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THE OSCILLATIONS OF A SPHERE IN A MICROPOLAR FLUID

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Abstract—The paper examines the rectilinear oscillation of a sphere along a dimeter and the rotary oscillation of a sphere about a diameter in Eringen's micropolar fluid. The physical quantities like the velocity, micro-rotation and the stress and couple stress components are calculated. The drag on the rectilinearly oscillating sphere and the couple on the rotational oscillating sphere are calculated. It is observed that over any period of oscillation, the maximum drag or the maximum couple, as the case may be, is larger in the case of micropolar fluids as compared to the Newtonian fluid.

1. INTRODUCTION

THE STUDY of micropolar fluids was put forward by A. C. Eringen in 1966[1]. These are a sub-class of microfluids earlier introduced by Eringen himself[2], which exhibit certain microscopic effects arising from the local structure and micromotions of the fluid elements. In the simplified theory of micropolar fluids also, as in the more complicated theory of microfluids, there is the possibility of sustenance of the couple stress. The mathematical model underlying micropolar fluids is described, by the field equations expressing the conservation of mass, the balance of the linear momentum and the balance of the first stress moments in terms of the two basic vectors $\bar{q}$, the velocity, and $\bar{\nu}$, the micro-rotation velocity. For incompressible fluids, which we consider in this paper, ρ, the density, is constant and the unknown pressure, p, in the momentum equation is to be determined from the boundary conditions.

In this paper we examine the unsteady flow of micropolar fluid (i) due to the oscillation of a sphere rectilinearly along the vertical diameter and (ii) due to the rotary oscillation of a sphere about a diameter. The magnitude of oscillation in both the cases is assumed so small that terms of second order in the amplitudes of oscillation can be neglected. The velocity and micro-rotation and the surface and the couple stress components are calculated in both the cases. In (i) the sphere experiences only a drag and does not receive any contribution from the couple stresses even though they are sustained in the fluid motion. In (ii) the sphere experiences only a couple. Both these quantities are calculated for several fixed values of the 'micropolarity coefficient', k/μ, allowing the frequency of the oscillating system to vary. The above quantities are conveniently expressed in terms of two parameters K and K', whose variations for different values of 'micropolarity coefficient', k/μ, as well as for different imposed frequencies of oscillations are numerically computed. It is seen that in both the drag and the couple in (i) and (ii) respectively, the parameters K and K' have the tendency to increase steadily with increase of the 'micropolarity coefficient', k/μ. Further this trend is observed over the entire range of the imposed frequency values considered in our numerical computations. In particular these parameters for the micropolar fluid are always larger than their values in the case of the (non-polar) Newtonian fluid.

2. FIELD EQUATIONS

The equations of motion for incompressible micropolar fluids are given by

$$\frac{\partial \rho}{\partial t}+\text{div}\,(\rho \bar{q})=0 \tag{2.1}$$

$$\rho \frac{\mathrm{d}\bar{q}}{\mathrm{d}t}=\rho \bar{f}-\text{grad}\,p+k\,\text{curl}\,\bar{\nu}-(\mu+k)\,\text{curl curl}\,\bar{q}+(\lambda+2\mu+k)\,\text{grad div}\,\bar{q} \tag{2.2}$$

$$\rho j\frac{\mathrm{d}\bar{\nu}}{\mathrm{d}t}=\rho \bar{l}-2k\bar{\nu}+k\,\text{curl}\,\bar{q}-\gamma\,\text{curl curl}\,\bar{\nu}+(\alpha+\beta+\gamma)\,\text{grad div}\,\bar{\nu}, \tag{2.3}$$

in which $\bar{q}$, $\bar{\nu}$ are respectively the velocity and micro-rotation vectors. The symbols ρ and j denote the density and micro-inertia and the coefficients λ, μ, k as well as α, β, γ are the viscosity parameters, which are regarded as constants in this investigation. The quantities p, $\bar{f}$ and $\bar{l}$ denote the pressure, body force per unit mass and body couple per unit mass respectively.

The shear stress t_{ij} and the couple stress m_{ij} are defined by the constitutive equations

$$t_{ij}=(-p+\lambda\,\text{div}\,\bar{q})\delta_{ij}+(2\mu+k)e_{ij}+k\epsilon_{ijm}(\omega_m-\nu_m) \tag{2.4}$$

$$m_{ij}=\alpha(\text{div}\,\bar{\nu})\delta_{ij}+\beta\nu_{i,j}+\gamma\nu_{j,i} \tag{2.5}$$

where e_{ij} is the strain velocity tensor and $\bar{\omega}$ is (1/2) curl $\bar{q}$. δ_{ij} and ϵ_{ijm} have their standard meanings.

PART A

3. RECTILINEAR OSCILLATIONS

A sphere of radius 'a' oscillates rectilinearly along the vertical diameter, $\theta=0$, in a spherical polar coordinate system (r, θ, ϕ). Let the velocity of the sphere be $U_0 \cos \sigma t$. We may choose the vectors $\bar{q}$ and $\bar{\nu}$ in the form

$$\bar{q}=u(r,\theta,t)\bar{e}_r+v(r,\theta,t)\bar{e}_\theta \tag{3.1}$$

and

$$\bar{\nu}=C(r,\theta,t)\bar{e}_\phi. \tag{3.2}$$

In equations (3.1) and (3.2) and in all subsequent equations, only, the real parts are to be taken, when the physical quantities are represented by the complex quantities.

Since div $\bar{q}=0$, we can write

$$u=-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\ v=\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{3.3}$$

The equations of motion now take the form

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}-\frac{v^2}{r}\right)=\rho f_r-\frac{\partial p}{\partial r}+\frac{k}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta C)-\frac{(\mu+k)}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi) \tag{3.4}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}\right)=\rho f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}-\frac{k}{r}\frac{\partial}{\partial r}(rC)+\frac{(\mu+k)}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi) \tag{3.5}$$

$$\rho j\left(\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial r}+\frac{v}{r}\frac{\partial C}{\partial \theta}\right)=\rho l_\phi-2kC+\frac{k}{r\sin\theta}E^2\psi+\gamma DC, \tag{3.6}$$

where

$$E^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} \tag{3.7}$$

and

$$D\equiv\nabla^2-\frac{1}{r^2\sin^2\theta}\equiv\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}-\frac{\operatorname{cosec}^2\theta}{r^2}. \tag{3.8}$$

Assuming that the amplitude of oscillation U_0 is small and neglecting the inertial terms in (3.4) and (3.5) and the bilinear terms and the external couple in (3.6), we have the linearized system of equations

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial}{\partial r}(p-\rho rg\cos\theta)+\frac{k}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta C)-\frac{(\mu+k)}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi) \tag{3.9}$$

$$\rho\frac{\partial v}{\partial t}=-\frac{1}{r}\frac{\partial}{\partial\theta}(p-\rho rg\cos\theta)-\frac{k}{r}\frac{\partial}{\partial r}(rC)+\frac{(\mu+k)}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi) \tag{3.10}$$

$$\rho j\frac{\partial C}{\partial t}=-2kC+\frac{k}{r\sin\theta}E^2\psi+\gamma DC. \tag{3.11}$$

On the boundary $r=a$, we have the adherence condition which can be expressed in the form

$$u=U_0\,e^{i\sigma t}\cos\theta \tag{3.12}$$

$$v=-U_0\,e^{i\sigma t}\sin\theta \tag{3.13}$$

$$C=0. \tag{3.14}$$

Eliminating the pressure term from (3.9) and (3.10), we obtain the equation

$$\rho\frac{\partial}{\partial t}E^2\psi-(\mu+k)E^2E^2\psi=-kr\sin\theta DC. \tag{3.15}$$

In view of the boundary conditions (3.12), (3.13) we choose

$$\psi=f(r)\,e^{i\sigma t}\sin^2\theta \tag{3.16}$$

and hence take

$$C=g(r)\,e^{i\sigma t}\sin\theta. \tag{3.17}$$

On substituting these in the equations (3.15) and (3.11), we get

$$\left\{(\mu+k)\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)^2 - i\rho\sigma\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\right\}f(r) = kr\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2}{r^2}\right)g(r) \tag{3.18}$$

$$\frac{k}{r}\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)f(r) = \left\{-\gamma\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2}{r^2}\right)+(\rho j i\sigma+2k)\right\}g(r). \tag{3.19}$$

If L denotes the operator $(d^2/dr^2)-(2/r^2)$, we see that the functions $f(r)$ and $h(r) = rg(r)$ are governed by the system of differential equations

$$\{(\mu+k)L^2 - i\rho\sigma L\}f(r) = kL(h(r)) \tag{3.20}$$

and

$$kL(f(r)) = \{-\gamma L + (\rho j i\sigma + 2k)\}h(r). \tag{3.21}$$

Hence from (3.20), we obtain

$$\{(\mu+k)L - i\rho\sigma\}f(r) - kh(r) = A_1 r^2 + \frac{B_1}{r} \tag{3.22}$$

where A_1 and B_1 are arbitrary (complex) constants. Eliminating the function $h(r)$ from (3.21) and (3.22), we get the following equation for the function $f(r)$.

