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UNIQUENESS OF COMPRESSIBLE MICROPOLAR FLUID FLOWS

S. K. LAKSHMANA RAO, K. VENKATAPATHI RAJU and T. K. V. IYENGAR
Department of Mathematics, Regional Engineering College, Warangal-506004, India

Abstract—The paper examines the uniqueness of compressible micropolar fluid flows over an arbitrary region $R(t)$ with a smooth boundary $\partial R(t)$. It is shown that there is at most one solution of the flow equations and boundary conditions which corresponds to suitably assigned initial values of the density, velocity, microrotation and temperature fields. The analysis rests on the use of differential inequalities involving the time derivatives of certain energy integrals.

INTRODUCTION

In this paper, we examine the uniqueness of compressible micropolar fluid flows[1] in an arbitrary bounded region. The question of uniqueness of viscous fluid flows has a long history and the uniqueness of incompressible viscous fluids was examined by Foá[2] in 1929. Extension of the enquiry on the uniqueness of flows to compressible viscous fluids was first considered by Graffi[3] who restricted the investigation to fluids satisfying the piezotropic relation $p = f(\rho)$ connecting the fluid density ρ and the pressure p. The uniqueness of viscous compressible flows with a general equation of state was taken up by Serrin[4] in a remarkable paper in 1959. Under very general conditions, Serrin was able to establish the uniqueness of viscous compressible flows over an arbitrary region R under diverse boundary conditions. His investigations cover also the nonviscous fluids and the results in both the cases are obtained by the use of the energy method. This method essentially consists of evolving differential inequalities for certain energy functionals over a time interval and it is interesting to note that such a method can be developed also for compressible micropolar fluids.

The theory of micropolar fluids[1] introduced by Eringen in 1966 differs from the classical theory of Navier–Stokes viscous fluids in two important features, viz. the sustenance of the couple stress in the fluid medium and the nonsymmetry of the force stress tensor. In this theory, the fluid element has the usual translatory degrees of freedom reckoned by the velocity vector $\mathbf{q}$ and has, in addition, degrees of freedom, enabling the intrinsic rotary motions of the fluid element. The latter motion is reckoned by the microrotation vector $\boldsymbol{\nu}$. The constitutive equations for the stress and couple stress in the case of a non-heat conducting micropolar fluid medium have been presented by Eringen[1] and when the model is assumed to be linear, these involve, in all, six material constants. In the present investigation, we have to take note of the heat conduction in the compressible micropolar fluid medium and accordingly modifications are necessary in the constitutive equations for the force stress tensor and the couple stress tensor given in[1]. The constitutive equations for heat-conducting micropolar fluids have already been given by Cowin[5, 6] as well as Kline and Allen[7].

For the linear model of heat-conducting micropolar fiuids, the constitutive equations for the force stress tensor t_{ij}, the couple stress tensor m_{ij} and the heat flux vector $\mathbf{h}$ are given by[5–7]

$$t_{ij} = (-p + \lambda_1 \text{Ⓗ})\delta_{ij} + (2\mu + k)d_{ij} + k\epsilon_{ijm}(\omega_m - \nu_m) \tag{1}$$

$$m_{ij} = \alpha(\text{div}\, \boldsymbol{\nu})\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i} + \sigma\epsilon_{ijm}\theta_{,m} \tag{2}$$

$$\mathbf{h} = -\chi_1 \,\text{grad}\, \theta + \chi_2 \,\text{curl}\, \boldsymbol{\nu} \tag{3}$$

where $\mathbf{q}$ is the fluid velocity vector, $\boldsymbol{\nu}$ is the microrotation vector, $\boldsymbol{\omega} = (1/2)\,\text{curl}\,\mathbf{q}$, d_{ij} is the rate of deformation tensor, $\nu_{i,j}r$ is the matrix gradient of the microrotation vector, Ⓗ $= \text{div}\,\mathbf{q}$, p is the thermodynamic pressure, θ is the temperature and ϵ_{ijm} is the alternating tensor. The material constants $\{\lambda_1, \mu, k\}$ are the viscosity coefficients, $\{\alpha, \beta, \gamma\}$ are the gyroviscosity coefficients, χ_1

is the thermal conductivity and $\{\chi_2, \sigma\}$ are measures of thermodynamic coupling. These material constants conform to the following inequalities [1, 6, 7]

$$3\lambda_1 + 2\mu + k \geq 0, \quad 2\mu + k \geq 0, \quad k \geq 0$$
$$3\alpha + \beta + \gamma \geq 0, \quad \gamma \geq 0, \quad |\beta| \leq \gamma \tag{4}$$

$$\chi_1 \geq 0, (\chi_2 + \theta\sigma)^2 \leq 2(\gamma - \beta)\chi_1\theta. \tag{5}$$

