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STABILITY OF MICROPOLAR FLUID MOTIONS

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Abstract—The paper employs the energy method for obtaining criteria for the stability of the motion of an incompressible micropolar fluid in an arbitrary domain. A formula is obtained for the time-rate of change of the kinetic energy of the difference of two flows and it is shown that the original flow is stable when $Re + 0{\cdot}5Rm < 80$ and $Rm < 2m_2 + 6\pi^2 m_9$. The quantities m_2 and m_9 are material constants of the fluid and Re, Rm denote the Reynolds number and microrotational Reynolds number respectively. A different form of stability criterion is also noticed and a theorem is deduced concerning the uniqueness of steady, incompressible micropolar flows. Finally, a variational algorithm is established for the stability of a micropolar flow and this can be employed to sharpen the estimate of the Reynolds number below which the flow is stable.

IN THIS paper we employ the energy method to examine the stability of flows of micropolar incompressible fluids. This method has the advantage that it is applied to a difference of two flows, rather than to a perturbation over another flow. The special features and the generality of conclusions regarding stability that are possible by this method have been clearly exposed by James Serrin[1] in his classic paper on the stability of viscous fluid motions and subsequent to this work, the energy method of stability has indeed gained an extraordinary resonance. In [1] Serrin obtains sufficiency conditions ensuring stability of flows for *arbitrary* disturbances in bounded domains of *arbitrary geometry* and the results are applicable also to cases of unbounded geometry if the disturbances are periodic in the corresponding variables. The stability criterion is expressed as a Reynolds number estimate and the method also leads to a uniqueness theorem for steady bounded flows. Serrin has also given a variational algorithm for improving the Reynolds number estimate and by an application of the energy method, he obtains stability criterion for laminar Couette flow between rotating coaxial cylinders. Daniel D. Joseph[2] has investigated the stability of the Boussinesq equations using the energy method initiated by Serrin in [1].

The micropolar fluid flow discussed in this paper has two prominent departures from the case of Navier–Stokes theory, viz., the sustenance of the couple stress and the nonsymmetry of the stress tensor. Polar fluids have been the subject of study in recent years by several authors, prominent among them being A. C. Eringen[3, 4] and J. L. Bleustein and A. E. Green[5]. The constitutive equations of the linear micropolar flow involve six constants and the field equations of this theory as presented by Eringen [4] consist of a coupled system of differential equations for the two basic vectors of the theory, viz., the velocity and microrotation of the fluid elements. The six constants (three viscosity coefficients and three gyroviscosity coefficients) conform to inequalities forced by thermodynamic considerations and at a boundary the two field vectors satisfy the *hyper-stick* or adherence condition.

The stability criterion for micropolar fluid flow developed in this paper rests on the formula for the rate of change of the kinetic energy of the difference motion. A stability estimate expressed in terms of Reynolds numbers is obtained for arbitrary disturbances in confined regions of unspecified geometry and the criterion is valid for periodic disturbances in an unbounded fluid layer or cylinder. A uniqueness theorem is proved in the case of steady micropolar flows in a bounded domain. This is followed by a variational algorithm for obtaining the stability criterion.

1. REYNOLDS-ORR-SERRIN ENERGY EQUATION

The field equations of micropolar fluid flow are given by

$$\rho\left\{\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\,.\,\mathrm{grad})\mathbf{v}\right\}=-\mathrm{grad}\,p+k\,\mathrm{curl}\,\boldsymbol{\nu}-(\mu+k)\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{v}$$
$$+(\lambda_1+2\mu+k)\,\mathrm{grad}\,(\mathrm{div}\,\mathbf{v}) \qquad (1)$$

$$\rho j\left\{\frac{\partial \boldsymbol{\nu}}{\partial t}+(\mathbf{v}\div\mathrm{grad})\,\mathbf{v}\right\}=-2k\boldsymbol{\nu}+k\,\mathrm{curl}\,\mathbf{v}-\gamma\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\nu}+(\alpha+\beta+\gamma)\,\mathrm{grad}\,(\mathrm{div}\,\boldsymbol{\nu}) \qquad (2)$$

in which the vectors $\mathbf{v}, \boldsymbol{\nu}$ denote respectively the velocity and microrotation. The constants λ_1, μ, k are the viscosity coefficients while α, β, γ are the gyroviscosity coefficients. The terms representing the body force and couple are omitted. The density ρ and the gyration parameter j are constants and the velocity is a solenoidal field.

