \tag{60}$$

where

$$\delta = (C_3/C_4)\,R^2,\quad D_0 = (1 + 2\rho\cos\theta + \rho^2). \tag{60.1}$$

Cross-section a circular profile with two opposite flat sides (ovaloid form). In this case we use the mapping function

$$z = w(\zeta) = a(\zeta + t\zeta^3 + m\zeta^5 + n\zeta^7 + p\zeta^9 + q\zeta^{11}), \tag{61}$$

where $a > 0$, $t = .1206$, $m = -.0363$, $n = -.0227$, $p = .0118$, $q = .0107$.

From (61), (38), (47.1) to (47.8) we easily find

$$\begin{aligned}
&a_1 = a, \quad a_3 = .1206a, \quad a_5 = -.0363a, \quad a_7 = -.0227a,\\
&a_9 = .0118a, \quad a_{11} = .0107a, \quad b_0 = 1.0165a^2, \quad b_2 = .774a^2,\\
&b_4 = -.0396a^2, \quad b_6 = -.0216a^2, \quad b_8 = .0131a^2,\\
&b_{10} = .0107a^2, \quad c_2 = 0.774a^3, \quad c_4 = .2404a^3, \quad c_6 = -.064a^3,\\
&c_8 = .0002a^3, \quad c_{10} = .1036a^3, \quad c_{12} = .0918a^3,\\
&c_{14} = -.0109a^3, \quad c_{16} = -.0073a^3, \quad c_{18} = .0024a^3,\\
&c_{20} = .0011a^3, \quad f_1 = -1203a^4, \quad f_3 = -.1419a^4,\\
&f_5 = -.043a^4, \quad f_7 = .0995a^4, \quad f_9 = .0707a^4, \quad t_2 = .0489a^4,\\
&t_4 = -.0255a^4, \quad t_6 = -.0074a^4, \quad t_8 = .0103a^4,\\
&t_{10} = .0064a^4, \quad e_1 = .5082a^3, \quad e_3 = .3193a^3, \quad e_5 = .0295a^3,\\
&e_7 = -.0186a^3, \quad e_9 = .0061a^3, \quad e_{13} = .0071a^3,\\
&e_{11} = .0085a^3, \quad e_{15} = -.0007a^3, \quad e_{17} = -.0002a^3,\\
&e_{19} = .0001a^3, \quad e_{21} = .0001a^3, \quad d_0 = 1.055a^2,\\
&d_2 = .1468a^2, \quad d_4 = -.2062a^2, \quad d_6 = -.1504a^2,\\
&d_8 = .1204a^2, \quad d_{10} = .1177a^2, \quad d_{-2} = -.008a^2,\\
&d_{-4} = -.0478a^2, \quad d_{-6} = -.0188a^2. \qquad (62)
\end{aligned}$$

$$\begin{aligned}
&h_0 = 1.0399a^4, \quad h_2 = .1527a^4, \quad h_4 = -.108a^4,\\
&h_6 = .0523a^4, \quad h_8 = -.1425a^4, \quad h_{10} = .2549a^4,\\
&h_{12} = .0009a^4, \quad h_{14} = -.0015a^4, \quad h_{16} = -.0003a^4,\\
&h_{18} = .0003a^4, \quad h_{20} = .0001a^4, \quad t_0 = .545a^4, \quad t_2 = .3236a^4,\\
&t_4 = .0268a^4, \quad t_6 = -.0184a^4, \quad t_8 = .0141a^4,\\
&t_{10} = .0093a^4, \quad t_{12} = .007a^4, \quad t_{14} = -.0008a^4,\\
&t_{16} = -.0002a^4, \quad t_{18} = .0001a^4, \quad t_{20} = .0001a^4, \quad \text{etc.} \qquad (63)
\end{aligned}$$

From (39), (43), (40), (42) and (45) we easily obtain

$$A = 1.0555\pi a^2, \qquad (64)$$

$$u_m = -0.1496C_1a^2, \qquad (65)$$

$$X(\zeta) = -(C_4/16)\, a^2[.5083 + .774\zeta^2 - .0396\zeta^4 - .0216\zeta^6 + + .0131\zeta^8 + .0107\zeta^{10}], \qquad (66)$$

$$\Phi(\zeta) = -(C_4/16)\, a^3[.5082\zeta + .3193\zeta^3 + .0295\zeta^5 - .0186\zeta^7 + + 0061\zeta^9 + .0085\zeta^{11} + .0071\zeta^{13} - .0007\zeta^{15} - .0002\zeta^{17} + + .0001\zeta^{19} + .0001\zeta^{21}], \qquad (67)$$

$$\Psi(\zeta) = (C_4/64)\, a^4[1.66 + 1.337\zeta^2 + .1132\zeta^4 - .1555\zeta^6 + \\ + .2401\zeta^8 - .2121\zeta^{10} + .027\zeta^{12} - .0017\zeta^{14} + \\ - .0005\zeta^{16} + .0001\zeta^{18} + .0001\zeta^{20}]. \quad (68)$$

The expression for the velocity u, average temperature T_m and the mixed mean temperature T_M can be calculated.

Half-sections. Suppose we have a cross-section for which the x-axis is an axis of symmetry. This section can be mapped onto the unit circle in a t-plane by the transformation $z = z(t)$ in such a manner that the axis of symmetry of the section maps onto the diameter on the real axis in the t-plane. Then the same mapping formula maps the half-section in the z-plane on the semi-circle in the t-plane. This semi-circle can be mapped onto the unit circle in the ζ-plane by the mapping formula

$$\left(\frac{t-1}{t+1}\right)^2 = i\left(\frac{\zeta - 1}{\zeta + 1}\right) \quad (69)$$

and these two mappings map the half-section in the z-plane on the unit circle in the ζ-plane. Hence we can solve the problem for the half-section.

Received 12th February, 1963.

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