$$\left\{\frac{\gamma(\mu+k)}{k}L^2 - \left[(2\mu+k) + i\rho\sigma\left(j+\frac{\mu j}{k}+\frac{\gamma}{k}\right)\right]L + \left(-\frac{j\rho^2\sigma^2}{k}+i2\rho\sigma\right)\right\}f(r)$$
$$= -\frac{2k+i\rho j\sigma}{k}\left(A_1 r^2 + \frac{B_1}{r}\right). \tag{3.23}$$

Let α_1, β_1, be the complex numbers defined by means of the equations

$$\alpha_1^2+\beta_1^2 = \frac{k(2\mu+k)}{\gamma(\mu+k)} + \frac{i\rho\sigma(jk+j\mu+\gamma)}{\gamma(\mu+k)} \tag{3.24}$$

$$\alpha_1^2\beta_1^2 = -\frac{j\rho^2\sigma^2}{\gamma(\mu+k)} + i\frac{(2k\rho\sigma)}{\gamma(\mu+k)}. \tag{3.25}$$

We can see that neither of the constants α_1, β_1 can be purely imaginary. We choose them such that their real parts are positive. The solution of (3.23) is found to be

$$f(r) = C_1\left(\frac{1}{r}+\alpha_1\right)e^{-\alpha_1 r} + C_2\left(\frac{1}{r}+\beta_1\right)e^{-\beta_1 r}$$
$$+ C_3\left(\frac{1}{r}-\alpha_1\right)e^{\alpha_1 r} + C_4\left(\frac{1}{r}-\beta_1\right)e^{\beta_1 r} + \frac{iA_1}{\rho\sigma}r^2 + \frac{iB_1}{\rho\sigma}\frac{1}{r} \tag{3.26}$$

where C_1 to C_4 are a further batch of arbitrary (complex) constants. Since the sphere has only a small amplitude of oscillation, the velocity components, as well as the micro-rotation have to vanish as $r \to \infty$. For this reason we choose the constants C_3, C_4 and A_1 in (3.26) to be zero. Then the term A_1 in equation (3.22) also is to be dropped. We now have

$$f(r) = C_1\left(\frac{1}{r}+\alpha_1\right) e^{-\beta_1 r} + C_2\left(\frac{1}{r}+\beta_1\right) e^{-\beta_1 r} + \frac{iB_1}{\rho\sigma}\frac{1}{r} \tag{3.27}$$

and from equation (3.22), we see that

$$h(r) = \left(\frac{\mu+k}{k}L - \frac{i\rho\sigma}{k}\right)f(r) - \frac{B_1}{kr}$$

$$= \frac{(\mu+k)\alpha_1^2 - i\rho\sigma}{k} C_1\left(\frac{1}{r}+\alpha_1\right)e^{-\alpha_1 r} + \frac{(\mu+k)\beta_1^2 - i\rho\sigma}{k} C_2\left(\frac{1}{r}+\beta_1\right) e^{-\beta_1 r}. \tag{3.28}$$

The velocity and micro-rotation are expressible in the form

$$u = -\frac{2f(r)}{r^2} e^{i\sigma t}\cos\theta \tag{3.29}$$

$$v = \frac{f'(r)}{r} e^{i\sigma t}\sin\theta \tag{3.30}$$

$$C = \frac{h(r)}{r} e^{i\sigma t}\sin\theta \tag{3.31}$$

and from the boundary condition

$$f(a) = -\tfrac{1}{2}U_0 a^2 \tag{3.32}$$

$$f'(a) = -U_0 A \tag{3.33}$$

$$h(a) = 0 \tag{3.34}$$

the constants C_1, C_2 and B_1 in $f(r)$ and $h(r)$ are determined. They are found to be

$$C_1 = \frac{3U_0 a(1+a\beta_1)[(\mu+k)\beta_1^2 - i\rho\sigma]\, e^{\alpha_1 a}}{2(\beta_1-\alpha_1)[a(\mu+k)\alpha_1^2\beta_1^2 + i\rho\sigma(a\alpha_1\beta_1+\alpha_1+\beta_1)]} \tag{3.35}$$

$$C_2 = \frac{-3U_0 a(1+a\alpha_1)[(\mu+k)\alpha_1^2 - i\rho\sigma]\, e^{\beta_1 a}}{2(\beta_1-\alpha_1)[a(\mu+k)\alpha_1^2\beta_1^2 + i\rho\sigma(a\alpha_1\beta_1+\alpha_1+\beta_1)]} \tag{3.36}$$

$$B_1 = \frac{i\rho\sigma U_0 a}{2}\left\{a^2 + \frac{3(\mu+k)(1+a\alpha_1)(1+a\beta_1)(\alpha_1+\beta_1)}{a(\mu+k)\alpha_1^2\beta_1^2 + i\rho\sigma(a\alpha_1\beta_1+\alpha_1+\beta_1)}\right\}. \tag{3.37}$$

The graphs 1, 2 and 3 show the variation of the absolute values of the two velocity components and the single micro-rotation component with the distance from the sphere.

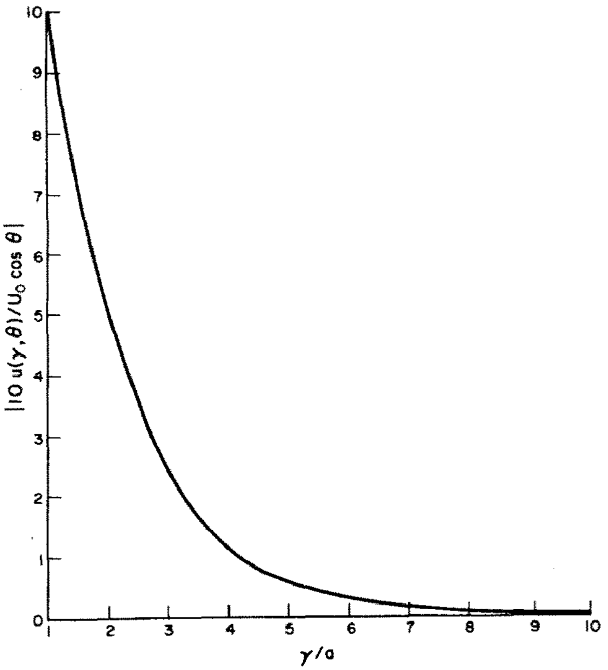

|10 u(γ,θ)/U₀ cos θ|
0
1
2
3
4
5
6
7
8
9
10
γ/a

Fig. 1.

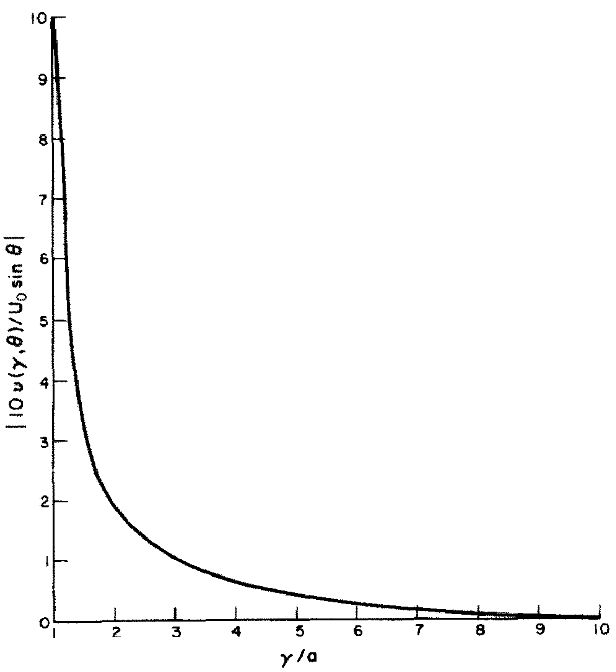

|10 v(γ,θ)/U₀ sin θ|
0
1
2
3
4
5
6
7
8
9
10
γ/a

Fig. 2.

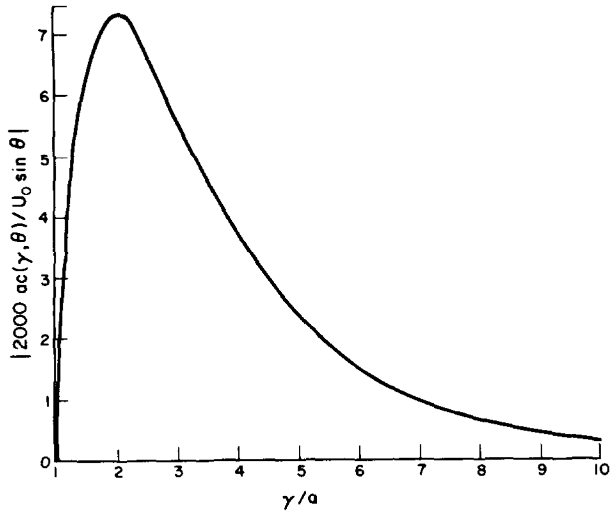


Fig. 3.