We consider a basic motion $(\mathbf{v}, \boldsymbol{\nu})$ of the micropolar fluid in the region $\mathrm{N}=\mathrm{N}(t)$ of space and on the boundary of N, we have the hyperstick condition. At an instant $(t=0)$ the basic motion is altered to the (starred) motion $(\mathbf{v}^*, \boldsymbol{\nu}^*)$ and the latter flow satisfies the same conditions at the boundary as the unstarred or basic flow. To determine if the altered flow approaches the basic flow asymptotically as $t\to\infty$ or differs radically from it, we consider the kinetic energy of the difference motion with its velocity $\mathbf{u}=\mathbf{v}^*-\mathbf{v}$ (3) and the microrotation $\boldsymbol{\vartheta}=\boldsymbol{\nu}^*-\boldsymbol{\nu}$ (4). The kinetic energy is given by

$$T=T_1+T_2 \qquad (5)$$

in which†

$$T_1=\tfrac{1}{2}\int \rho u^2 \qquad (6)$$

and

$$T_2=\tfrac{1}{2}\int \rho j\vartheta^2 \qquad (7)$$

and on the boundary φ we have $\mathbf{u}=\boldsymbol{\nu}=0$.

The rates of change of T_1 and T_2 are governed by the formulae

$$\frac{\mathrm{d}T_1}{\mathrm{d}t}=-\int \rho\mathbf{u}\,.\,D\,.\,\mathbf{u}+k\int \boldsymbol{\vartheta}\,.\,\mathrm{curl}\,\mathbf{u}-(\mu+k)\int (\mathrm{curl}\,\mathbf{u})^2 \qquad (8)$$

and

$$\frac{\mathrm{d}T_2}{\mathrm{d}t}=-\int \rho j\mathbf{u}\,.\,E\,.\,\boldsymbol{\vartheta}+k\int \boldsymbol{\vartheta}\,.\,\mathrm{curl}\,\mathbf{u}-2k\int (\boldsymbol{\vartheta})^2-\gamma\int (\mathrm{curl}\,\boldsymbol{\vartheta})^2$$
$$-(\alpha+\beta+\gamma)\int (\mathrm{div}\,\boldsymbol{\vartheta})^2 \qquad (9)$$

†The conventional volume infinitesimal is omitted consistently in the integrals, which are extended over the volume N.

where D is the strain rate matrix of the velocity ($\mathbf{v}$) and E is the matrix gradient of the microrotation ($\boldsymbol{\nu}$) of the basic flow. In an obvious notation

$$(D)_{ik} = \tfrac{1}{2}(v_{i,k} + v_{k,i}), E_{ik} = \nu_{k,i}. \tag{10}$$

To prove (8) and (9) we take the differences of the field equations (1) and (2) for the starred and unstarred flows. The difference fields $\mathbf{u}$ and $\boldsymbol{\vartheta}$ are then governed by the equations

$$\rho\left\{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \,.\, \mathrm{grad})\mathbf{v} + (\mathbf{v}^* \,.\, \mathrm{grad})\mathbf{u}\right\} = \mathrm{grad}\,(p - p^*) + k\,\mathrm{curl}\,\boldsymbol{\vartheta} - (\mu + k)\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} \tag{11}$$

$$\rho j\left\{\frac{\partial \boldsymbol{\vartheta}}{\partial t} + (\mathbf{u} \,.\, \mathrm{grad})\boldsymbol{\nu} + (\mathbf{v} \,.\, \mathrm{grad})\boldsymbol{\vartheta}\right\} = k\,\mathrm{curl}\,\mathbf{u} - 2k\boldsymbol{\vartheta} - \gamma\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\vartheta} + (\alpha + \beta + \gamma)\,\mathrm{grad}\,(\mathrm{div}\,\boldsymbol{\vartheta}). \tag{12}$$

Scalar product of (11) with $\mathbf{u}$ yields

$$\rho\frac{\partial}{\partial t}(u^2/2) = -\rho\mathbf{u} \,.\, D \,.\, \mathbf{u} - \rho\,\mathrm{div}\,[(\tfrac{1}{2}u^2)\mathbf{v}^*] + \mathrm{div}\,[(p - p^*)\mathbf{u}] + k\mathbf{u} \,.\, \mathrm{curl}\,\boldsymbol{\vartheta} - (\mu + k)(\mathrm{curl}\,\mathbf{u})^2 + (\mu + k)\,\mathrm{div}\,(\mathbf{u} x\,\mathrm{curl}\,\mathbf{u}) \tag{13}$$

on using the conditions $\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{v}^* = 0$. Integrating this equation over the region N, using the divergence theorem selectively and invoking the boundary conditions, we obtain (8). In a similar way, the scalar product of (12) with $\boldsymbol{\vartheta}$ yields

