Int. J. Engng Sci. Vol. 7, pp. 905–916. Pergamon Press 1969. Printed in Great Britain

SLOW STEADY ROTATION OF A SPHERE IN A MICRO-POLAR FLUID

S. K. LAKSHMANA RAO, N. C. PATTABHI RAMACHARYULU and P. BHUJANGA RAO
Department of Mathematics, Regional Engineering College, Warangal-4, Andhra Pradesh, India

Abstract—The Stokes' flow due to the slow steady rotation of a sphere in a micro-polar liquid which can sustain anti-symmetric stress and couple stress is investigated. Expressions are obtained for the velocity and the micro-rotation as well as the couple on the body. It is found that the couple is more in the present theory than in that of classical viscous fluids. The figures in the end show the distribution of the physical quantities near the body.

1. INTRODUCTION

IN A RECENT paper, Eringen[1] has given a theory of 'simple micro-fluids' characterized by properties and behaviour of the fluent medium that are affected by the local motions of the material particles in each volume element. These fluids possess local inertia and new principles of continuum mechanics are needed for their study. The new concept of the inertial spin, body moments, micro-stress averages and stress moments are necessarily introduced in this theory along with the principles of conservation of micro-inertia moments as well the balance of first stress moments. Simple micro-fluids are viscous fluids in which the constitutive equations express the stress tensor (t_{kl}), the micro-stress average (S_{kl}) and the first stress moments (λ_{klm}) as functions of the velocity gradients (d_{kl}), the gyration tensor (ν_{kl}) and its gradient. These relations are subject to spacial and material objectivity. In the somewhat simplified case of the linear theory of simple-micro-fluids, the stress and micro-stress average are linear expressions in the rate of deformation tensor (d) and micro-deformation rate tensor (b) and there are in all twenty two viscosity coefficients in this model.

A simple micro-fluid is called 'micro-polar' if for all motions (the first stress moments) λ_{klm} and the gyration tensor ν_{kl} satisfy the condition (i) $\lambda_{klm} = -\lambda_{kml}$ and (ii) $\nu_{kl} = -\nu_{lk}$. This is a further specialization of the linear theory of simple micro-fluids and A. C. Eringen has proposed a theory of such fluids also[2]. These exhibit micro-rotational effects and can support surface and body couples. Some anisotropic fluids such as animal blood and liquid crystals made up of bar-like or dumb-bell-shaped molecules seem to fall within the micro-polar fluid theory. This is a theory of structured continua obtained by assigning continuous fields to the average per unit volume of the respective moments of the dynamical, kinematical, and structural properties of the molecular distributions. The flow field at a point is defined as the local average translational velocity of the molecules, and the local average rotational velocity of the molecules is defined as the spin field in the micro-polar theory. This spin field is dynamically coupled with the fluid velocity by means of the collisional interactions of the molecules. The coupling between these two fields is a manifestation of non-central inter-molecular forces.

In the theory of micro-polar fluids, the field equations are presentable in terms of the velocity vector (q_i) and the micro-rotation vector $(\nu_i = (1/2)\,\epsilon_{ikl}\nu_{kl})$. The field equations involve only six material constants. Eringen has examined the steady flow of a micro-polar liquid in a straight circular tube under the influence of a constant pressure gradient and shown that the velocity profile is no longer parabolic[2]. T. Ariman and A. S. Cakmak have given one dimensional steady flow profile for a micro-polar fluid between two parallel plates[3].

In this paper, we examine the problem of slow steady rotation of a sphere about its diameter in a micro-polar fluid. This corresponds to the well known Stokesian problem for the rotation of a sphere in a classical viscous liquid[4].