4. STRESS COMPONENTS

We find that the strain components are

$$e_{rr} = -2e_{\theta\theta} = -2e_{\phi\phi} = \left\{\frac{6}{r^3}f(r) + \frac{2}{r^2}\left(C_1\alpha_1^2 e^{-\alpha_1 r} + C_2\beta_1^2 e^{-\beta_1 r}\right)\right\}\cos\theta\, e^{i\sigma t} \tag{4.1}$$

$$e_{r\theta} = \left\{\frac{1}{2}C_1\left(\frac{6}{r^4} + \frac{6\alpha_1}{r^3} + \frac{3\alpha_1^2}{r^2} + \frac{\alpha_1^3}{r}\right)e^{-\alpha_1 r} + \frac{1}{2}C_2\left(\frac{6}{r^4} + \frac{6\beta_1}{r^3} + \frac{3\beta_1^2}{r^2} + \frac{\beta_1^3}{r}\right)e^{-\beta_1 r} + i\frac{3B_1}{\rho\sigma r^4}\right\}\sin\theta\, e^{i\sigma t} \tag{4.2}$$

and

$$e_{r\phi} = e_{\theta\phi} = 0. \tag{4.3}$$

The vorticity vector, $\bar{\omega}$ is

$$\bar{\omega} = \frac{1}{2r}Lf(r)\, e^{i\sigma t}\sin\theta\, \bar{e}_\phi$$

$$= \frac{1}{2r}\left\{C_1\alpha_1^2\left(\frac{1}{r} + \alpha_1\right)e^{-\alpha_1 r} + C_2\beta_1^2\left(\frac{1}{r} + \beta_1\right)e^{-\beta_1 r}\right\}\sin\theta\, e^{i\sigma t}\bar{e}_\phi. \tag{4.4}$$

The shear stress components t_{ij} are now calculated from the constitutive equation (2.4). We find that

$$t_{rr} = -p' + 2\frac{(2\mu + k)}{r^2}\left\{\frac{3}{r}f(r) + C_1\alpha_1^2 e^{-\alpha_1 r} + C_2\beta_1^2 e^{-\beta_1 r}\right\}\cos\theta\, e^{i\sigma t} \tag{4.5}$$

$$t_{\theta\theta} = t_{\phi\phi} = -p' - \frac{(2\mu + k)}{r^2}\left\{\frac{3}{r}f(r) + C_1\alpha_1^2 e^{-\alpha_1 r} + C_2\beta_1^2 e^{-\beta_1 r}\right\}\cos\theta\, e^{i\sigma t} \tag{4.6}$$

$$t_{r\theta}=\Big[(2\mu+k)\Big\{\Big(\frac{3}{r^4}+\frac{3\alpha_1}{r^3}+\frac{\alpha_1^2}{r^2}\Big)C_1\,e^{-\alpha_1 r}+\Big(\frac{3}{r^4}+\frac{3\beta_1}{r^3}+\frac{\beta_1^2}{r^2}\Big)C_2\,e^{-\beta_1 r}+i\frac{3B_1}{\rho\sigma}\frac{1}{r^4}\Big\}+i\rho\sigma\Big\{C_1\Big(\frac{1}{r^2}+\frac{\alpha_1}{r}\Big)\,e^{-\alpha_1 r}+C_2\Big(\frac{1}{r^2}+\frac{\beta_1}{r}\Big)\,e^{-\beta_1 r}\Big\}\Big]\sin\theta\,e^{i\sigma t} \quad (4.7)$$

$$t_{\theta r}=\Big[(2\mu+k)\Big\{\Big(\frac{3}{r^4}+\frac{3\alpha_1}{r^3}+\frac{2\alpha_1^2}{r^2}+\frac{\alpha_1^3}{r}\Big)C_1\,e^{-\alpha_1 r}+\Big(\frac{3}{r^4}+\frac{3\beta_1}{r^3}+\frac{2\beta_1^2}{r^2}+\frac{\beta_1^3}{r}\Big)C_2\,e^{-\beta_1 r}+i\frac{3B_1}{\rho\sigma}\frac{1}{r^4}\Big\}+i\rho\sigma\Big\{C_1\Big(\frac{1}{r^2}+\frac{\alpha_1}{r}\Big)\,e^{-\alpha_1 r}+C_2\Big(\frac{1}{r^2}+\frac{\beta_1}{r}\Big)\,e^{-\beta_1 r}\Big\}\Big]\sin\theta\,e^{i\sigma t} \quad (4.8)$$

and

$$t_{\theta\phi}=t_{\phi\theta}=t_{r\phi}=t_{\phi r}=0, \quad (4.9)$$

where

$$p'=p-\rho rg\cos\theta.$$

From equations (3.9) and (3.10), we get after an integration

$$p'=B_1r^{-2}\cos\theta\,e^{i\sigma t}+\text{const.} \quad (4.10)$$

5. COUPLE STRESS

The couple stress components m_{ij} are obtained from equation (12.5). We find that

$$m_{rr}=m_{r\theta}=m_{\theta r}=m_{\theta\theta}=m_{\phi\phi}=0 \quad (5.1)$$

while

$$m_{r\phi}=-\Big\{\frac{(\beta+2\gamma)}{kr^2}\Big[\Big\{(\mu+k)\alpha_1^2-i\rho\sigma\Big\}C_1\Big(\alpha_1+\frac{1}{r}\Big)\,e^{-\alpha_1 r}+\Big\{(\mu+k)\beta_1^2-i\rho\sigma\Big\}C_2\Big(\beta_1+\frac{1}{r}\Big)\,e^{-\beta_1 r}\Big]+\frac{\gamma}{kr}[\{(\mu+k)\alpha_1^2-i\rho\sigma\}C_1\alpha_1^2\,e^{-\alpha_1 r}+\{(\mu+k)\beta_1^2-i\rho\sigma\}C_2\beta_1^2\,e^{-\beta_1 r}]\Big\}\sin\theta\,e^{i\sigma t} \quad (5.2)$$

$$m_{\phi r}=-\Big\{\frac{(\gamma+2\beta)}{kr^2}\Big[\{(\mu+k)\alpha_1^2-i\rho\sigma\}C_1\Big(\alpha_1+\frac{1}{r}\Big)\,e^{-\alpha_1 r}+\{(\mu+k)\beta_1^2-i\rho\sigma\}C_2\Big(\beta_1+\frac{1}{r}\Big)\,e^{-\beta_1 r}\Big]+\frac{\beta}{kr}[\{(\mu+k)\alpha_1^2-i\rho\sigma\}C_1\alpha_1^2\,e^{-\alpha_1 r}+\{(\mu+k)\beta_1^2-i\rho\sigma\}C_2\beta_1^2\,e^{-\beta_1 r}]\Big\}\sin\theta\,e^{i\sigma t} \quad (5.3)$$

and

$$m_{\theta\phi} = -m_{\phi\theta} = \frac{(\gamma-\beta)}{kr^2}\Big[\{(\mu+k)\alpha_1^2 - i\rho\sigma\}C_1\Big(\frac{1}{r}+\alpha_1\Big)\,\mathrm{e}^{-\alpha_1 r}$$

$$+\{(\mu+k)\beta_1^2 - i\rho\sigma\}C_2\Big(\frac{1}{r}+\beta_1\Big)\,\mathrm{e}^{-\beta_1 r}\Big]\cos\theta\,\mathrm{e}^{i\sigma t}. \quad (5.4)$$

6. DRAG

The drag on the sphere due to surface stresses is given by

$$D_S = 2\pi a^2 \int_0^\pi [t_{rr}\cos\theta - t_{r\theta}\sin\theta]_{r=a}\sin\theta\,\mathrm{d}\theta \quad (6.1)$$

and this is found to be

$$D_S = \frac{4}{3}\pi a^2\Big(i\rho\sigma a U_0 - \frac{3B_1}{a^2}\Big)\,\mathrm{e}^{i\sigma t}$$

$$= \Big\{-\frac{2}{3}\pi i\rho\sigma a^3 U_0 - \frac{6\pi i\rho\sigma a U_0(\mu+k)(1+a\alpha_1)(1+a\beta_1)(\alpha_1+\beta_1)}{a(\mu+k)\alpha_1^2\beta_1^2 + i\rho\sigma(a\alpha_1\beta_1+\alpha_1+\beta_1)}\Big\}\,\mathrm{e}^{i\sigma t}. \quad (6.2)$$

The contribution of the surface stresses to the couple on the body (if any) is given by

$$\int_0^\pi\int_0^{2\pi} a\bar{e}_r X[t_{rr}\bar{e}_r + t_{r\theta}\bar{e}_\theta]_{r=a} a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = a^3\int_0^\pi\Big\{\int_0^{2\pi}[t_{r\theta}]_{r=a}\bar{e}_\phi\,\mathrm{d}\phi\Big\}\,\mathrm{d}\theta \quad (6.3)$$

and this is found to be zero.

The couple vector on the sphere $r = a$ arising from the couple stresses is

$$m_{rr}\bar{e}_r + m_{r\theta}\bar{e}_\theta + m_{r\phi}\bar{e}_\phi \quad (6.4)$$

and this reduces to

$$\frac{\gamma h'(a)}{a}\sin\theta\,\mathrm{e}^{i\sigma t}\bar{e}_\phi. \quad (6.5)$$

Hence the couple on the sphere due to the couple stresses in the fluid equals

$$\int_0^\pi\Big\{\int_0^{2\pi}\gamma h'(a)a\sin\theta\,\mathrm{e}^{i\sigma t}\bar{e}_\phi\,\mathrm{d}\phi\Big\}\,\mathrm{d}\theta \quad (6.6)$$

and this is zero.