$$\rho j\frac{\partial}{\partial t}(\vartheta^2/2) = -\rho j\mathbf{u} \,.\, E \,.\, \boldsymbol{\vartheta} - \rho j\,\mathrm{div}\,[(\tfrac{1}{2}\vartheta^2)\mathbf{v}^*] + k\boldsymbol{\vartheta} \,.\, \mathrm{curl}\,\mathbf{u} + 2k\vartheta^2 - \gamma\{\mathrm{div}\,[(\mathrm{curl}\,\boldsymbol{\vartheta})x\boldsymbol{\vartheta}] + (\mathrm{curl}\,\boldsymbol{\vartheta})^2\} + (\alpha + \beta + \gamma)\{\mathrm{div}\,[(\mathrm{div}\,\boldsymbol{\vartheta})\boldsymbol{\vartheta}] - (\mathrm{div}\,\boldsymbol{\vartheta})^2\} \tag{14}$$

on using the conditions $\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{v}^* = 0$. Integrating this result over the region N and invoking the divergence theorem selectively, we obtain (9).

The proofs of the formulae (8), (9) demand the use of the divergence theorem and hence do not hold when the region N is unbounded. However, under suitably restrictive assumptions on the asymptotic character of the functions, (8) and (9) can be validated for unbounded regions. Further when the flow geometry permits disturbances which are spatially periodic at each instant, the formulae (8) and (9) continue to be valid.

By addition of (8) and (9) we have

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\int \rho\mathbf{u} \,.\, D \,.\, \mathbf{u} - \int \rho j\mathbf{u} \,.\, E \,.\, \boldsymbol{\vartheta} - k\int (\boldsymbol{\vartheta} - \mathrm{curl}\,\mathbf{u})^2 - \mu\int (\mathrm{curl}\,\mathbf{u})^2 - k\int \vartheta^2 - \gamma\int (\mathrm{curl}\,\boldsymbol{\vartheta})^2 - (\alpha + \beta + \gamma)\int (\mathrm{div}\,\boldsymbol{\vartheta})^2. \tag{15}$$

The first term on the right side in the above is replaceable by $+\int \rho\mathbf{u}\,.\,(\mathrm{grad}\,\mathbf{u})\,.\,\mathbf{v}$. We shall refer to (15) as Reynolds–Orr–Serrin energy equation for micropolar fluids.

The viscosity coefficients k, μ and the gyroviscosity coefficients γ, $\alpha+\beta+\gamma$ are each non-negative and from the formula (15) we see that viscosity has the tendency to stabilize the basic flow. In (8) and (9) the cross term $\int \boldsymbol{\vartheta}\,.\,\mathrm{curl}\,\mathbf{u}$ is indefinite and this may inhibit stability as an interaction effect, but any such effects are rounded in the combined equation (15). The Reynolds stress term $\int \rho\mathbf{u}\,.\,D\,.\,\mathbf{u}$ and the couple stress term $\int \rho j\mathbf{u}\,.\,E\,.\,\boldsymbol{\nu}$ can be destabilizing.

The symbols appearing in the paper are defined below:

d diameter of a sphere which includes the bounded region $\mathrm{N}=\mathrm{N}(t)$

v_0 maximum speed of the basic flow over the time interval $(0,t)$

$-m$ a lower bound for the eigenvalues of the strain rate matrix D of the basic flow over $(0,t)$

n an upper bound for the magnitude of the matrix E over $(0,t)=u\,.\,b\,.\,\{\mathrm{tr}\,(EE^T)\}^{1/2}$ over $(0,t)$

a $\min\,(\alpha+\beta+\gamma,\gamma)$

$j/d^2=m_1$, $k/\mu=m_2$, $\gamma/\mu d^2=m_3$, $\rho m d^2/\mu=m_4=Re$, $\alpha/\gamma=m_5$, $\beta/\gamma=m_6$, $(\alpha+\beta+\gamma)/\gamma=1+m_5+m_6=m_7$, $\rho jnd/\mu=m_8=Rm$, $a/\mu d^2=m_9$, $\rho v_0 d/\mu=m_{10}=Ro$.

Here d is a characteristic geometrical length of the problem and v_0, m, n depend on the basic flow over the interval $(0,t)$. The quantities $m_1, m_2, \ldots, m_{10}$ are dimensionless and barring m_5, m_6 the rest are all non-negative. At any instant we fix the values of $m_1, m_2, m_3, m_7, m_9=\min\,(m_3, m_3m_7)$. The Reynolds number of the flow is $m_4=Re$ or $m_{10}=Ro$ as may be the case and Rm is the microrotational Reynolds number.