2. BASIC EQUATIONS

The field equations of micro-polar fluid dynamics are

$$\frac{d\rho}{dt}+\rho \operatorname{div} \mathbf{q}=0 \tag{1}$$

$$\rho\frac{d\mathbf{q}}{dt}=(\lambda_1+2\mu+k)\operatorname{grad}(\operatorname{div}\mathbf{q})-(\mu+k)\operatorname{curl}\operatorname{curl}\mathbf{q}+k\operatorname{curl}\boldsymbol{\nu}-\operatorname{grad}p+\rho\mathbf{f} \tag{2}$$

and

$$j\rho\frac{d\nu}{dt}=(\alpha+\beta+\gamma)\operatorname{grad}(\operatorname{div}\boldsymbol{\nu})-\gamma\operatorname{curl}\operatorname{curl}\mathbf{q}+k\operatorname{curl}\mathbf{q}-2k\boldsymbol{\nu}+\mathbf{l} \tag{3}$$

in which $\mathbf{q}$, ν, $\mathbf{f}$ and $\mathbf{l}$ represent respectively the velocity, micro-rotation, body force, and body couple vectors. The constants ρ and j are the density and gyration parameters while (μ, k, λ_1) and (α, β, γ) are material constants. The constants in the first group are viscosity coefficients with the dimensions M/LT and those in the second group have the dimensions ML/T. These constants conform to the inequalities:

$$\mu\geq 0;k\geq 0;\gamma\geq 0;|\beta|\leq\gamma;3\lambda_1+2\mu+k>0;3\alpha+\beta+\gamma\geq 0;3\alpha+2\gamma\geq 0;$$
$$\alpha+\beta+\gamma\geq 0. \tag{4}$$

The stress tensor t_{kl} and the couple stress tensor $m_{kl}=-\epsilon_{lij}\lambda_{kij}$ are given by

$$t_{kl}=\lambda_1 u_{r,r}\delta_{kl}+\tfrac{1}{2}(2\mu+k)(u_{k,l}+u_{l,k})+k\epsilon_{klm}(\omega_m-\nu_m) \tag{5}$$

and

$$M_{kl}=\alpha\nu_{r,r}\delta_{kl}+\beta\nu_{k,l}+\gamma\nu_{l,k} \tag{6}$$

wherein u_i and ν_i denote the components of the velocity $\mathbf{q}$ and the micro-rotation $\boldsymbol{\nu}$. Also $2\omega_i$ are the components of the vorticity vector.

When $k=0$, the field equations (2) and (3) are decoupled and the stress tensor is symmetric. The global motion of the fluid is then unaffected by the micro-rotation of the fluid particles.

3. SLOW ROTATION OF A SPHERE ABOUT A DIAMETER

Let (r, θ, ϕ) denote spherical polar coordinates and $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ represent unit vectors in the directions of (r, θ, ϕ) respectively and the sphere $r=a$ rotate with the angular velocity Ω about its diameter along $\theta=0$ in a micro-polar liquid. The velocity $\mathbf{q}$ is then in the direction of the $\mathbf{e}_\phi$ and the micro-rotation vector $\boldsymbol{\nu}$ is in the meridian plane.

Let

$$\mathbf{q} = W(r, \theta)\mathbf{e}_\phi$$

and

$$\boldsymbol{\nu} = A(r, \theta)\mathbf{e}_r + B(r, \theta)\mathbf{e}_\theta. \tag{7}$$

Under the assumption of the Stokesian flow (i.e. with the omission of the inertial terms) and neglecting the body forces and couples, the flow is described by the following equations:

$$\frac{\partial p}{\partial r} = 0; \quad \frac{\partial p}{\partial \theta} = 0; \tag{8}$$

$$0 = (\mu + k)DW + kg(r, \theta), \tag{9}$$

$$0 = (\alpha+\beta+\gamma)\frac{\partial f}{\partial r} - \frac{\gamma}{r}\left(\frac{\partial g}{\partial \theta} + g\cot\theta\right) + \frac{k}{r}\left(\frac{\partial W}{\partial \theta} + W\cot\theta\right) - 2kA \tag{10}$$

$$0 = (\alpha+\beta+\gamma)\frac{1}{2}\frac{\partial f}{\partial \theta} + \gamma\left(\frac{\partial g}{\partial r} + \frac{g}{r}\right) - k\left(\frac{\partial W}{\partial r} + \frac{W}{r}\right) - 2kB \tag{11}$$

