Int. J. Engng Sci., 1971, Vol. 9, pp. 1151–1156. Pergamon Press. Printed in Great Britain

KINETIC ENERGY OF INCOMPRESSIBLE MICROSTRETCH FLUID IN A DOMAIN BOUNDED BY RIGID WALLS

S. K. LAKSHMANA RAO
Department of Mathematics, Regional Engineering College, Warangal-4 (A.P.), India

Abstract—The basic physical quantities of microstretch flow are the velocity vector ($\bar{q}$), the microrotation vector ($\bar{\nu}$) and the microstretch (ν), the last quantity being a scalar field signifying the stretch or contraction experienced by the local fluid element. The kinetic energy T of the flow over a domain has contributions T_1, T_2, T_3 one from each of the above three quantities $\bar{q}$, $\bar{\nu}$, and ν. It is shown that Sgn $(dT/dt) = -1$ and that $T(t) \leq T(t_0) \exp[-\sigma(t-t_0)]$ for $0 < t_0 \leq t$. The (positive) number σ depends on the material constants of the flow and also on the geometry of the domain.

INTRODUCTION

THE kinetic energy of a viscous incompressible Navier–Stokes fluid in a domain D with fixed rigid boundary is known to decay and estimates for the rate of decay are given by Leray[1] for the case of plane flows and for spatial flows by Kampé de Fériet[2] as well as Berker[3]. The fact of the decay and estimates for the rate of decay of the kinetic energy are consequences of employing certain functional inequalities and the thermodynamic restriction on the kinematic viscosity coefficient. The motion as such is restricted only in a general way by the requirement of the validity of the Navier–Stokes differential equations of motion and enough smoothness of the functions to permit the use of Gauss's theorem rendering volume integrals into surface integrals.

Serrin[4] has examined the more general question of the stability of Navier–Stokes viscous flows and from his analysis it emerges that for the mean stability of flows in a bounded region of space the Reynolds number (suitably defined) must be less than 5·71.

Some years ago Eringen[5] developed the theory of simple micro-fluids in which the local effects arising from the local structure and intrinsic motions of the fluid elements are taken into account. In this theory the fluid element has the usual translatory degrees of freedom reckoned by the velocity vector $\bar{q}$ and has, in addition, degrees of freedom enabling the intrinsic rotatory motions as well as stretch of the fluid element. These latter are reckoned by three gyration vector fields $\bar{\nu}_K$. The linear model of this theory has 22 constants in its constitutive equations and these are subject to inequalities forced by thermodynamic considerations. Even this linear model is complicated for systematic theoretical investigations. A special case of the theory of simple microfluids is that of micropolar fluids[6] where the gyration tensor is presumed antisymmetric and the fluid elements can perform only rigid rotations without stretch. The linear model in this case has only six constants and the fluid motion is representable in terms of two vectors, viz. the velocity $\bar{q}$ and the microrotation $\bar{\nu}$. The six constants are subject to inequalities dictated by thermodynamics. It has been noticed by Lakshmana Rao[7] that the kinetic energy of an incompressible micropolar fluid occupying a spatial domain bound by rigid walls decays even as in the classical case of Navier–Stokes fluids and an estimate of the rate of decay has been determined. It is to be noted that here again, as in the classical case, these conclusions follow as long as the functions involved (like velocity, microrotation and fields derived from these) are smooth enough to allow transformation of volume integrals to surface integrals. The boundary condition needed for the above conclusions is the adherence or hyper-stick condition so that $\bar{q}$ and $\bar{\nu}$ vanish on rigid fixed boundaries. In a forthcoming paper [8] a general criterion has been noticed for the mean stability of incompressible micropolar fluids in a spatial domain, extending some of the results of Serrin [4] to the micropolar case.

This paper aims at similar study concerning the kinetic energy of incompressible *microstretch* fluids. The theory of microstretch fluids is put forward by Eringen [9] as a class more general than that of micropolar fluids [6] and here the local fluid elements can undergo stretch or contraction besides rotation. The basic physical quantities of the flow here are two vectors and a scalar viz. the velocity ($\bar{q}$), the microrotation ($\bar{\nu}$) and the microstretch (ν). The linear model here has ten constants which are subject to inequalities conforming to the principle of entropy. At rigid boundaries, all these field quantities satisfy the hyper-stick or adherence condition.