2. CRITERIA FOR UNIVERSAL STABILITY

It is clear from the definition of m that

$$\mathbf{u}\,.\,D\,.\,\mathbf{u}\geqslant -mu^2. \tag{16}$$

Also n is the upper bound over $(0,t)$ of the quantity

$$\{\mathrm{tr}\,(EE^T)\}^{1/2}=\left\{\sum_{k,l=1}^{3}\left(\frac{\partial \nu_k}{\partial x_l}\right)^2\right\}^{1/2} \tag{17}$$

and hence we have

$$\mathbf{u}\,.\,E\,.\,\boldsymbol{\vartheta}\geqslant -n|\mathbf{u}|\,|\boldsymbol{\vartheta}|\geqslant -(n/2)(d\boldsymbol{\vartheta}^2+u^2/d). \tag{18}$$

Using (16) and (18) in the energy equation (15) we have the result

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t}\leqslant \int\left(\rho m+\frac{\rho jn}{2d}\right)u^2-\mu\int(\mathrm{curl}\,\mathbf{u})^2+\int(\tfrac{1}{2}\rho jnd-k)\boldsymbol{\vartheta}^2-k\int(\boldsymbol{\vartheta}-\mathrm{curl}\,\mathbf{u})^2\\ -\gamma\int(\mathrm{curl}\,\boldsymbol{\vartheta})^2-(\alpha+\beta+\gamma)\int(\mathrm{div}\,\boldsymbol{\vartheta})^2.\end{aligned} \tag{19}$$

Since $\mathrm{div}\,\mathbf{u}=0$ we have [1]†

†The constant on the right hand side of (20) as available in [1] is $(3+\sqrt{13})\pi^2/2$. I thank Professor James B. Serrin for the information that this value is improved to 80 by Professors L. E. Payne and H. F. Weinberger.

$$\int (\text{curl}\,\mathbf{u})^2 = \int (\text{grad}\,\mathbf{u})^2 \geqslant \frac{80}{d^2}\int \mathbf{u}^2 \tag{20}$$

and

$$\int \{(\text{div}\,\boldsymbol{\vartheta})^2 + (\text{curl}\,\boldsymbol{\vartheta})\} \geqslant \frac{3\pi^2}{d^2}\int \boldsymbol{\vartheta}^2. \tag{21}$$

Combining (19), (20), (21) we find that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leqslant \int (\rho m + \rho jn/2d - 80\mu/d^2)\,u^2 + \int (\rho jnd/2 - k - 3\pi^2 a/d^2)\,\boldsymbol{\vartheta}^2 \tag{22}$$

$$= (Re + Rm/2 - 80)(2m/m_4)T_1 + (Rm - 2m_2 - \ \pi^2 m_9)(m/m_1 m_4)T_2. \tag{23}$$

From (23) we can deduce the stability criterion and the result is Theorem 1.

Theorem 1. If the Reynolds number Re and the microrotational Reynolds number Rm of a micropolar flow in a bounded region $\mathrm{N} = \mathrm{N}(t)$ of space satisfy the restrictions

$$Re + Rm/2 < 80 \tag{24}$$

$$Rm < 2m_2 + 6\pi^2 m_9 \tag{25}$$

the kinetic energy of the difference motion tends to zero as $t \to \infty$ and the unstarred motion is stable.

Whenever the conditions (24), (25) are valid we see that the energy T of the difference flow $(\mathbf{u}, \boldsymbol{\vartheta})$ decays faster than the exponential. Specifically, we have

$$T(t) \leqslant T(0)\exp(-bt) \tag{26}$$

where

$$-b = \max\{(Re + Rm/2 - 80)(2m/m_4),\ (Rm - 2m_2 - 6\pi^2 m_9)(m/m_1 m_4)\}. \tag{27}$$

As mentioned earlier, the energy equation (15) can also be expressed in the form

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \int \rho\mathbf{u}\,.\,(\text{grad}\,\mathbf{u})\,.\,\mathbf{v} - \int \rho j\mathbf{u}\,.\,E\,.\,\boldsymbol{\vartheta} - \mu\int (\text{curl}\,\mathbf{u})^2 - k\int \boldsymbol{\vartheta}^2 -$$

$$-k\int (\boldsymbol{\vartheta} - \text{curl}\,\mathbf{u})^2 - \gamma\int (\text{div}\,\boldsymbol{\vartheta})^2 - (\alpha + \beta + \gamma)\int (\text{curl}\,\boldsymbol{\vartheta})^2. \tag{28}$$