where the functions $f(r, \theta)$ and $g(r, \theta)$ represent

$$f(r, \theta) = \operatorname{div}\boldsymbol{\nu} = \frac{\partial A}{\partial r} + \frac{2A}{r} + \frac{1}{r}\frac{\partial B}{\partial \theta} + \frac{B\cot\theta}{r} \tag{12}$$

and

$$g(r, \theta) = [\operatorname{curl}\boldsymbol{\nu}]_\phi = \frac{\partial B}{\partial r} + \frac{B}{r} - \frac{1}{r}\frac{\partial A}{\partial \theta}$$

and

$$D = \nabla^2 - \frac{1}{r^2\sin^2\theta} \tag{13}$$

with ∇^2 representing the Laplacian operator in spherical polar coordinates viz.,

$$\nabla^2 = \frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}.$$

Equation (8) yields a constant pressure p throughout the flow-region.

From the above set of equations, it is noted that the velocity satisfies the equation

$$D\left(D - \frac{\lambda^2}{a^2}\right)W = 0 \tag{14}$$

which is of the 4th order reducible in terms of the operator D and the function f satisfies the relation:

$$\left(\nabla^2-\frac{c^2}{a^2}\right)f=0 \tag{15}$$

wherein

$$\frac{\lambda^2}{a^2}=\frac{k(2\mu+k)}{\gamma(\mu+k)}\text{ and }\frac{c^2}{a^2}=\frac{2k}{\alpha+\beta+\gamma}. \tag{16}$$

It is interesting to note that $g(r,\theta)=[\text{curl}\,\boldsymbol{\nu}]_\phi$ satisfies the relation

$$\left(D-\frac{\lambda^2}{a^2}\right)g=0. \tag{17}$$

From equations (15) and (17), both the divergence and the only non-vanishing component of the curl of the micro-rotation vector $\boldsymbol{\nu}$ can be easily determined after obtaining the solution of (14).

The functions A and B are obtained from equations (10) and (11) in terms of two functions W and f. Hence the problem now reduces to the solution of the two differential equations (14) and (15) with the regularity of the flow functions and the conditions of zero slip and spin on the boundary.

The solution of (14) can be expressed in the form

$$W=W_1+W_2 \tag{18}$$

where

$$DW_1=0 \tag{19a}$$

and

$$\left(D-\frac{\lambda^2}{a^2}\right)W_2=0. \tag{19b}$$

We find that

$$W_1=\langle r^n;r^{-(n+1)}\rangle\cdot\langle P_n^{(1)}(\cos\theta);Q_n^{(1)}(\cos\theta)\rangle \tag{20}$$

and

$$W_2=\left\langle r^{-(1/2)}I_{n+1/2}\left(\frac{\lambda r}{a}\right);\quad r^{-(1/2)}K_{n+1/2}\left(\frac{\lambda r}{a}\right)\right\rangle\cdot\langle P_n^{(1)}(\cos\theta);Q_n^{(1)}(\cos\theta)\rangle, \tag{21}$$

$$n=1,2,3,\ldots$$

The parenthesis $\langle\ \rangle$ indicates a linear combination of the arguments concerned and $P_n^{(1)}(\cos\theta)$ and $Q_n^{(1)}(\cos\theta)$ denote the associated Legendre functions. The portion W_1 of (18) given in (20) occurs in the corresponding problem in the classical viscous theory, i.e. the solution of equation (2) with the suppression of the term containing $\boldsymbol{\nu}$.

Equation (15) yields the solution

$$f=\left\langle r^{-(1/2)}I_{n+1/2}\left(\frac{cr}{a}\right);\quad r^{-(1/2)}K_{n+1/2}\left(\frac{cr}{a}\right)\right\rangle\langle P_n(\cos\theta);\quad Q_n(\cos\theta)\rangle \tag{22}$$