The motion of micropolar fluids is governed by the laws of conservation of mass, conservation of microinertia, the balance of momentum, the balance of first stress moments and the energy balance. The equations of motion are given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div} \mathbf{q} = 0 \tag{6}$$

$$\frac{\mathrm{d}j}{\mathrm{d}t} = 0 \tag{7}$$

$$\rho \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \rho \mathbf{f} - \operatorname{grad} p + k \operatorname{curl} \boldsymbol{\nu} + (\mu + k) \operatorname{curl} \operatorname{curl} \mathbf{q} + (\lambda_1 + 2\mu + k) \operatorname{grad} (\operatorname{div} \mathbf{q}) \tag{8}$$

$$\rho j \frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}t} = \rho \mathbf{l} - 2k\boldsymbol{\nu} + k \operatorname{curl} \mathbf{q} - \gamma \operatorname{curl} \operatorname{curl} \boldsymbol{\nu} + (\alpha + \beta + \gamma) \operatorname{grad} (\operatorname{div} \boldsymbol{\nu}) \tag{9}$$

$$\rho \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = t_{ij}d_{ij} + t_{ij}\epsilon_{ijm}(\omega_m - \nu_m) + m_{ij}\nu_{j,i} - \operatorname{div} \mathbf{h} + \rho r. \tag{10}$$

In the above equations, the scalars ρ, j, ϵ, r denote the fluid density, the gyration parameter, the internal energy density per unit mass and the heat source per unit mass, respectively while the vectors $\mathbf{f}$, $\mathbf{l}$ denote the body force per unit mass and the body couple per unit mass. In the sequel, the term ρr in the energy balance will be omitted as no heat sources are assumed to be present in the domain of the flow. The equation of state relating the thermodynamic variables ρ, p, ϵ and θ are taken to be

$$p = p(\rho, \theta) \tag{11}$$

$$\epsilon = \epsilon(\rho, \theta) \tag{12}$$

and both the functions p and ϵ are assumed to be functions of class $C^{(2)}$. To simplify the discussion in the sequel, the energy function $\epsilon(\rho, \theta)$ is assumed in the special form

$$\epsilon(\rho, \theta) = c_v(\theta) \tag{13}$$

and the specific heat $c_v = (\partial\epsilon/\partial\theta)_\rho$ is assumed to be a positive constant. The energy balance (10) can, therefore, be written in the form

$$\rho \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \rho c_v \frac{\mathrm{d}\theta}{\mathrm{d}t}$$
$$= \psi - \operatorname{div} \mathbf{h} \tag{14}$$

where

$$\psi = t_{ij}d_{ij} + t_{ij}\epsilon_{ijm}(\omega_m - \nu_m) + m_{ij}\nu_{j,i}. \tag{15}$$

We consider the fluid flow over the finite region $R = R(t)$ and the boundary ∂R of the region is assumed to be smooth. The flow velocity $\mathbf{q}$ and the microrotation vector $\boldsymbol{\nu}$ have prescribed values on the boundary $\partial R(t)$. The relative normal speed U of fluid particles at the boundary is defined in the form

$$U = \mathbf{q} \cdot \mathbf{n} - G \tag{16}$$

when $\mathbf{n}$ denotes the outer normal vector and G is the outward normal velocity of the boundary $\partial R(t)$.

BOUNDARY CONDITIONS

At all points of the boundary $\partial R(t)$ and at all instants of time in the range of interest, the velocity vector $\mathbf{q}$ and the microrotation vector $\boldsymbol{\nu}$ are specified. Further (a) at points where $U < 0$, the density ρ and the absolute temperature θ are prescribed; (b) at points where $U > 0$, the absolute temperature θ is prescribed; (c) at points where $U = 0$, any one of the following three alternative conditions is assumed: (i) the temperature θ is prescribed; (ii) the heat flux $\mathbf{h} \cdot \mathbf{n}$ is prescribed; (iii) the heat flux is assumed proportional to the difference between the fluid temperature θ and a given wall temperature θ_0, i.e.

$$\mathbf{h} \cdot \mathbf{n} = k_0(\theta - \theta_0),\ k_0 \geq 0. \tag{17}$$

The above conditions barring those pertaining to the microrotation vector $\boldsymbol{\nu}$, are the same as those adopted by Serrin[4] for proving the uniqueness of viscous compressible flows. Serrin's proof of the uniqueness theorem which we follow in general here rests on the use of the energy integrals and systematically employs the transport formula which can be expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R \rho F\,\mathrm{d}\tau = \int_R \rho \frac{\mathrm{d}F}{\mathrm{d}t}\,\mathrm{d}\tau - \oint_{\partial R} \rho U F\,\mathrm{d}a \tag{18}$$

where F is a continuously differentiable function over $R(t)$. In the sequel, the volume infinitesimal $\mathrm{d}\tau$ in the volume integrals and the surface area infinitesimal $\mathrm{d}a$ in the surface integrals are omitted.

STATEMENT OF THE UNIQUENESS THEOREM

Let the viscosity coefficients $\{\lambda_1, \mu, k\}$, the gyroviscosity coefficients $\{\alpha, \beta, \gamma\}$ the thermal conductivity χ_1 and the thermomechanical coupling coefficients $\{\chi_2, \sigma\}$ satisfy the inequalities (4) and (5) and let the relations (11) and (13) be valid. Then there can be at most one set of continuously differentiable functions $\{\rho, \mathbf{q}, \boldsymbol{\nu}, \theta\}$ satisfying the differential eqns (6)–(9) and the boundary conditions stated in the preceding paragraph and reducing when $t = 0$ to the assigned initial distributions.