Using the inequality (cf. [1])

$$\mathbf{u}\,.\,(\text{grad}\,\mathbf{u})\,.\,\mathbf{v} \leqslant (\mu/2\rho)(\text{grad}\,\mathbf{u})^2 + \rho u^2 v^2/2\mu \tag{29}$$

to replace the first term on the right side of (28) and dealing with the other terms in the same way as before, we see that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leqslant [Ro + (Rm - 80)/Ro](v_0/d)T_1 + (Rm - 2m_2 - 6\pi^2 m_9)(v_0/d)(1/m_1 Ro)T_2. \tag{30}$$

From this we can deduce the stability criterion stated here as Theorem 2.

Theorem 2. If the Reynolds number Ro and the microrotational Reynolds number Rm of a micropolar flow in a bounded domain $\mathrm{N} = \mathrm{N}(t)$ of space satisfy the restriction

$$Ro^2 + Rm < 80 \tag{31}$$

$$Rm < 2m_2 + 6\pi^2 m_9 \tag{32}$$

the kinetic energy of the difference motion tends to zero as $t \to \infty$ and the unstarred or basic flow is stable.

When the conditions (31), (32) are valid we can deduce the law of decay of the kinetic energy of the difference flow. If

$$-c = \max\{(Ro^2 + Rm - 80)(v_0/R_0 d), (Rm - 2m_2 - 6\pi^2 m_9)(v_0/d)(1/m_1 Ro)\} \tag{33}$$

we have

$$T(t) \leqslant T(0)\exp(-ct). \tag{34}$$

Theorem 3. Let $(\mathbf{v}, \boldsymbol{\nu})$ and $(\mathbf{v}^*, \boldsymbol{\nu}^*)$ be two steady flows in the domain N subject to the hyperstick boundary conditions. If the Reynolds numbers Re, Ro and the microrotational Reynolds number Rm satisfy the conditions (24), (25) or (31), (32) the two flows are identical.

This result is seen easily from the fact that the difference motion $\mathbf{u} = \mathbf{v}^* - \mathbf{v}$, $\boldsymbol{\vartheta} = \boldsymbol{\nu}^* - \boldsymbol{\nu}$ also being steady, has constant kinetic energy and the laws of decay of the kinetic energy in (26), (34) must hold true. This can happen only if $T_1 = T_2 = 0$ and these imply that $\mathbf{u} = 0$, $\boldsymbol{\vartheta} = 0$ or $\mathbf{v}^* = \mathbf{v}$, $\boldsymbol{\nu}^* = \boldsymbol{\nu}$.

Let us now suppose that the boundary of N consists of *rigid fixed* surfaces. Any motion initially prevailing would presumably die out as there is no supply of energy. If the basic motion $(\mathbf{v}, \boldsymbol{\nu})$ is chosen to be the trivial one so that $\mathbf{v} = \boldsymbol{\nu} = 0$, we have from (22) that

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leqslant -\left\{\left(\frac{160\mu}{\rho d^2}\right)T_1 + \left(\frac{2k + (6\pi^2 a/d^2)}{\rho j}\right)T_2\right\}. \tag{35}$$

From this we see that the energy T of an arbitrary flow $(\mathbf{u}, \boldsymbol{\vartheta})$ tends to zero according to the law

$$T \leqslant T(0)\exp(-st) \tag{36}$$

where

$$-s = \max\left\{-\frac{160\mu}{\rho d^2}, -\frac{2k + (6\pi^2 a/d^2)}{\rho j}\right\}. \tag{37}$$

A similar but different estimate has been obtained earlier by the author[6].

3. VARIATIONAL ALGORITHM

The theorems concerning the stability criterion noticed in the above section are universal in the sense that they do not depend on the special features of either the geometry or the distribution of the basic field variables. This generality cuts across the sharpness of the limits for stability of the flow. It is however evident from the nature of the results that there must exist a finite Reynolds number and a finite microrotational Reynolds number which are sharp estimates for the stability criterion. In Newtonian or non-polar flows the variation technique has been employed [1] for obtaining (theoretically at least) the precise estimate for stability and this technique has been extended also to the case of thermally-driven convective flows [2]. For the micropolar flows also we can develop the variational technique.