$n=0,1,2,3,\ldots$

Since the velocity W equals $\Omega a \sin\theta$ on the boundary $r = a$ and is regular at infinity, we take only $r^{-(n+1)} P_n^{(1)}(\cos\theta)$ in (20) and $r^{-(1/2)} K_{n+1/2}\left(\frac{\lambda r}{a}\right) P_n^{(1)}(\cos\theta)$ in (21) with $n = 1$. The regularity for f at infinity and its connection with W as seen from (10–12) require the omission of $I_{n+1/2}(cr/a)$ and the choice of $n = 1$ in (22) also. The functions $Q_n^{(1)}(\cos\theta)$ in (20–22) are, of course, omitted by the consideration of regularity on the axis of rotation for which $\cos\theta = \pm 1$. Thus we have

$$W_1 = \frac{P_1}{r^2}\sin\theta; \tag{23}$$

$$W_2 = \frac{P_2}{r^{1/2}} K_{3/2}\left(\frac{\lambda r}{a}\right)\sin\theta; \tag{24}$$

$$W = W_1 + W_2 \tag{25}$$

and

$$f = \frac{P_3}{r^{1/2}} K_{3/2}\left(\frac{cr}{a}\right)\cos\theta, \tag{26}$$

where P_1, P_2 and P_3 are constants to be determined.

Determination of A(r, θ) *and* B(r, θ)

From equation (9), we get

$$(\text{curl}\,\boldsymbol{\nu})_\phi = g(r, \theta) = -\frac{\mu + k}{k} DW = -\frac{\lambda^2}{a^2}\frac{\mu + k}{k} W_2. \tag{27}$$

When (27) is substituted in (10) and (11), we obtain that

$$\begin{aligned} A &= \frac{1}{2r}\left(\frac{\partial}{\partial\theta} + \cot\theta\right) W_1 + \frac{1}{r}\frac{\mu + k}{k}\left(\frac{\partial}{\partial\theta} + \cot\theta\right) W_2 + \frac{a^2}{c^2}\frac{\partial f}{\partial r} \\ &= \left[\frac{P_1}{r^3} + \frac{2(\mu + k)}{k r^{3/2}} P_2 K_{3/2}\left(\frac{\lambda r}{a}\right) - \frac{P_3 a^2}{c^2 r^{3/2}}\left\{2K_{3/2}\left(\frac{cr}{a}\right) + \frac{cr}{a} K_{1/2}\left(\frac{cr}{a}\right)\right\}\right]\cos\theta \end{aligned} \tag{28}$$

and

$$\begin{aligned} B &= -\frac{1}{2}\left(\frac{\partial W_1}{\partial r} + \frac{W_1}{r}\right) - \frac{\mu + k}{k}\left(\frac{\partial W_2}{\partial r} + \frac{W_2}{r}\right) + \frac{a^2}{c^2 r}\frac{\partial f}{\partial\theta} = \left[\frac{P_1}{2r^3} + \frac{\mu + k}{k} P_2 \right. \\ &\quad \left. \times\left\{r^{-3/2} K_{3/2}\left(\frac{\lambda r}{a}\right) + \frac{r^{-(1/2)}}{a} K_{1/2}\left(\frac{\lambda r}{a}\right)\right\} - \frac{P_3 a^2}{c^2 r^{3/2}} K_{3/2}\left(\frac{cr}{a}\right)\right]\sin\theta. \end{aligned} \tag{29}$$

The conditions of zero-slip and spin on the boundary $r = a$ lead to the equations

$$W(a, \theta) = \Omega a \sin\theta; \quad A(a, \theta) = 0; \quad B(a, \theta) = 0 \tag{30}$$

from which the constants P_1, P_2, P_3 in the equations (23), (24) and (26) are determined.

We find that

$$P_1 = 2\Omega a^3(\mu+k)\,[(2+2c+c^2)\lambda^2+c^2(1+\lambda)]/D;$$
$$P_2 = -\Omega k\lambda a^{3/2}c^2/[DK_{1/2}(\lambda)] \tag{31}$$

and

$$P_3 = 2\Omega a^{-1/2}c^3\lambda^2(\mu+k)/[DK_{1/2}(c)]$$

where

$$D = 2(\mu+k)(2+2c+c^2)\lambda^2+c^2(2\mu+k)(1+\lambda).$$

Stresses

The stress tensor t_{kl} is given by (5). In the present problem, the strain-velocity tensor has its only non-vanishing component

$$d_{r\theta} = \frac{1}{2}\left(\frac{\partial W}{\partial r}-\frac{W}{r}\right) \tag{32}$$

and the vorticity components are

$$\omega_r = \frac{W\cot\theta}{r};\quad \omega_\theta = \frac{1}{2}\left(\frac{\partial W}{\partial r}+\frac{W}{r}\right);\quad \omega_\phi = 0. \tag{33}$$