Thus the sphere experiences only a drag in the direction of the oscillation given above in (6.2) and no couple acts on the body though the fluid sustains a couple stress distribution.

The case of the (non-polar) Newtonian viscous liquid with the single viscosity coefficient, μ, is recoverable from the above analysis by letting the parameters k and γ to zero and carrying out the appropriate limiting process. Then in the equations

(3.24) and (3.25) one 'root' (say, α_1^2) becomes infinity and the finite root is given by

$$\beta^2 = \operatorname*{Lt}_{|\alpha_1|\to\infty} \frac{\alpha_1^2\beta_1^2}{\alpha_1^2+\beta_1^2} = \frac{i\rho\sigma}{\mu}. \tag{6.7}$$

The expression for the drag then becomes

$$D_N = \left\{-\frac{2}{3}\pi i\rho\sigma a^3 U_0 - \frac{6\pi i\rho\sigma a U_0(1+a\beta_1)}{\beta_1^2}\right\} e^{i\sigma t} \tag{6.8}$$

and in view of (6.7), this becomes

$$D_N = \left[-\frac{2}{3}\pi i\rho\sigma a^3 U_0 - 6\pi\mu U_0 a\left\{1+a\sqrt{\left(\frac{i\rho\sigma}{\mu}\right)}\right\}\right] e^{i\sigma t}. \tag{6.9}$$

If M denotes the mass of the fluid displaced by the sphere, the drag, D, on it as given in (6.2) can also be put in the form

$$D = MU_0\sigma(-K'-iK)\, e^{i\sigma t} \tag{6.10}$$

and the real part of this expression is seen to be

$$D_R = MU_0\sigma(K \sin \sigma t - K' \cos \sigma t). \tag{6.11}$$

The graphs 4 and 5 show the distribution of the parameters K and K' over a range of the imposed frequencies. It is clear from these graphs that the parameters K and K' increase steadily with increase of k/μ and the curves for the polar case ($k/\mu > 0$) are naturally above the curve for the (non-polar) Newtonian case, as far as is evidenced from the numerical values noticed by us. In particular it follows therefore that $\sqrt{K^2+K'^2}$ is larger in the micropolar flow than in the Newtonian flow. Hence the maximum drag experienced by the sphere over any period of oscillation is larger in the case of micropolar fluids as compared to the Newtonian fluids. Also the larger the 'micropolarity coefficient', k/μ, the greater is this maximum drag value.

Analysis, analogous to the above, has been worked out for a three constant model of Oldroyd elastico viscous fluids by K. R. Frater[3].

PART B

7. ROTARY OSCILLATION

A sphere performs slow oscillations about a diameter, which is chosen along the axis $\theta = 0$ of the coordinate system (r, θ, ϕ). The motion is then represented by the velocity,

$$\bar{q} = \frac{\Omega(r, \theta, t)}{r \sin\theta}\,\bar{e}_\phi \tag{7.1}$$

and the micro-rotation vector $\bar{\nu}$ will now have the form

$$\bar{\nu} = A(r, \theta, t)\bar{e}_r + B(r, \theta, t)\bar{e}_\theta. \tag{7.2}$$

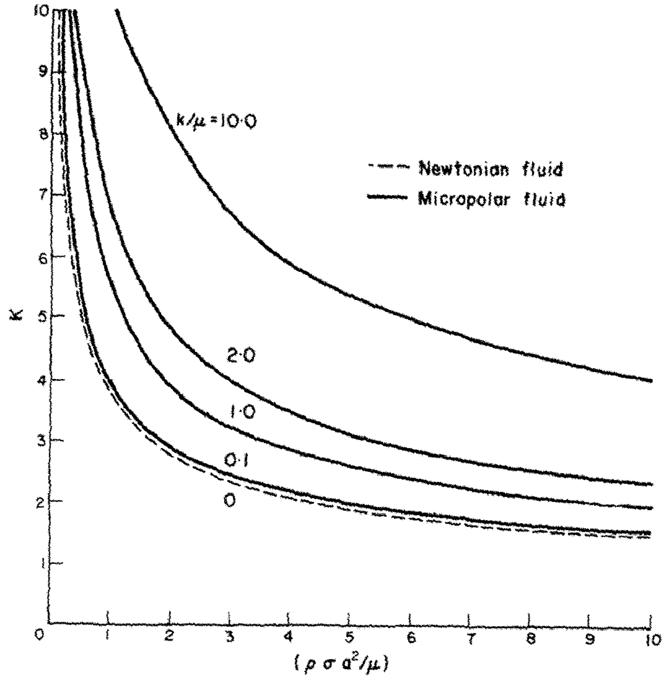

k/μ = 10·0
2·0
1·0
0·1
0
Newtonian fluid
Micropolar fluid
K
(ρ σ a²/μ)

Fig. 4.

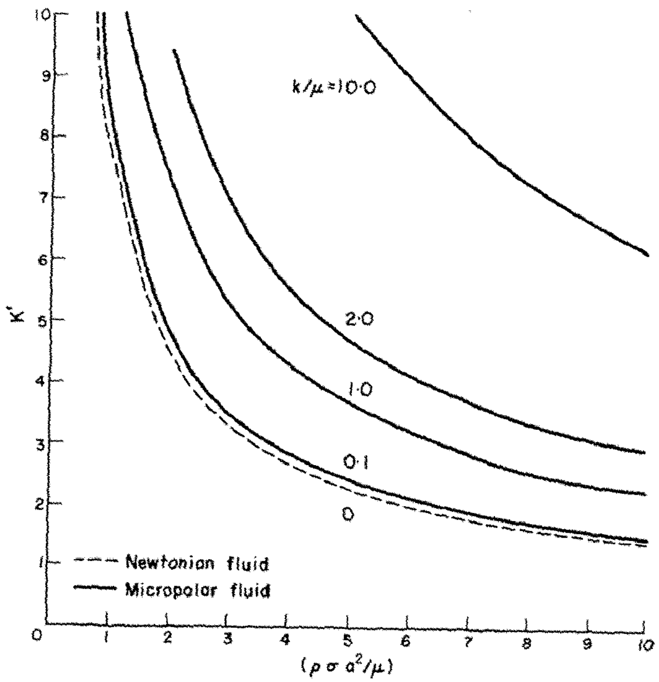

k/μ = 10·0
2·0
1·0
0·1
0
Newtonian fluid
Micropolar fluid
K′
(ρ σ a²/μ)

Fig. 5.

The problem has axial symmetry about the axis of rotation and the partial derivatives with respect to ϕ do not appear in this analysis. Since the oscillations are presumed to be slow, we neglect the nonlinear terms on the left hand side of the equations (2.2) and (2.3). The body force and the body couple are also neglected. The problem is then governed by the following equations,

$$\rho\frac{\partial\Omega}{\partial t} = kr\sin\theta\left(\frac{\partial B}{\partial r}+\frac{B}{r}-\frac{1}{r}\frac{\partial A}{\partial\theta}\right)+(\mu+k)E^2\Omega \tag{7.3}$$

$$\rho j\frac{\partial A}{\partial t} = -2kA+\frac{k}{r^2\sin\theta}\frac{\partial\Omega}{\partial\theta}-\frac{\gamma}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[r\sin\theta\left(\frac{\partial B}{\partial r}+\frac{B}{r}-\frac{1}{r}\frac{\partial A}{\partial\theta}\right)\right]+(\alpha+\beta+\gamma)\frac{\partial f}{\partial r}. \tag{7.4}$$

$$\rho j\frac{\partial B}{\partial t} = -2kB-\frac{k}{r\sin\theta}\frac{\partial\Omega}{\partial r}+\frac{\gamma}{r\sin\theta}\frac{\partial}{\partial r}\left[r\sin\theta\left(\frac{\partial B}{\partial r}+\frac{B}{r}-\frac{1}{r}\frac{\partial A}{\partial\theta}\right)\right]+(\alpha+\beta+\gamma)\frac{1}{r}\frac{\partial f}{\partial\theta}. \tag{7.5}$$

In the above equation

$$f=f(r,\theta,t)=\operatorname{div}\bar{v} \tag{7.6}$$

and

$$E^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}. \tag{7.7}$$

On the boundary $r=a$, we have the adherence condition and this can be taken in the form

$$\Omega(a,\theta,t) = \Omega_0 a^2\sin^2\theta\, e^{i\sigma t} \tag{7.8}$$

$$A(a,\theta,t) = 0 \tag{7.9}$$

$$B(a,\theta,t) = 0. \tag{7.10}$$

At $r=\infty$, we have the conditions

$$\Omega = A = B = 0.$$

Let

$$h(r,\theta,t) = r\sin\theta\left(\frac{\partial B}{\partial r}+\frac{B}{r}-\frac{1}{r}\frac{\partial A}{\partial\theta}\right). \tag{7.11}$$