We shall alter the scales in the variables and the starred symbols below denote the non-dimensionalized quantities. Soon after their introduction the star superfixes are removed and all the equations below will be in their non-dimensional forms even though without the asterisks. The scale alterations are

$$t^* = [m(1+m_2)/m_4]t \tag{38}$$

$$\mathbf{u}^* = [m_4/(1+m_2)md]\mathbf{u} \tag{39}$$

$$D^* = D/m, E^* = E/n \tag{40}$$

$$\boldsymbol{\vartheta}^* = [Rm/(1+m_2)nd]\boldsymbol{\vartheta}. \tag{41}$$

From the earlier equations (8) and (9) in section 1, we can deduce their non-dimensional equivalents in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u^2/2 = -[Re/(1+m_2)]\int \mathbf{u}\,.\,D\,.\,\mathbf{u} + [m_2/(1+m_2)m_1]\int \boldsymbol{\vartheta}\,.\,\mathrm{curl}\,\mathbf{u} - \int (\mathrm{curl}\,\mathbf{u})^2 \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \vartheta^2/2 = -[Rm/(1+m_2)]\int \mathbf{u}\,.\,E\,.\,\boldsymbol{\vartheta} + [m_2/(1+m_2)]\int \boldsymbol{\vartheta}\,.\,\mathrm{curl}\,\mathbf{u} - [2m_2/(1+m_2)m_1]\int \vartheta^2 - [m_3/(1+m_2)m_1]\int (\mathrm{curl}\,\boldsymbol{\vartheta})^2 - [m_3m_7/(1+m_2)m_1]\int (\mathrm{div}\,\boldsymbol{\vartheta})^2. \tag{43}$$

To determine the best possible estimate for the Reynolds number to ensure the stability of the basic flow, we now observe that the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (u^2/2 + \lambda\vartheta^2/2) = \frac{\mathrm{d}}{\mathrm{d}t}(T_1 + \lambda T_2) \leqslant 0 \tag{44}$$

must hold for all $t > 0$ and for *every* fixed positive value of λ. This criterion provides for possible inducement of instability by the interaction effects between the velocity and microrotation fields when viewed in terms of the individual energy terms T_1 and T_2. However the combination $T_1 + \lambda T_2$ which indeed plays the role of an energy for the entire system of the field equations, does decrease monotonically when the condition (44) holds for all $t > 0$. We choose

$$\lambda = [(3m_2+4)-\{8(m_2+1)(m_2+2)\}^{1/2}]/m_2 \tag{45}$$

so that λ is fixed and $0 < \lambda < 1$. Introduce the quantity

$$f = (1+m_1\lambda)m_2/2(1+m_2)m_1. \tag{46}$$

The energy equation then takes the form

$$\begin{aligned}\frac{d}{dt}\int (u^2/2+\lambda\boldsymbol{\vartheta}^2/2) = &-[Re/(1+m_2)]\int \mathbf{u}\,.\,D\,.\,\mathbf{u}-[\lambda Rm/(1+m_2)]\int \mathbf{u}\,.\,E\,.\,\boldsymbol{\vartheta}\\ &-\int(\operatorname{curl}\mathbf{u}-f\boldsymbol{\vartheta})^2-[m_3\lambda/(1+m_2)m_1]\int(\operatorname{curl}\boldsymbol{\vartheta})^2\\ &-[m_3m_7\lambda/(1+m_2)m_1]\int(\operatorname{div}\boldsymbol{\vartheta})^2.\end{aligned} \tag{47}$$

The statement in (44) will be true if and only if we have

$$Re\,\{I_1(\mathbf{u})+I_2(\mathbf{u},\boldsymbol{\vartheta})\}+D(\mathbf{u},\boldsymbol{\vartheta}) \geqslant 0 \tag{48}$$

for every fixed value of Rm/Re, the quantities I_1, I_2, D being defined by

$$I_1(\mathbf{u}) = [1/(1+m_2)]\int \mathbf{u}\,.\,D\,.\,\mathbf{u} \tag{49}$$

$$I_2(\mathbf{u}\ \boldsymbol{\vartheta}) = [\lambda Rm/Re(1+m_2)]\int \mathbf{u}\,.\,E\,.\,\boldsymbol{\vartheta} \tag{50}$$

$$\begin{aligned}D(\mathbf{u},\boldsymbol{\vartheta}) = &\int(\operatorname{curl}\mathbf{u}-f\boldsymbol{\vartheta})^2+[m_3\lambda/(1+m_2)m_1]\int(\operatorname{curl}\boldsymbol{\vartheta})^2\\ &+[m_3m_7\lambda/(1+m_2)m_1]\int(\operatorname{div}\boldsymbol{\vartheta})^2.\end{aligned} \tag{51}$$