Hence we have the stress-distribution

$$t_{rr} = t_{\theta\theta} = t_{\phi\phi} = -p \text{ (a constant)}; \tag{34}$$

$$t_{r\theta} = t_{\theta r} = 0; \tag{35}$$

$$t_{\theta\phi} = -t_{\phi\theta} = k\left(\frac{W\cot\theta}{r}-A\right) = \left[-(2\mu+k)P_2r^{-3/2}\left(1+\frac{a}{\lambda r}\right)K_{1/2}\left(\frac{\lambda r}{a}\right)\right.$$
$$\left.+kP_3\frac{a}{c}r^{-1/2}\left(1+\frac{2a}{cr}+\frac{2a^2}{c^2r^2}\right)K_{1/2}\left(\frac{cr}{a}\right)\right]\cos\theta; \tag{36}$$

$$t_{\phi r} = -\left[\frac{3(2\mu+k)}{2r^3}P_1+(2\mu+k)P_2r^{-3/2}\left(\frac{\lambda r}{a}+2+\frac{2a}{\lambda r}\right)K_{1/2}\left(\frac{\lambda r}{a}\right)\right.$$
$$\left.-kP_3\frac{a^2}{c^2}r^{-3/2}\left(1+\frac{a}{cr}\right)K_{1/2}\left(\frac{cr}{a}\right)\right]\sin\theta; \tag{37}$$

and

$$t_{r\phi} = -\left[\frac{3(2\mu+k)}{2r^3}P_1+(2\mu+k)P_2r^{-3/2}\left(1+\frac{a}{\lambda r}\right)K_{1/2}\left(\frac{\lambda r}{a}\right)\right.$$
$$\left.+kP_3\frac{a^2}{c^2}r^{-3/2}\left(1+\frac{a}{cr}\right)K_{1/2}\left(\frac{cr}{a}\right)\right]\sin\theta. \tag{38}$$

The couple stress m_{kl} defined in (6) has the components

$$m_{rr}=\alpha f+(\beta+\gamma)\frac{\partial A}{\partial r}$$

$$=\alpha P_3 r^{-1/2}K_{3/2}(cr/a)\cos\theta+(\beta+\gamma)\Big[-\frac{3P_1}{r^4}-2\frac{\mu+k}{k}P_2 r^{-5/2}\Big(\frac{\lambda r}{a}+3+\frac{3a}{\lambda r}\Big)K_{1/2}\Big(\frac{\lambda r}{a}\Big)$$

$$+P_3 r^{-1/2}\Big(1+\frac{3a}{cr}+\frac{6a^2}{c^2r^2}+\frac{6a^3}{c^3r^3}\Big)K_{1/2}\Big(\frac{cr}{a}\Big)\Big]\cos\theta; \tag{39}$$

$$m_{\theta\theta}=m_{\phi\phi}=\alpha f+(\beta+\gamma)\frac{A+B\cot\theta}{r}=\alpha P_3 r^{-1/2}K_{3/2}(cr/a)\cos\theta+(\beta+\gamma)\Big[\frac{3P_1}{2r^4}$$

$$+\frac{\mu+k}{k}P_2 r^{-5/2}\Big(\frac{\lambda r}{a}+3+\frac{3a}{\lambda r}\Big)K_{1/2}\Big(\frac{\lambda r}{a}\Big)-P_3\frac{a}{c}r^{-3/2}\Big(1+\frac{3a}{cr}+\frac{3a^2}{c^2r^2}\Big)K_{1/2}(cr/a)\Big]\cos\theta; \tag{40}$$

$$m_{r\theta}=-(\beta+\gamma)\Big[\frac{3P_1}{2r^4}+\frac{\mu+k}{k}P_2 r^{-5/2}\Big(\frac{\lambda r}{a}+3+\frac{3a}{\lambda r}\Big)K_{1/2}\Big(\frac{\lambda r}{a}\Big)-P_3\frac{a}{c}r^{-3/2}\Big(1+\frac{3a}{cr}+\frac{3a^2}{c^2r^2}\Big)$$