The equations (7.3), (7.4) and (7.5) can be put in the form

$$\rho\frac{\partial\Omega}{\partial t} = kh+(\mu+k)E^2\Omega \tag{7.12}$$

$$\rho j\frac{\partial A}{\partial t} = -2kA+\frac{k}{r^2\sin\theta}\frac{\partial\Omega}{\partial\theta}-\frac{\gamma}{r^2\sin\theta}\frac{\partial h}{\partial\theta}+(\alpha+\beta+\gamma)\frac{\partial f}{\partial r} \tag{7.13}$$

$$\rho j \frac{\partial B}{\partial t} = -2kB - \frac{k}{r\sin\theta}\frac{\partial \Omega}{\partial r} + \frac{\gamma}{r\sin\theta}\frac{\partial h}{\partial r} + (\alpha+\beta+\gamma)\frac{1}{r}\frac{\partial f}{\partial \theta}. \tag{7.14}$$

From the last two equations, we see that

$$\left(\rho j \frac{\partial}{\partial t} + 2k\right) f = (\alpha+\beta+\gamma)\nabla^2 f \tag{7.15}$$

where ∇^2 denotes the Laplacian operator in spherical polar coordinates and also

$$\left(\rho j \frac{\partial}{\partial t} + 2k\right) h = \gamma E^2 h - kE^2 \Omega. \tag{7.16}$$

From the equations (7.15), (7.16) and (7.12) we see that

$$\left\{E^4 - \frac{k(2\mu+k)}{\gamma(\mu+k)} E^2 - \frac{\rho(\gamma+j\mu+jk)}{\gamma(\mu+k)}\frac{\partial}{\partial t} E^2 + \frac{2k\rho}{\gamma(\mu+k)}\frac{\partial}{\partial t} + \frac{\rho^2 j}{\gamma(\mu+k)}\frac{\partial^2}{\partial t^2}\right\}\Omega = 0. \tag{7.17}$$

In view of the time-dependence in boundary-condition (7.8) we have to obtain the solution Ω in the form

$$\Omega = \Omega(r, \theta)\, e^{i\sigma t} \tag{7.18}$$

and so the quantities f, A, B, h must also be chosen in the form

$$f = f(r, \theta)\, e^{i\sigma t} \tag{7.19}$$

$$A = A(r, \theta)\, e^{i\sigma t} \tag{7.20}$$

$$B = B(r, \theta)\, e^{i\sigma t} \tag{7.21}$$

$$h = h(r, \theta)\, e^{i\sigma t}. \tag{7.22}$$

The equation (7.17) will now become

$$\left[E^4 - \left\{\frac{k(2\mu+k) + i\rho\sigma(\gamma+j\mu+jk)}{\gamma(\mu+k)}\right\} E^2 + \frac{\rho\sigma(2ik - j\rho\sigma)}{\gamma(\mu+k)}\right]\Omega = 0. \tag{7.23}$$

In view of the boundary condition (7.18) we may take

$$\Omega(r, \theta) = F(r)\sin^2\theta \tag{7.24}$$

and then

$$f(r, \theta) = g(r)\cos\theta. \tag{7.25}$$

The function $F(r)$ is then to be determined from the equation

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}-\alpha_1^2\right)\left(\frac{d^2}{dr^2}-\frac{2}{r^2}-\beta_1^2\right)F(r)=0 \tag{7.26}$$

where the complex constants α_1, β_1 are given by equations (3.24) and (3.25).

The solution of (7.26) is seen to be

$$F(r)=A_1\left(\frac{1}{r}+\alpha_1\right)e^{-\alpha_1 r}+B_1\left(\frac{1}{r}+\beta_1\right)e^{-\beta_1 r}+C_1\left(\frac{1}{r}-\alpha_1\right)e^{\alpha_1 r}+D_1\left(\frac{1}{r}-\beta_1\right)e^{\beta_1 r} \tag{7.27}$$

and the function $f(r,\theta)$ is found from (7.15) to be

$$f(r,\theta)=\frac{1}{r}\left\{E_1\left(\frac{1}{r}+\frac{c}{a}\right)e^{-cr/a}+F_1\left(\frac{1}{r}-\frac{c}{a}\right)e^{cr/a}\right\}\cos\theta \tag{7.28}$$

where c/a is the complex root, with positive real part, of the equation

$$c^2/a^2=(2k+i\rho j\sigma)/(\alpha+\beta+\gamma). \tag{7.29}$$

In view of the regularity of the solution at $r=\infty$, the divergent terms in Ω and $f(r,\theta)$ are to be deleted and this means that the arbitrary constants C_1, D_1, F_1 are zero. We therefore have

$$\Omega(r,\theta,t)=\left\{A_1\left(\frac{1}{r}+\alpha_1\right)e^{-\alpha_1 r}+B_1\left(\frac{1}{r}+\beta_1\right)e^{-\beta_1 r}\right\}\sin^2\theta\, e^{i\sigma t} \tag{7.30}$$

$$f(r,\theta,t)=E_1\left(\frac{1}{r^2}+\frac{c}{ar}\right)e^{-cr/a}\cos\theta\, e^{i\sigma t}. \tag{7.31}$$

From equation (7.12) we get

$$h(r,\theta,t)=\left\{\frac{i\rho\sigma-(\mu+k)\alpha_1^2}{k}A_1\left(\frac{1}{r}+\alpha_1\right)e^{-\alpha_1 r}\right.$$
$$\left.+\frac{i\rho\sigma-(\mu+k)\beta_1^2}{k}B_1\left(\frac{1}{r}+\beta_1\right)e^{-\beta_1 r}\right\}\sin^2\theta\, e^{i\sigma t}. \tag{7.32}$$

The functions A and B are to be found from equations (7.13) and (7.14). In view of the expressions for $\Omega(r,\theta,t)$ and $f(r,\theta,t)$ we have to take

$$A(r,\theta,t)=A(r)\cos\theta\, e^{i\sigma t} \tag{7.33}$$

and

$$B(r,\theta,t)=B(r)\sin\theta\, e^{i\sigma t}. \tag{7.34}$$

It is seen that

$$A(r)=\{2/(2k+i\rho j\sigma)\}\left\{\left(k+\frac{\gamma(\mu+k)\alpha_1^2}{k}-\frac{i\gamma\rho\sigma}{k}\right)A_1\left(\frac{1}{r^3}+\frac{\alpha_1}{r^2}\right)e^{-\alpha_1 r}\right.$$
$$\left.+\left(k+\frac{\gamma(\mu+k)}{k}\beta_1^2-\frac{i\gamma\rho\sigma}{k}\right)B_1\left(\frac{1}{r^3}+\frac{\beta_1}{r^2}\right)e^{-\beta_1 r}-(\alpha+\beta+\gamma)E_1\left(\frac{1}{r^3}+\frac{c}{ar^2}+\frac{c^2}{2a^2 r}\right)e^{-cr/a}\right\} \tag{7.35}$$

$$B(r) = \{1/(2k+i\rho j\sigma)\}\Big\{\Big(k+\frac{\gamma(\mu+k)\alpha_1^2}{k}-\frac{i\gamma\rho\sigma}{k}\Big)A_1\Big(\frac{1}{r^3}+\frac{\alpha_1}{r^2}+\frac{\alpha_1^2}{r}\Big)\,e^{-\alpha_1 r}$$

$$+\Big(k+\frac{\gamma(\mu+k)\beta_1^2}{k}-\frac{i\gamma\rho\sigma}{k}\Big)B_1\Big(\frac{1}{r^3}+\frac{\beta_1}{r^2}+\frac{\beta_1^2}{r}\Big)\,e^{-\beta_1 r}-(\alpha+\beta+\gamma)E_1\Big(\frac{1}{r^3}+\frac{c}{ar^2}\Big)\,e^{-cr/a}\Big\}. \tag{7.36}$$

From the boundary conditions, viz., $\Omega = \Omega_0 a^2 \sin^2\theta\, e^{i\sigma t}$, $A = 0$, $B = 0$ on $r = a$, the constants A_1, B_1, and E_1 are determined. They are found to be

$$A_1 = \Big(\frac{-\Omega_0 a^3}{D}\Big)\{k^2+\gamma(\mu+k)\beta_1^2-i\gamma\rho\sigma\}\{c^2(1+a\beta_1)+a^2\beta_1^2(c^2+2c+2)\}\,e^{\alpha_1 a} \tag{7.37}$$

$$B_1 = \Big(\frac{\Omega_0 a^3}{D}\Big)\{k^2+\gamma(\mu+k)\alpha_1^2-i\gamma\rho\sigma\}\{c^2(1+a\alpha_1)+a^2\alpha_1^2(c^2+2c+2)\}\,e^{\beta_1 a} \tag{7.38}$$

$$E_1 = \Big(\frac{\Omega_0 a^3}{D}\Big)2(\mu+k)kc^2\{\alpha_1^2(1+a\beta_1)-\beta_1^2(1+a\alpha_1)\}\,e^{c} \tag{7.39}$$

where

$$D = -c^2(1+a\alpha_1)(1+a\beta_1)[\{k(2\mu+k)-i\rho\sigma(\gamma-j\mu-jk)\}^2-4i\gamma\rho\sigma k^2]^{1/2}$$

$$+(c^2+2c+2)(2k+i\rho j\sigma)a^2[(\mu+k)\{\alpha_1^2(1+a\beta_1)-\beta_1^2(1+a\alpha_1)\}+ai\rho\sigma(\alpha_1-\beta_1)]. \tag{7.40}$$

The graphs 6, 7, 8 and 9 show the variation of the magnitude of the velocity component, the two micro-rotation components and the angular momentum for increasing distance from the sphere.