We can now connect the problem of finding sharp estimates for the stability criterion to the variational problem of finding the maximum of

$$-\{I_1(\mathbf{u})+I_2(\mathbf{u},\boldsymbol{\vartheta})\} \tag{52}$$

among the class of vector functions $\mathbf{u}, \boldsymbol{\vartheta}$ which satisfy the constraint and the normalization condition

$$\operatorname{div}\mathbf{u} = 0 \tag{53}$$

$$D(\mathbf{u},\boldsymbol{\vartheta}) = 1 \tag{54}$$

in the region N and the boundary conditions

$$\mathbf{u} = 0,\ \boldsymbol{\vartheta} = 0 \text{ on } \varphi. \tag{55}$$

This gives rise to the variational problem

$$\delta\left\{I_1(\mathbf{u})+I_2(\mathbf{u},\boldsymbol{\vartheta})-P\operatorname{div}\mathbf{u}+\frac{1}{R}D(\mathbf{u},\boldsymbol{\vartheta})\right\} = 0 \tag{56}$$

in which $P = P(x, y, z, t)$ and R are the Lagrange multipliers. The Euler–Lagrange equations corresponding to (56) are

$$[2/(1+m_2)]\,\mathbf{u}\,.\,D + [\lambda Rm/Re(1+m_2)]\,\boldsymbol{\vartheta}\,.\,E^T - \operatorname{grad} P - (2f/R)\operatorname{curl}\boldsymbol{\vartheta} - (2/R)\nabla^2\mathbf{u} = 0 \quad (57)$$

and

$$[\lambda Rm/Re(1+m_2)]\,\mathbf{u}\,.\,E - (2f/R)\operatorname{curl}\mathbf{u} + (2f^2/R)\,\boldsymbol{\vartheta} + \frac{2m_3(1+m_7)\lambda}{m_1(1+m_2)R}\operatorname{grad}(\operatorname{div}\boldsymbol{\vartheta}) - [2m_3\lambda/m_1(1+m_2)R]\,\nabla^2\boldsymbol{\vartheta} = 0. \quad (58)$$

The scalar product of (57) with $\mathbf{u}$ and integration over the region yields the result

$$[2/(1+m_2)]\int \mathbf{u}\,.\,D\,.\,\mathbf{u} + [\lambda Rm/Re(1+m_2)]\int \mathbf{u}\,.\,E\,.\,\boldsymbol{\vartheta} - (2f/R)\int \mathbf{u}\,.\operatorname{curl}\boldsymbol{\vartheta} + (2/R)\int (\operatorname{curl}\mathbf{u})^2 = 0 \quad (59)$$

on applying the divergence theorem, using the constraint $\operatorname{div}\mathbf{u} = 0$ and the boundary conditions $\mathbf{u} = \boldsymbol{\vartheta} = 0$ on φ. Likewise, the scalar product of (58) with $\boldsymbol{\vartheta}$ and integration over the region yields the following result

$$[\lambda Rm/Re(1+m_2)]\int \mathbf{u}\,.\,E\,.\,\boldsymbol{\vartheta} - (2f/R)\int \boldsymbol{\vartheta}\,.\operatorname{curl}\mathbf{u} + (2f^2/R)\int \boldsymbol{\vartheta}^2 - \frac{2m_3(1+m_7)\lambda}{m_1(1+m_2)R}\int (\operatorname{div}\boldsymbol{\vartheta})^2 + 2m_3\lambda/m_1(1+m_2)R\int \{(\operatorname{div}\boldsymbol{\vartheta})^2 + (\operatorname{curl}\boldsymbol{\vartheta})^2\} = 0. \quad (60)$$

Addition of (59), (60) and the use of (54) show that any solution of the system comprising the variational equations (57), (58) and the ancillary conditions (53), (54), (55) will satisfy the relation

$$-\{I_1(\mathbf{u}) + I_2(\mathbf{u}\ \boldsymbol{\vartheta})\} = 1/R. \quad (61)$$

It is already known that for the maximization problem stated in (52), (53), (54), (55) solutions $\tilde{u}$, $\tilde{\boldsymbol{\vartheta}}$ exist and these are also eigen-functions of the corresponding system of variational equations with the eigen value

$$1/\tilde{R} = -\{I_1(\tilde{u}) + I_2(\tilde{u}, \tilde{\vartheta})\} = \max\{I_1(\mathbf{u}) + I_2(\mathbf{u}, \boldsymbol{\vartheta})]\}. \quad (62)$$

From (61) and (62) we have

$$R \geqslant \tilde{R}$$

for any eigenvalue $\tilde{R}$. We can now deduce the following theorem and the proof is identical with that of a corresponding result (theorem 3) in reference [2].