The field equations for the flow of incompressible microstretch fluids (cf. ref. [9]) are

$$\operatorname{div} \bar{q} = 0 \tag{1}$$

$$\frac{\partial j}{\partial t} + (\bar{q} \,.\, \operatorname{grad}) j - 2\nu j = 0 \tag{2}$$

$$\rho \left\{ \frac{\partial \bar{q}}{\partial t} - \bar{q} x \operatorname{curl} \bar{q} + \operatorname{grad}\left(\frac{1}{2} q^2\right)\right\} = \rho \bar{f} - \operatorname{grad} p + \lambda_0 \operatorname{grad} \nu + k \operatorname{curl} \bar{\nu}$$
$$- (\mu + k) \operatorname{curl}\operatorname{curl} \bar{q} + (\lambda + 2\mu + k) \operatorname{grad}(\operatorname{div} \bar{q}) \tag{3}$$

$$\rho j \left\{ \frac{\partial \bar{\nu}}{\partial t} + (\bar{q} \,.\, \operatorname{grad}) \bar{\nu} \right\} = \rho \bar{l} - 2k\bar{\nu} + k \operatorname{curl} \bar{q} - \gamma \operatorname{curl}\operatorname{curl} \bar{\nu} + (\alpha + \beta + \gamma) \operatorname{grad}(\operatorname{div} \bar{\nu}) \tag{4}$$

$$\frac{1}{2} \rho j \left\{ \frac{\partial \nu}{\partial t} + (q \,.\, \operatorname{grad}) \nu \right\} = \rho l + \alpha_0 \nabla^2 \nu - (\eta_0 - \lambda_0) \nu . \tag{5}$$

In the above equations ρ is the (real constant > 0) density of the fluid and j (real, > 0) denotes the gyration parameter, $\bar{f}$ and $\bar{l}$ are respectively the body force and body couple per unit mass and $3l$ is the trace of the first body moment per unit mass. The vectors $\bar{q}$, $\bar{\nu}$ are respectively the velocity and microrotation and the scalar ν denotes the microstretch of the fluid elements. The viscosity coefficients λ, μ, κ, λ_0, η_0 and the gyroviscosity coefficients α, β, γ, α_0 are constant and are subject to the following inequalities (cf. ref. [9]):

$$3\lambda + 2\mu + k \geqslant 0, 2\mu + k \geqslant 0, k \geqslant 0,$$
$$\eta_0 - \lambda_0 \geqslant 0, (\eta_0 - \lambda_0)(3\lambda + 2\mu + k) \geqslant \lambda_0^2/4,$$
$$3\alpha + \beta + \gamma \geqslant 0, \gamma \geqslant 0, |\beta| \leqslant \gamma . \tag{6}$$

We presume the body force $\bar{f}$ to be an irrotational vector and neglect the quantities $\bar{l}$ and l in the equations (4) and (5).

Let D be a spatial domain bounded by the rigid wall and occupied by an incompressible microstretch fluid and let T_1, T_2, T_3 be the contributions to the kinetic energy T of the fluid from the velocity $\bar{q}$, the microrotation $\bar{\nu}$ and the microstretch ν. We have then

$$T_1 = \tfrac{1}{2} \int \rho(\bar{q})^2, T_2 = \tfrac{1}{2} \int \rho j(\bar{\nu})^2,$$
$$T_3 = \tfrac{3}{4} \int \rho j \nu^2 \tag{7, 8, 9}$$

and

$$T = T_1 + T_2 + T_3. \tag{10}$$

(All the integrals are over the volume of the domain and the conventional volume element $d\tau$ is omitted.) The gyration parameter j is positive throughout the region of the flow and over the time interval $(0, t)$ while the microstretch can be of either sign in the flow region.

Let us introduce the quantities

$$J = \max(j), M = \max(|\nu|) \tag{11}$$

over the region and over the time interval $(0, t)$ and let d be the maximum diameter of a ball in which the domain D is embeddable. We have assumed the hyper-stick boundary condition, so that on the boundary of D

$$\bar{q} = 0, \bar{\nu} = 0, \nu = 0 \text{ for } t > 0. \tag{12}$$

If f is a continuously differentiable function over the domain D, we have the transport formula

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \rho f = \int \rho \frac{\mathrm{d}f}{\mathrm{d}t} \tag{13}$$

and in view of the continuity equation (1) and the hyper-stick boundary conditions (12), we see by an application of the divergence theorem that we may rewrite (13) also in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \rho f = \int \rho \frac{\partial f}{\partial t}. \tag{13'}$$

From (7) and (13′) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} T_1 = \int \rho \bar{q} \cdot \frac{\partial \bar{q}}{\partial t}. \tag{14}$$

Using the momentum equation (3) and employing Gauss's divergence theorem selectively and invoking the boundary conditions (12) we arrive at the result

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = k \int \bar{q} \cdot \operatorname{curl} \bar{\nu} - (\mu + k) \int (\operatorname{curl} \bar{q})^2. \tag{15}$$

From (8) and (13′) we have

$$\frac{dT_2}{dt}=\frac{1}{2}\int \rho\frac{\partial}{\partial t}[j(\bar{\nu})^2].$$