8. STRESSES

From the expressions for $\bar{q}$ and $\bar{\nu}$ obtained in the previous section and equation (2.4), we see that

$$t_{rr} = t_{\theta\theta} = t_{\phi\phi} = -p \tag{8.1}$$

$$t_{r\theta} = t_{\theta r} = 0 \tag{8.2}$$

$$t_{\theta\phi} = -t_{\phi\theta}$$

$$= \{1/(2k+i\rho j\sigma)\}\Big\{[i\rho\sigma(jk+2\gamma)-2\gamma(\mu+k)\alpha_1^2]$$

$$\times A_1\Big(\frac{1}{r^3}+\frac{\alpha_1}{r^2}\Big)\,e^{-\alpha_1 r}+[i\rho\sigma(j+k+2\gamma)-2\gamma(\mu+k)\beta_1^2]$$

$$\times B_1\Big(\frac{1}{r^3}+\frac{\beta_1}{r^2}\Big)\,e^{-\beta_1 r}+k(\alpha+\beta+\gamma)E_1\Big(\frac{2}{r^3}+\frac{2c}{ar^2}+\frac{c^2}{a^2 r}\Big)\,e^{-cr/a}\Big\}\cos\theta\, e^{i\sigma t}. \tag{8.3}$$

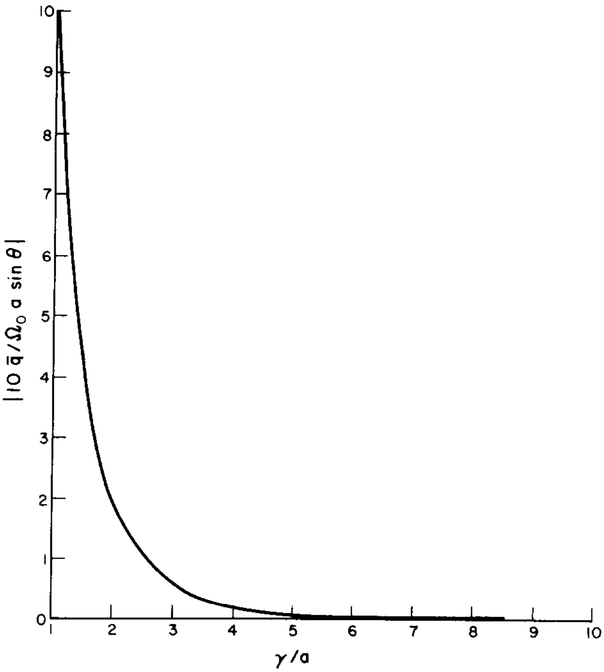


Fig. 6.

$$t_{\phi r}=\left[-\frac{2\mu+k}{2}\left\{A_1\left(\frac{\alpha_1^2}{r}+\frac{3\alpha_1}{r^2}+\frac{3}{r^3}\right)\mathrm{e}^{-\alpha_1 r}+B_1\left(\frac{\beta_1^2}{r}+\frac{3\beta_1}{r^2}+\frac{3}{r^3}\right)\mathrm{e}^{-\beta_1 r}\right\}+\left(\frac{1}{4k+i2\rho j\sigma}\right)\right.$$
$$\times\left\{[i\rho\sigma(jk+2\gamma)-2\gamma(\mu+k)\alpha_1^2]A_1\left(\frac{\alpha_1^2}{r}+\frac{\alpha_1}{r^2}+\frac{1}{r^3}\right)\mathrm{e}^{-\alpha_1 r}\right.$$
$$+[i\rho\sigma(jk+2\gamma)-2\gamma(\mu+k)\beta_1^2]B_1\left(\frac{\beta_1^2}{r}+\frac{\beta_1}{r^2}+\frac{1}{r^3}\right)\mathrm{e}^{-\beta_1 r}$$
$$\left.\left.+k(\alpha+\beta+\gamma)E_1\left(\frac{c}{ar^2}+\frac{1}{r^3}\right)\mathrm{e}^{-cr/a}\right\}\right]\sin\theta\,\mathrm{e}^{i\sigma t}=\{[P]+[Q]\}\sin\theta\,\mathrm{e}^{i\sigma t} \tag{8.4}$$

$$t_{r\phi}=\{[P]-[Q]\}\sin\theta\,\mathrm{e}^{i\sigma t}. \tag{8.5}$$

9. COUPLE STRESS

From equation (2.5), we get

$$m_{rr}=\left[(\alpha+\beta+\gamma)E_1\left(\frac{1}{r^2}+\frac{c}{ar}\right)\mathrm{e}^{-(cr/a)}+2(\beta+\gamma)E_1\left(\frac{1}{r^2}+\frac{3a}{cr^3}+\frac{3a^2}{c^2r^4}\right)\mathrm{e}^{-(cr/a)}\right.$$
$$-\frac{2(\beta+\gamma)}{k(2k+i\rho\sigma j)}\left\{[k^2-i\gamma\rho\sigma+\gamma(\mu+k)\alpha_1^2]\,A_1\left(\frac{\alpha_1^2}{r^2}+\frac{3\alpha_1}{r^3}+\frac{3}{r^4}\right)\right.$$
$$\left.\left.\times\mathrm{e}^{-\alpha_1 r}+[k^2-i\gamma\rho\sigma+\gamma(\mu+k)\beta_1^2]B_1\left(\frac{\beta_1^2}{r^2}+\frac{3\beta_1}{r^3}+\frac{3}{r^4}\right)\mathrm{e}^{-\beta_1 r}\right\}\right]\cos\theta\,\mathrm{e}^{i\sigma t} \tag{9.1}$$

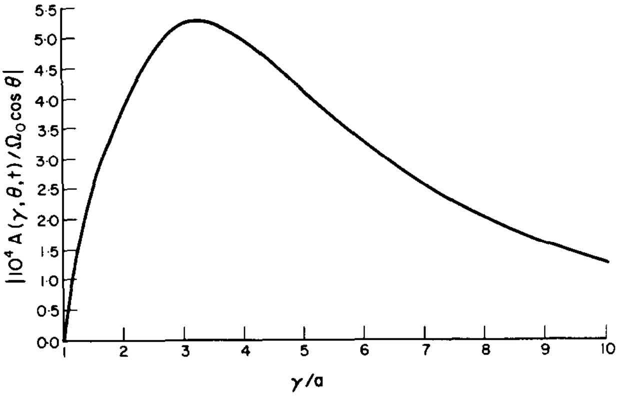


Fig. 7.

$$m_{\theta\theta} = m_{\phi\phi}$$

$$= \Bigg[\Bigg\{\alpha\left(\frac{1}{r^2}+\frac{c}{ar}\right)-(\beta+\gamma)\left(\frac{1}{r^2}+\frac{3a}{cr^3}+\frac{3a^2}{c^2r^4}\right)E_1\Bigg\}\,e^{-(cr/a)}$$

$$+\frac{(\beta+\gamma)}{k(2k+i\rho j\sigma)}\Bigg\{[k^2-i\gamma\rho\sigma+\gamma(\mu+k)\alpha_1^2]A_1\left(\frac{\alpha_1^2}{r^2}+\frac{3\alpha_1}{r^3}+\frac{3}{r^4}\right)e^{-\alpha_1 r}$$

$$+[k^2-i\gamma\rho\sigma+\gamma(\mu+k)\beta_1^2]B_1\left(\frac{\beta_1^2}{r^2}+\frac{3\beta_1}{r^3}+\frac{3}{r^4}\right)e^{-\beta_1 r}\Bigg\}\Bigg]\cos\theta\, e^{i\sigma t} \tag{9.2}$$

$$m_{r\theta} = \Bigg\{(\beta+\gamma)\Bigg[E_1\left(\frac{1}{r^2}+\frac{3a}{cr^3}+\frac{3a^2}{c^2r^4}\right)e^{-(cr/a)}$$

$$-\{1/[k(2k+i\rho\sigma j)]\}\Bigg\{[k^2-i\gamma\rho\sigma+\gamma(\mu+k)\alpha_1^2]A_1\left(\frac{\alpha_1^2}{r^2}+\frac{3\alpha_1}{r^3}+\frac{3}{r^4}\right)e^{-\alpha_1 r}$$

$$+[k^2-i\gamma\rho\sigma+\gamma(\mu+k)\beta_1^2]B_1\left(\frac{\beta_1^2}{r^2}+\frac{3\beta_1}{r^3}+\frac{3}{r^4}\right)e^{-\beta_1 r}\Bigg\}\Bigg]$$