$$\times K_{1/2}\Big(\frac{cr}{a}\Big)\Big]\sin\theta-\gamma\frac{\mu+k}{k}P_2\frac{\lambda^2}{a^2}r^{-1/2}\Big(1+\frac{a}{\lambda r}\Big)K_{1/2}\Big(\frac{\lambda r}{a}\Big)\sin\theta; \tag{41}$$

$$m_{\theta r}=-(\beta+\gamma)\Big[\frac{3P_1}{2r^4}+\frac{\mu+k}{k}P_2 r^{-5/2}\Big(\frac{\lambda r}{a}+3+\frac{3a}{\lambda r}\Big)K_{1/2}\Big(\frac{\lambda r}{a}\Big)-P_3\frac{a}{c}r^{-3/2}\Big(1+\frac{3a}{cr}+\frac{3a^2}{c^2r^2}\Big)$$

$$\times K_{1/2}\Big(\frac{cr}{a}\Big)\Big]\sin\theta-\beta\frac{\mu+k}{k}P_2\frac{\lambda^2}{a^2}r^{-1/2}\Big(1+\frac{a}{\lambda r}\Big)K_{1/2}(\lambda r/a)\sin\theta; \tag{42}$$

and

$$m_{\theta\phi}=m_{\phi\theta}=m_{r\phi}=m_{\phi r}=0. \tag{43}$$

Couple on the body

The couple on the sphere has contributions from the stress t_{kl} as well as from the couple stress m_{kl}. The contribution from the stress is given by

$$N_s=\int \mathbf{r}\times(\mathbf{n}:t)\,.\,\mathbf{K}\,\mathrm{d}s \tag{44}$$

wherein $\mathbf{r}=a\mathbf{e}_r$; $\mathbf{n}:t=t_{rr}\mathbf{e}_r+t_{r\theta}\mathbf{e}+t_{r\phi}\mathbf{e}_\phi$ and $\mathbf{K}$ is the unit vector in the direction of the axis of rotation. Also the integral extends over the surface of the sphere. We find that

$$N_S=2\pi a^3\int_0^\pi t_{r\phi}\Big|_{r=a}\sin^2\theta\,\mathrm{d}\theta$$

$$=-\frac{8\pi}{3}\Big[(2\mu+k)\Big\{\frac{3P_1}{2}+P_2 a^{3/2}\frac{(1+\lambda)}{\lambda}K_{1/2}(\lambda)+kP_3 a^{7/2}\frac{(1+c)}{c^3}K_{1/2}(c)\Big]. \tag{45}$$

The contribution from the couple stress is given by

$$N_c = \int \mathbf{n} : m|_{(r=a)} . \mathbf{K}\, ds \tag{46}$$

$$= 2\pi a^2 \int_0^\pi [m_{rr} \cos\theta - m_{r\theta} \sin\theta]_{r=a} \sin\theta\, d\theta$$

$$= \frac{8\pi}{3}\left[\frac{(\mu+k)\gamma\lambda(\lambda+1)}{k} P_2 a^{-1/2} K_{1/2}(\lambda) + \frac{ka^{7/2}}{c^3}(1+c) P_3 K_{1/2}(c)\right]. \tag{47}$$

The total couple on the body can now be obtained as

$$N_\lambda = N_S + N_c = -4\pi(2\mu+k)P_1$$

$$= \frac{-8\pi\Omega a^3(2\mu+k)(\mu+k)[(2+2c+c^2)\lambda^2 + c^2(1+\lambda)]}{2(\mu+k)(2+2c+c^2)\lambda^2 + c^2(2\mu+k)(1+\lambda)}. \tag{48}$$

The results for the case of the classical viscous flow can be recovered in the limit as $k \to 0$. In that case both λ and $c \to 0$ and it is seen that

$$P_1 \to \Omega a^3. \tag{49}$$

Hence the velocity is given by

$$W = \frac{\Omega a^3 \sin\theta}{r^2}. \tag{50}$$

The couple on the body can now be given by

$$N_0 = -8\pi\mu\Omega a^3. \tag{51}$$

These results agree with the well-known classical result of Stokes [4].