Theorem 4. Let $\mathbf{u}$, $\boldsymbol{\vartheta}$ be solutions of the variational problem for the system defined in (52), (53), (53), (54), (55) above for fixed values of m_1, m_2, m_3, m_7 and Rm/Re. The eigenvalue problem stated in (53), (55), (57), (58) has then a least eigenvalue $\tilde{R}$ and the basic flow will be stable whenever $Re < \tilde{R}$.

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Résumé – Cet article expose la méthode de l'énergie pour obtenir des critères de stabilité du mouvement d'un fluide micropolaire incompressible dans un domaine arbitraire. Une formule est obtenue pour la variation en fonction du temps de l'énergie cinétique de la différence de deux écoulements et il est montré que l'écoulement initial est stable lorsque $Re+0{,}5\ Rm < 80$ et $Rm < 2m_2+6\pi^2 m_9$. Les quantités m_2 et m_9 sont des constantes matérielles du fluide et Re, Rm représentent respectivement le nombre de Reynolds et le nombre de Reynolds microrotationnel. Une forme différente du critère de stabilité est également relevée et un théorème est déduit concernant l'unicité des écoulements micropolaires incompressibles stables. Finalement un algorithme varationnel est établi pour la stabilité d'un écoulement micropolaire et il peut être utilisé pour préciser l'évaluation du nombre de Reynolds en dessous duquel l'écoulement est stable.

Zusammenfassung – Die Arbeit benützt die Energiemethode um Kriterien für die Stabilität der Bewgung einer inkompressiblen mikropolaren Flüssigkeit in einem willkürlichen Gebiet zu erhalten. Eine Formel für das Zeitmass der Änderung der kinetischen Energie der Differenz zweier Strömungen wird erhalten und es wird gezeigt, dass die ursprüngliche Strömung stabil ist wenn $Re+0{,}5\ Rm < 80$ und $Rm < 2m_2+6\pi^2 m_9$. Die Grössen m_2 und m_9 sind Materialkonstanten der Flüssigkeit und Re, Rm bezeichnen die Reynolds-Zahl und die Mikrodreh-Reynolds-Zahl, beziehungsweise. Eine verschiedene Form des Stabilitätskriteriums wird auch bemerkt und ein Theorem betreffend die Einzigkeit stetiger, inkompressibler mikropolarer Strömungen wird abgeleitet. Letzlich wird ein Variationsalgorithmus für die Stabilität einer mikropolaren Strömung aufgestellt, und dieser kann dazu verwendet werden, die Schätzung der Reynolds-Zahl zu verfeinern, unter der die Strömung stabil ist.

Sommario – Nell'articolo l'A. adopera il metodo d'energia per ottenere criteri sulla stabilità del moto di un fluido micropolare incomprimibile in un campo arbitrario. Ricava una formula per il tempo-ritmo di cambio dell'energia cinetica della differenza di due flussi e dimostra che il flusso originale è stabile quando $Re+0{,}5\ Rm < 80$ e $Rm < 2m_2+6\pi^2 m_9$. Le quantità m_2 ed m_9 sono costanti materiali del fluido e Re, Rm denotano il numero di Reynolds e il numero microrotativo di Reynolds rispettivamente. Si nota anche una forma diversa di criterio di stabilità e si ricava un teorema riguardante l'unicità dei flussi micropolari costanti e incomprimibili. Per ultimo, stabilisce un algoritma variazionale per 1_a stabilità di un flusso micropolare, che può essere impiegato per rendere piú precisa la stima del numero di Reynolds sotto il quale si ha stabilità di flusso.

Абстракт—Изложено применение энергетического метода, чтобы получивь критерии устойчивости для движения несжимаемой, микрополярной жидкости в произвольной области. Получена формула для темпа изменения во времени по кинетической знергии разницы двух потоков, показано что оригинальный поток устойчиво, когда $Re + 0.5\ Rm < 80$, $Rm < 2m_2 + 6\pi^2 m_9$, где m_2, m_9 — материальные постоянные жидкости, Re и Rm— число Рейнольдса и микроротационное число Рейнольдса соответсвенно. Замечена критерия устойчивости различного вида, дан вывод теорема о единственности невозмущенных, несжинаемых микрополярных потоков. Окончательно, установлен вариационный алгоритм для уетойчивости микрополярного потока, что можно применить, чтобы уточнить оценку числа Рейнольдса, ниже которого поток устойчиво.