Using the equations (2) and (4) and by a selective use of Gauss's theorem and the boundary conditions (12) we can see that

$$\frac{dT_2}{dt}=-2k\int(\bar{\nu})^2+k\int\bar{\nu}\,.\,\mathrm{curl}\,\bar{q}-\gamma\int(\mathrm{curl}\,\bar{\nu})^2 -(\alpha+\beta+\gamma)\int(\mathrm{div}\,\bar{\nu})^2+\int\rho j\nu(\bar{\nu})^2. \quad (16)$$

From (9), (13′) and (5) we can deduce that

$$\frac{dT_3}{dt}=\frac{3}{4}\int\rho\frac{\partial}{\partial t}(j\nu^2) =-3\alpha_0\int(\mathrm{grad}\,\nu)^2-3(\eta_0-\lambda_0)\int\nu^2+\frac{3}{2}\int\rho j\nu^3. \quad (17)$$

Adding the equations (15), (16), (17) we see after some simplification on the right hand side that

$$\frac{dT}{dt}=-\frac{1}{2}(2\mu+k)\int(\mathrm{curl}\,\bar{q})^2-\frac{1}{2}k\int(\mathrm{curl}\,\bar{q}-2\bar{\nu})^2 -3(\eta_0-\lambda_0)\int\nu^2-\gamma\int(\mathrm{curl}\,\bar{\nu})^2-(\alpha+\beta+\gamma)\int(\mathrm{div}\,\bar{\nu})^2 -3\alpha_0\int(\mathrm{grad}\,\nu)^2+\int\rho j\nu(\bar{\nu})^2+\frac{3}{2}\int\rho j\nu^3. \quad (18)$$

In view of the inequalities (6) above, we see that the first six terms on the right side of (18) are negative and the expression made up of these six terms is thus negative definite. The last two integrals constitute an indefinite expression as the microstretch may be either an extension or a contraction and thus may not be of constant sign. Using the quantities J, M defined above in (11) we see easily from (8), (9) and (18) that

$$\frac{dT}{dt}\leqslant-\frac{1}{2}(2\mu+k)\int(\mathrm{curl}\,\bar{q})^2-3(\eta_0-\lambda_0)\int\nu^2 -\gamma\int(\mathrm{curl}\,\bar{\nu})^2-(\alpha+\beta+\gamma)\int(\mathrm{div}\,\bar{\nu})^2 -3\alpha_0\int(\mathrm{grad}\,\nu)^2+2M(T_2+T_3). \quad (19)$$

The following functional inequalities

$$\int(\mathrm{curl}\,\bar{q})^2\geqslant\frac{80}{d^2}\int(\bar{q})^2, \quad (20)$$

$$\int \left\{(\operatorname{div} \bar{\nu})^2 + (\operatorname{curl} \bar{\nu})^2 \geq \frac{3\pi^2}{d^2} \int (\bar{\nu})^2, \right. \tag{21}$$

$$\int (\operatorname{grad} \nu)^2 \geq \frac{3\pi^2}{d^2} \int \nu^2 \tag{22}$$

which are valid in the present problem (cf. refs. [4] and [10]) will be used below to majorize the right hand expression in (19). If $\epsilon = \min(\gamma, \alpha+\beta+\gamma)$ we can easily deduce from (7–9) and (19–22), the inequality

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq -\frac{80(2\mu+k)}{\rho d^2} T_1 + \left(2M - \frac{6\pi^2\epsilon}{\rho J d^2}\right) T_2$$
$$+ \{2M - [4(\eta_0 - \lambda_0)d^2 + 12\pi^2\alpha_0]/\rho J d^2\}\, T_3 . \tag{23}$$

Let

$$\min\{3\pi^2\epsilon/\rho J d^2, [2(\eta_0 - \lambda_0)d^2 + 6\pi^2\alpha_0]/\rho J d^2\} = m. \tag{24}$$

We see from (23) and (24) that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq -\frac{80(2\mu+k)}{\rho d^2} T_1 + 2(M-m)(T_2 + T_3). \tag{25}$$

Hence we have

$$\operatorname{Sgn}\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right) = -1 \tag{26}$$

for the class of incompressible microstretch flows in the domain for which

$$M = \max.|\nu| < m. \tag{27}$$

Thus the kinetic energy decreases with time in this case. We see that the decay is indeed faster than the exponential. If

$$\min\{80(2\mu+k)/\rho d^2, 2(m-M)\} = \sigma, \tag{28}$$

we have from (27) and (28) that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq -\sigma T \tag{29}$$

and now we can deduce that

$$T(t) \leq T(t_0) \exp[-\sigma(t-t_0)] \tag{30}$$

for $0 < t_0 \leq t$.

It is worthy of note that the decay constant σ depends not only on the geometry of the flow and the material constants of the fluid but also on the flow field in view of (27), (28). Whether the field variables $\bar{q}$, $\bar{w}$, ν themselves tend to zero as $t \to \infty$ is a question which needs to be examined.