From equations (48) and (51) we can write

$$N_\lambda / N_0 = (R+S)/[R/(1+m) + S/(1+2m)] \tag{52}$$

where

$$R = (2+2c+c^2)\lambda^2; \quad S = (1+\lambda)c^2 \quad \text{and} \quad m = k/2\mu. \tag{53}$$

In view of equations (4) and (16), the constants R, S and m are positive and then from equation (52) we see that

$$N_\lambda / N_0 > 1. \tag{54}$$

This shows that the couple experienced by the sphere rotating in a micro-polar fluid is greater than that in the classical viscous liquid. More specifically we can see that

$$1 + \frac{k}{2\mu} < \frac{N_\lambda}{N_0} < 1 + \frac{k}{\mu}. \tag{55}$$

The following pictures show the distribution of the velocity, microrotation components, the shear stress differences and the couple stress components for the values

$$\frac{k}{\mu} = 10; \quad \lambda = 1; \quad c = 4 \quad \text{and} \quad \alpha/\gamma = \beta/\gamma = 0{\cdot}5. \tag{56}$$

It is clear from Fig. 1 that the velocity in the non-polar fluid is less than that in the micro-polar fluid.

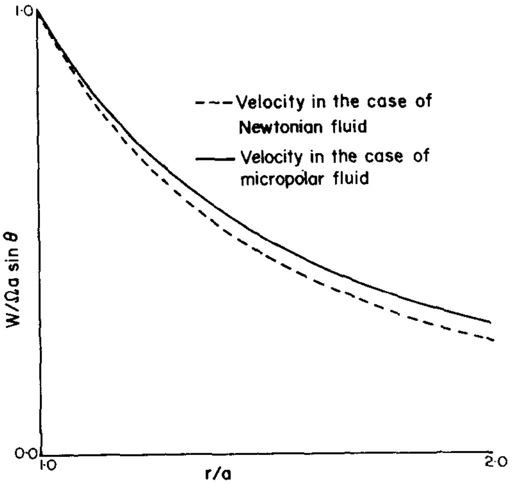

Fig. 1. Velocity profile.

Fig. 2. Micro-rotation, *A*.

Fig. 3. Micro-rotation, *B*.

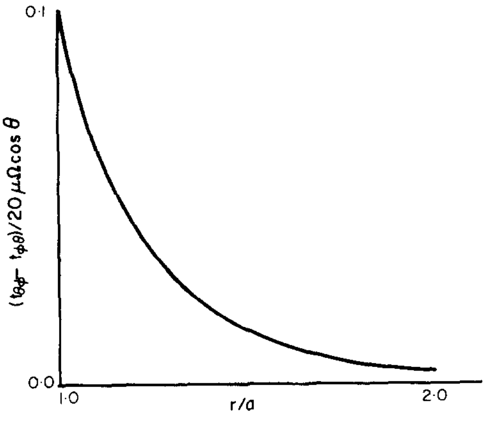

Fig. 4. Shear stress difference $t_{\theta\phi} - t_{\phi\theta}$.

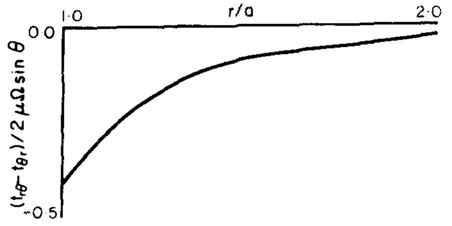

Fig. 5. Shear stress difference $t_{r\phi} - t_{\phi r}$.

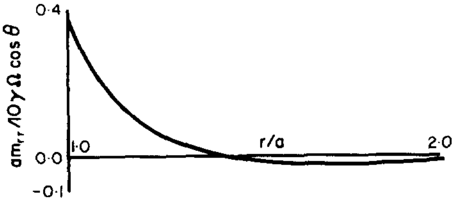

Fig. 6. Couple stress m_{rr}.