$$-\gamma\Bigg[\{1/[k(2k+i\rho j\sigma)]\}\Bigg\{[k^2-i\gamma\rho\sigma+\gamma(\mu+k)\alpha_1^2]A_1\left(\frac{\alpha_1^3}{r}+\frac{\alpha_1^2}{r^2}\right)e^{-\alpha_1 r}$$

$$+[k^2-i\gamma\rho\sigma+\gamma(\mu+k)\beta_1^2]B_1\left(\frac{\beta_1^3}{r}+\frac{\beta_1^2}{r^2}\right)e^{-\beta_1 r}\Bigg\}\Bigg]\Bigg\}\sin\theta\, e^{i\sigma t}$$

$$= \{(\beta+\gamma)[P_1]-\gamma[Q_1]\}\sin\theta\, e^{i\sigma t} \tag{9.3}$$

$$m_{\theta r} = \{(\beta+\gamma)[P_1]-\beta[Q_1]\}\sin\theta\, e^{i\sigma t} \tag{9.4}$$

and

$$m_{r\phi} = m_{\phi r} = m_{\theta\phi} = m_{\phi\theta} = 0. \tag{9.5}$$

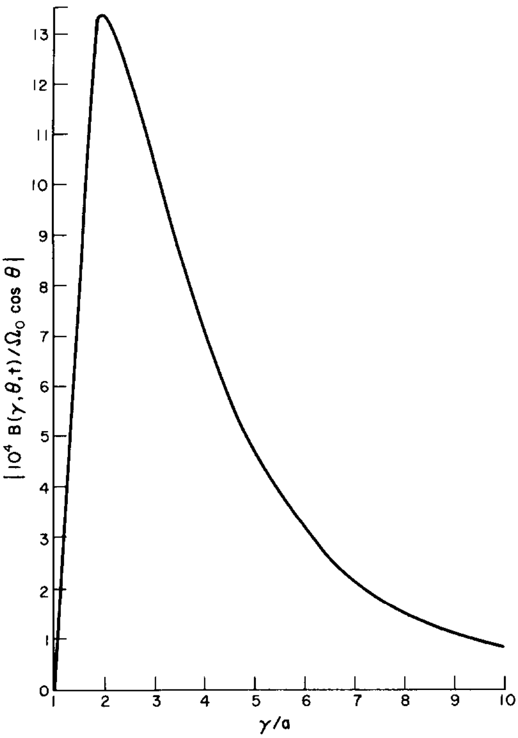
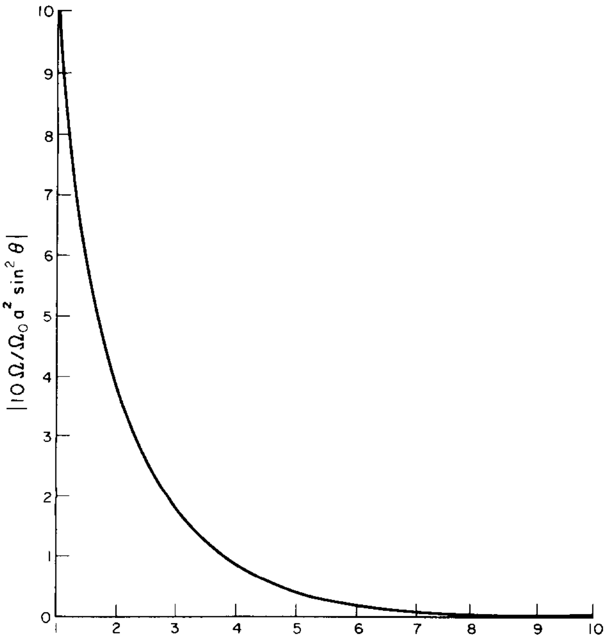

|10⁴ B(γ, θ, t)/Ω₀ cos θ|
0
1
2
3
4
5
6
7
8
9
10
11
12
13
1
2
3
4
5
6
7
8
9
10
γ/a

Fig. 8.


|10 Ω/Ω₀a² sin² θ|
0
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4
5
6
7
8
9
10
1
2
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4
5
6
7
8
9
10
γ/a

Fig. 9.

10. COUPLE ON THE SPHERE

The couple acting on the sphere is comprised of contribution from the surface stress t_{ij} and also the couple stress m_{ij}. The couple due to the surface stress t_{ij} is given by

$$\bar{N}_S = \int_0^{\pi}\int_0^{2\pi} a^3 \bar{e}_r x[t_{rr}\bar{e}_r + t_{r\theta}\bar{e}_\theta + t_{r\phi}\bar{e}_\phi]_{r=a} \sin\theta \, d\phi \, d\theta \, e^{i\sigma t}$$

$$= -a^3 \int_0^{\pi} d\theta \int_0^{2\pi} [t_{r\phi}]_{r=a}\bar{e}_\theta \sin\theta \, d$$

$$= 2\pi a^3 \int_0^{\pi} [t_{r\phi}]_{r=a} \sin^2\theta \, d\theta \, \bar{k} \, e^{i\sigma t} \tag{10.1}$$

where $\bar{k}$ is the unit vector in the direction of the axis of rotation. We get

$$N_S = -(8/3)\pi a^3[(3\mu+2k)\Omega_0 + \{(\mu+k)/a\}\{A_1\alpha_1^2 e^{-\alpha_1 a} + B_1\beta_1^2 e^{-\beta_1 a}\}] e^{i\sigma t}. \tag{10.2}$$

The contribution to the couple from the couple stress is given by

$$\bar{N}_C = \int_0^{\pi} \left\{ d\theta \int_0^{2\pi} [m_{rr}\bar{e}_r + m_{r\theta}\bar{e}_\theta + m_{r\phi}\bar{e}_\phi]_{r=a} a^2 \sin\theta \, d\phi \right\} e^{i\sigma t}$$

$$= \int_0^{\pi} \{ d\theta \int_0^{2\pi} [m_{rr}\bar{e}_r + m_{r\theta}\bar{e}_\theta]_{r=a} a^2 \sin\theta \, d\phi \} e^{i\sigma t}$$

$$= 2\pi a^2 \int_0^{\pi} [m_{rr}\cos\theta - m_{r\theta}\sin\theta]_{r=a} \sin\theta \, d\theta \, \bar{k} \, e^{i\sigma t} \tag{10.3}$$

and this becomes

$$\begin{aligned} N_C = {} & (4/3)\pi[\{(\alpha+\beta+\gamma)c + \alpha + \beta\}E_1 e^{-c} \\ & + \{(2\gamma a)/[k(2k+i\rho\sigma j)]\}\{[k^2 - i\gamma\rho\sigma + \gamma(\mu+k)\alpha_1^2]A_1\alpha_1^3 e^{-\alpha_1 a} \\ & + [k^2 - i\gamma\rho\sigma + \gamma(\mu+k)\beta_1^2]B_1\beta_1^3 e^{-\beta_1 a}\}] e^{i\sigma t}. \end{aligned} \tag{10.4}$$

On substituting for the constants A_1, B_1, E_1 in (10.3) and (10.4) we get

$$\begin{aligned} N_S = {} & -(8/3)\pi a^2\Omega_0\{3\mu + 2k + \{(\mu+k)a^2/D\}[(k^2 - i\gamma\rho\sigma)c^2\{\beta_1^2(1+a\alpha_1) - \alpha_1^2(1+a\beta_1)\} \\ & + i\rho\sigma(2k+i\rho\sigma j)a(\alpha_1-\beta_1)\{c^2 + a(\alpha_1+\beta_1)(c^2+2c+2)\}]\} e^{i\sigma t} \end{aligned} \tag{10.5}$$

$$\begin{aligned} N_C = {} & \left\{(8/3)\pi \frac{\Omega_0 a^3 k(\beta_1-\alpha_1)}{D}\right\}[\{(\alpha+\beta+\gamma)c + \alpha + \beta\}(\mu+k)c^2(-a\alpha_1\beta_1 - \alpha_1 - \beta_1) \\ & + a\gamma(\mu+k)c^2\alpha_1\beta_1(1+a\alpha_1+a\beta_1) + ac^2\{k(2\mu+k) + i\rho\sigma(\gamma+j\mu+jk)\} \\ & + a^3 i\rho\sigma(2k+i\rho\sigma j)(c^2+2c+2)] e^{i\sigma t} \end{aligned} \tag{10.6}$$

and D is already given in (7.40). The total couple is given by

$$N = N_S + N_C. \tag{10.7}$$

The case of slow steady rotation of a sphere is obtainable from the above analysis by allowing the period of oscillation $2\pi/\sigma$ tend to infinity. The expressions for N_S and N_C are then given by

$$N_S = (8/3)\pi a^3\Omega_0\left[-3\mu - 2k + \frac{\lambda^2 k(\mu+k)c^2}{(2\mu+k)c^2(1+\lambda)+2(\mu+k)\lambda^2(c^2+2c+2)}\right] \quad (10.8)$$

$$N_C = (8/3)\pi\Omega_0 a^3 k\left\{\frac{2(\mu+k)\lambda^2(c+1)-(2\mu+k)c^2(\lambda+1)}{(2\mu+k)c^2(1+\lambda)+2(\mu+k)\lambda^2(c^2+2c+2)}\right\} \quad (10.9)$$

and the couple on the sphere is

$$N = N_S + N_C = \frac{-8\pi\Omega_0 a^3(\mu+k)(2\mu+k)[c^2(1+\lambda)+\lambda^2(c^2+2c+2)]}{(2\mu+k)c^2(1+\lambda)+2(\mu+k)\lambda^2(c^2+2c+2)}. \quad (10.10)$$

This is in agreement with the couple calculated in the steady rotation of a sphere, S. K. Lakshmana Rao, N. Ch. Pattabhi Ramacharyulu and P. Bhujanga Rao[4], and as observed therein, the couple experienced by the sphere is larger in the case of micropolar fluid than in the Newtonian viscous liquid.