REFERENCES

[1] J. LERAY, *J. Math. Pures. Appl.* (9) **13**, 331 (1934).
[2] J. KAMPÉ de FÉRIET, *Ann. Soc. Sci. Bruxelles* **63**, 35 (1949).
[3] R. BERKER, *Bull. Tech. Univ. Instanbul* **2**, 41 (1949).
[4] JAMES SERRIN, *Arch. ration. Mech. Analysis.* **3**, 1 (1959).
[5] A. C. ERINGEN, *Int. J. Engng Sci.* **2**, 205 (1964).
[6] A. C. ERINGEN, *J. Math. Mech.* **16**, 1 (1966).
[7] S. K. LAKSHMANA RAO, *Q. Appl. Math.* **27**, 278 (1969).
[8] S. K. LAKSHMANA RAO, *Int. J. Engng Sci.* **8**, 753 (1970).
[9] A. C. ERINGEN, *Int. J. Engng Sci.* **7**, 115 (1969).
[10] L. E. PAYNE and H. F. WEINBERGER, *Nonlinear Problems*, pp. 311–312. The University of Wisconsin Press (1963).

(*Received* 11 *March* 1971)

Résumé – Les grandeurs physiques fondamentales d'un écoulement à micro allongements sont le vecteur vitesse ($\bar{q}$), le vecteur de microrotation ($\bar{\nu}$) et le micro allongement (ν), cette dernière grandeur étant un champ scalaire représentant l'allongement ou la contraction subi par l'élément local de liquide. L'énergie cinétique T de l'écoulement sur un domaine a les contributions T_1, T_2, T_3 chacune provenant de chacune des trois grandeurs ci-dessus $\bar{q}$, $\bar{\nu}$ et ν. Il est montré que Sgn $(\mathrm{d}T/\mathrm{d}t) = -1$ et que $T(t) \leq T(t_0)$ exp. $[-\sigma(t-t_0)]$ pour $0 < t_0 \leq t$. Le nombre (positif) σ dépend les constantes matérielles de l'écoulement et également de la géométrie du domaine.

Zusammenfassung – Die grundlegenden physikalischen Grössen der Mikrodehnunngesströnung sind der Geschwindigkeitsvektor ($\bar{q}$), der Mikrorotationsvektor ($\bar{\nu}$) und die Mikrodehnung (ν), wobei die letzte Grösse ein Skalarfeld ist, das die Dehnung oder Zusammenziehung anzeigt, die das örtliche fluide Element erfärt. Die kinetische Energie T der Strömung über ein Bereich hat die Beiträge T_1, T_2, T_3, einen von jeder der oben genannten Grössen $\bar{q}$, $\bar{\nu}$ and ν. Es wird gezeigt, dass Sgn $(\mathrm{d}T/\mathrm{d}t) = -1$ und dass $T(t) \leq T(t_0)$. $[-\sigma(t-t_0)]$ für $0 < t_0 \leq t$ ausdrückt. Die (positive) Zahl σ hängt von den Materialkonstanten der Strömung und auch von der Geometrie des Bereiches ab.

Sommario – I quantitativi fisici fondamentali del flusso di microallungamento sono il vettore di velocità ($\bar{q}$), il vettore di microrotazione ($\bar{\nu}$) e il microallungamento (ν); l'ultimo quantitativo à un campo scalare che significa l'allungamento o la contrazione verificatisi dall'elemento fluido locale. L'energia cinetica T del flusso lungo un campo ha contributi T_1, T_2 e T_3, uno da ciascuno dei tre quantitativi di cui sopra $\bar{q}$, $\bar{\nu}$ e ν. Si dimostra che Sgn $(\mathrm{d}T/\mathrm{d}t) = -1$ e che $T(t) \leq T(t_0)$ esp $[-\sigma(t-t_0)]$ per $0 < t_0 \leq t$. Il numero (positivo) σ dipende dalle costanti materiali del flusso ed anche dalla geometria del campo.

Абстракт — Для основных физических величин для микрорастяжимого потока Эрингена мы имеем: вектор скорости движения ($\bar{q}$), вектор микровращения ($\bar{\nu}$), микрорасширение (ν), причем последняя величина есть скалярное поле, которое означает расширение или сокращение локального элемента жидкости. Кинетическая энергия T потока по обдасти получает вклады T_1, T_2, T_3 от каждой из этих трех величин, именно, $\bar{q}, \bar{\nu}, \nu$. Показано, что Sgn $\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right) = -1$, и что $T(t) \leq T(t_0)\exp[-\sigma(t-t_0)]$ при $0 < t_0 \leq t$. При этом (положительное) число σ зависит от материальных постоянных потока и от геометрии области.