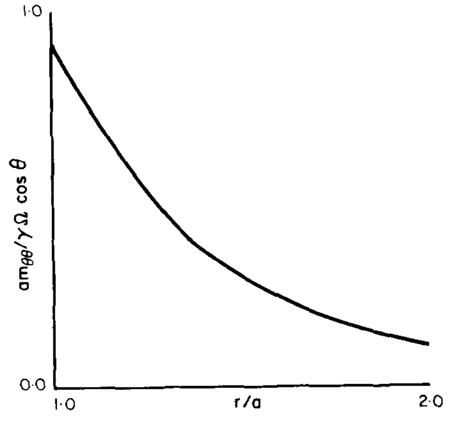

Fig. 7. Couple stress $m_{\theta\theta}$.

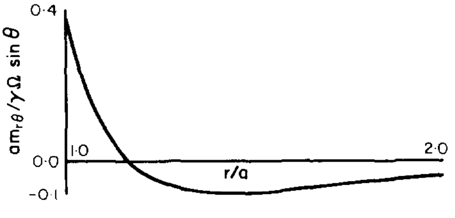

Fig. 8. Couple stress $m_{r\theta}$.

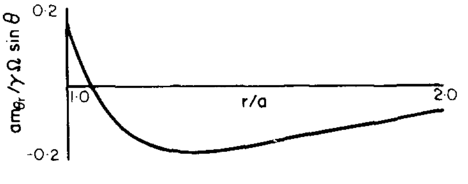

Fig. 9. Couple stress $m_{\theta r}$.

REFERENCES

[1] A. C. ERINGEN, *Int. J. Engng Sci.* **2**, 205 (1964).
[2] A. C. ERINGEN, *J. Math. Mech.* **16**, 1 (1966).
[3] T. ARIMAN and T. CAKMAK, *Physics Fluids* **10**, 2497 (1967).
[4] SIR H. LAMB, *Hydrodynamics*, p. 589. Dover.

(*Received* 14 *March* 1969)

Résumé—Dans le présent Rapport, on étudie l'écoulement de Stokes dû à la rotation lente et constante d'une sphère dans un liquide micropolaire qui peut subir une contrainte antisymétrique et une contrainte de couple. On obtient des expressions pour la vitesse et la microrotation, ainsi que pour le couple appliqué au corps. On trouve un couple plus élevé dans la présente théorie que dans celle des fluides visqueux classiques. Les chiffres donnés à la fin montrent quélle est la répartition des grandeurs physiques à proximité du corps.

Zusammenfassung—Die Stokes-Strömung, verursacht durch die stetige Drehung einer Kugel in einer mikropolaren Flüssigkeit, die antisymmetrische Spannung und Kräftepaarspannung aufrechterhalten kann, wird untersucht. Ausdrücke für die Geschwindigkeit und die Mikrodrehung sowohl wie für das Kräftepaar auf den Körper werden erhalten. Es wird gefunden, dass das Kräftepaar in der gegenwärtigen Theorie grösser ist als in der klassischer viskoser Flüssigkeiten. Die Abbildungen am Ende zeigen die Verteilung der physikalischen Grössen in der Nähe des Körpers.

Sommario—Si studia il flusso di Stokes causato dalla rotazione lenta e uniforme di una sfera in un liquido micropolare capace di sopportare sollecitazioni antisimmetriche e di coppia. Si ricavano espressioni per la velocità e la microrotazione, oltre che per la coppia sul corpo. Si scopre che la coppia è superiore nella presente teoria che in quella dei classici fluidi viscosi. Le cifre in calce indicano la distribuzione di quantitativi fisici in vicinanza del corpo.

Абстракт—Изучается течение Стоукса, вызываемое медленным постоянным вращением шара в микрополярной жидкости, в которой могут поддерживаться противосимметричные напряжения и моменты напряжений. Получаются выражения для скорости и микровращения, а также для момента приложенного к корпусу. Обнаруживается, что значение момента больше по настоящей теории, чем по теории для классических вязких жидкостей. В окончании приводятся данные, которые отображают распределение физических величин вблизи корпуса.