The couple obtained above in (10.5), (10.6) and (10.7) in the unsteady oscillation of a sphere can be employed for obtaining the limiting value when the fluid is Newtonian viscous and non-polar. This is done by allowing the viscous coefficients k and γ in Eringen's model to zero. We then notice that of the two quantities α_1^2, β_1^2 defined by (3.24) and (3.25), one (say, α_1) becomes infinite and the other has the limiting value

$$\beta_1^2 = \underset{|\alpha_1|\to\infty}{\mathrm{Lt}} \frac{\alpha_1^2\beta_1^2}{\alpha_1^2+\beta_1^2} = \frac{i\rho\sigma}{\mu}. \quad (10.11)$$

The couple N_C arising from the couple stresses is then zero in the limit and the limiting form of the couple is therefore

$$N = -(8/3)\pi\mu a^3\Omega_0[3 + a^2 i\rho\sigma/\{\mu(1+a\sqrt{i\rho\sigma/\mu})\}]$$

$$= -(8/3)\pi\mu a^3\Omega_0[3-\xi^2a^2/(1+i\xi a)] \quad (10.12)$$

where

$$\xi = (1-i)\sqrt{(\rho\sigma/2\mu)}. \quad (10.13)$$

which is a classical result[5].

The couple N given in (10.10) can be written in the form

$$N = M\Omega_0 a^2\sigma(-K'-iK)\,e^{i\sigma t} \quad (10.14)$$

where $M = 4\pi a^3\rho/3$ is the mass of the liquid displaced by the sphere. The real part of N is then given by

$$N_R = M\Omega_0 a^2\sigma(K\sin\sigma t - K'\cos\sigma t). \quad (10.15)$$

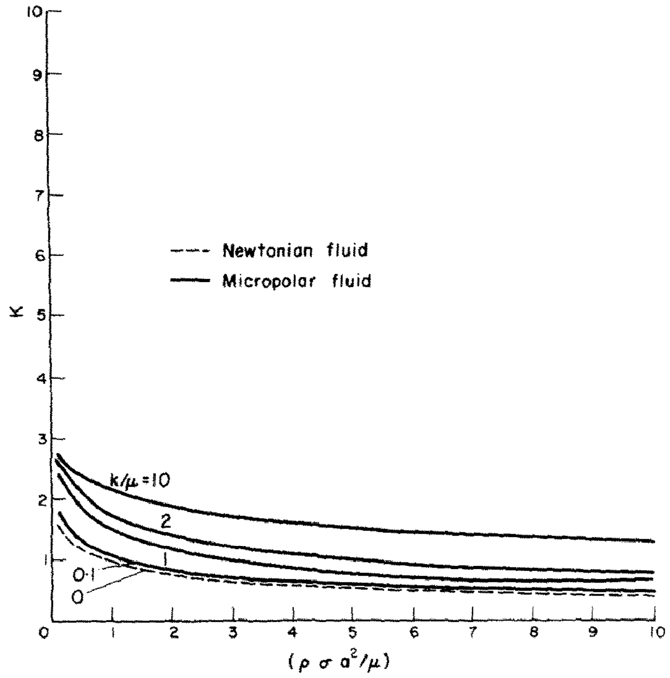
Newtonian fluid
Micropolar fluid
K
k/μ=10
2
1
0·1
0
(ρ σ a²/μ)

Fig. 10.

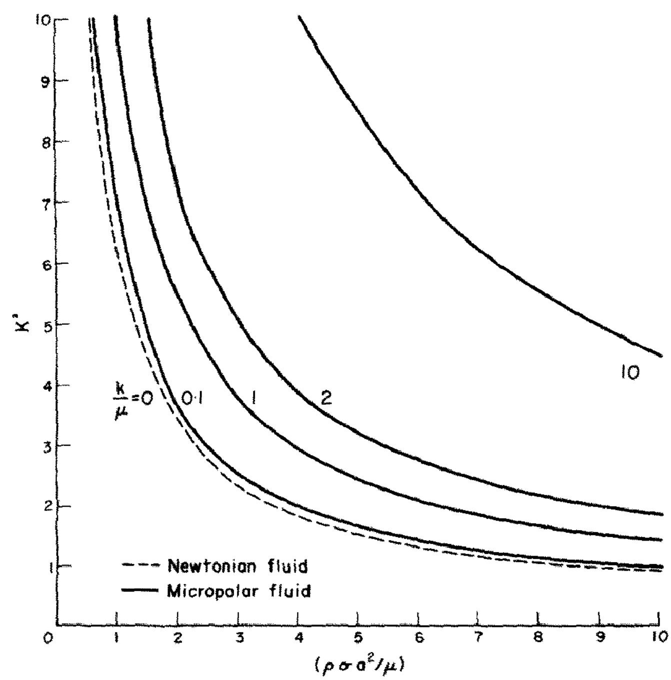
K'
k/μ=0
0·1
1
2
10
Newtonian fluid
Micropolar fluid
(ρ σ a²/μ)

Fig. 11.

The graphs 10 and 11 show the variation of the parameters K and K' introduced in the above equation for different frequencies of oscillation of the sphere. These graphs show increase in the values of these parameters when compared to the classical Newtonian fluid, over the entire frequency range considered. In fact it is observed from the graphs 10 and 11 that over the entire frequency range they increase steadily with increase of the 'micropolarity coefficient', k/μ. In particular it follows therefore that $\sqrt{K^2+K'^2}$ is larger in the micropolar flow than in the Newtonian flow. Hence the maximum couple experienced by the sphere over any period of oscillation is larger in the case of micropolar fluids as compared to the Newtonian fluids. Also the larger the 'micropolarity coefficient', k/μ, the greater is this maximum couple value.

The above analysis is entirely theoretical and we are not sure of the variables of the appropriate experimental results for the oscillating systems employed in this paper. In fact the values of the physical parameters or even their ranges do not seem to be available. In the numerical calculations the only guiding restrictions on the parameters are those mentioned in [1].

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Résumé—Cet article examine l'oscillation rectilinéaire d'une sphère le long d'un diamètre et l'oscillation de rotation d'une sphère autour d'un diamètre, dans un fluide micropolaire d'Eringen. Les grandeurs physiques telles que la vitesse, la micro-rotation, et les composantes de la contrainte et du couple de contrainte sont calculées. La trainée sur la sphère en oscillation rectilinéaire et le couple sur la sphère en oscillation de rotation sont calculés. Il est observé que sur une période quelconque d'oscillation, la trainée maximale ou le couple maximal, comme cela peut être le cas, est plus grand dans le cas d'un fluide micropolaire, par comparaison au fluide Necotonien.

Zusammenfassung—Diese Arbeit untersucht die geradlinige Schwingung einer Kugel entlang eines Durchmessers und die rotierende Schwingung einer Kugel um einen Durchmesser in Eringen's mikropolarer Flüssigkeit. Die physikalischen Grössen wie die Geschwindigkeit, Mikrorotation und Spannungs- und Paarspannungskomponenten werden berechnet. Der Rücktrieb auf die geradlinig schwingende Kugel und das Kräftepaar auf der rotationsschwingenden Kugel werden berechnet. Es wird beobachtet, dass über irgendeine Periode der Schwingung der maximale Rücktrieb oder das maximale Kräftepaar, wie der Fall liegt, im Falle der mikropolaren Flüssigkeit grösser ist, verglichen mit der Newton'schen Flüssigkeit.

Sommario—L'A. esamina l'oscillazione rettilinea di una sfera lungo un diametro e l'oscillazione rotatoria di una sfera sul diametro in un liquido micropolare di Eringen. Si calcolano le quantità fisiche quali la velocità, la microrotazione, sollecitazione e componenti la sollecitazione d'accoppiamento. La reazione sulla sfera oscillante rettilinearmente e la coppia sulla sfera oscillante rotazionalmente sono pure calcolate. Si osserva che lungo un periodo qualsiasi di oscillazione la reazione massima o la coppia massima, secondo il caso, sono piú grandi nel caso dei liquidi micropolari che non dei liquidi newtoniani.

Абстракт—Рассмотрены прямолинейная осцилляция шара вдоль диаметра и ротационная осцилляция шара о диаметре в эринговской микрополярной жидкости. Вычислятся физические величины как скорость, микроротация, составляющие напряжения и напряжения пары, а также сопротивление на прямолинейно колебающий шар и пара на ротационный колебающий шар. Уставлено, что в периоде осцилляции или максимальное сопротивление или максимальная пара больше для микрополярных жидкостей, чем в случае ньютонианской жидкости.