To proceed with the uniqueness theorem, it is necessary to consider the basic equations governing the flow and all the constitutive equations in nondimensional form. With appropriate scaling, each of the eqns (1)–(3) and (6)–(17) can be considered to be a nondimensional statement.

A set of continuously differentiable functions $(\rho, \mathbf{q}, \boldsymbol{\nu}, \theta)$ satisfying the flow equations and the boundary conditions and reducing to assigned initial distributions when $t = 0$ is designated as a solution of the initial value problem. The solutions are presumed to exist over the interval $0 \leq t \leq \tau$ but may not be continuable beyond the value τ. The solutions considered are assumed to have positive density ρ and positive gyration parameter j at every point of the closure of the region.

To prove the uniqueness of micropolar fluid flows it is necessary to compare two possible flows pertaining to the same initial data and force the conclusion that the flows cannot be distinct. Let $(\rho, j, \mathbf{q}, \boldsymbol{\nu}, \theta)$ and $(\rho^*, j, \mathbf{q}^*, \boldsymbol{\nu}^*, \theta^*)$ be two possible solutions of the equations of micropolar fluid flows. The gyration parameter j is assumed to be the same in both the flows as a simplifying assumption though it is also possible to consider greater generality and allow variation in this parameter also. The specific heat c_v has also the same value for the two motions in view of the earlier assumptions (cf. the statement immediately after eqn (13)). The functions involved as well as the boundary ∂R are assumed to be smooth enough to validate the divergence theorem.

For any arbitrary flow quantity F, let

$$F' = F^* - F \tag{19}$$

denote the difference in its value pertaining to the two flows. It then follows that (cf. [4])

$$(AB)^* - (AB) = AB' + A'B^* \tag{20}$$

$$\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^* - \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right) = \frac{\mathrm{d}F'}{\mathrm{d}t} + \mathbf{q}' \,.\, \mathrm{grad}\, F^* \tag{21}$$

$$\left(\rho\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)^* - \left(\rho\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right) = \rho\,\frac{\mathrm{d}\mathbf{q}'}{\mathrm{d}t} + \rho(\mathbf{q}' \,.\, \mathrm{grad})\mathbf{q}^* + \rho'\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)^* \tag{22}$$

$$\left(\rho j\,\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}t}\right)^* - \left(\rho j\,\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}t}\right) = \rho j\,\frac{\mathrm{d}\boldsymbol{\nu}'}{\mathrm{d}t} + \rho j(\mathbf{q}' \cdot \mathrm{grad})\,\boldsymbol{\nu}^* + \rho' j\left(\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}t}\right)^* \tag{23}$$

$$\left(\rho c_v\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^* = \rho c_v\,\frac{\mathrm{d}\theta'}{\mathrm{d}t} + \rho c_v \mathbf{q}' \,.\, \mathrm{grad}\,\theta^* + \rho' c_v\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^* + \rho c_v\,\frac{\mathrm{d}\theta}{\mathrm{d}t}. \tag{24}$$

Since the continuity eqn (6) is satisfied by the primary flow (ρ, j, $\mathbf{q}$, $\boldsymbol{\nu}$, θ) as well as the starred flow (ρ^*, j, $\mathbf{q}^*$, $\boldsymbol{\nu}^*$, θ^*), it follows from (6), (20) and (21) that

$$\frac{\mathrm{d}\rho'}{\mathrm{d}t} + \mathbf{q}' \,.\, \mathrm{grad}\,\rho^* + \rho\,\mathrm{div}\,\mathbf{q}' + \rho'\,\mathrm{div}\,\mathbf{q}^* = 0. \tag{25}$$

Let

$$J_1 = \int \frac{1}{2}\rho\rho'^2. \tag{26}$$

Multiplying eqn (25) by ρ' it can be noted that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho'^2\right) &= \rho'\,\frac{\mathrm{d}\rho'}{\mathrm{d}t}\\ &= -\,\rho'\mathbf{q}'\,\mathrm{grad}\,\rho^* - \rho\rho'Ⓗ' - \rho'^2Ⓗ^*\end{aligned} \tag{27}$$

where $Ⓗ' = \mathrm{div}\,\mathbf{q}'$, $Ⓗ^* = \mathrm{div}\,\mathbf{q}^*$. From the transport formula (18), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\rho\rho'^2 = \int \frac{1}{2}\rho\frac{\mathrm{d}}{\mathrm{d}t}(\rho'^2) - \frac{1}{2}\oint \rho U(\rho'^2) \tag{28}$$

and this can be rewritten as

$$\frac{\mathrm{d}J_1}{\mathrm{d}t} = -\int \rho\rho'\,\mathbf{q}' \,.\, \mathrm{grad}\,\rho^* - \int \rho^2\rho'Ⓗ' - \int \rho\rho'^2Ⓗ^* - \oint \rho U\left(\frac{1}{2}\rho'^2\right) \tag{29}$$

$$= -I_{11} - I_{12} - I_{13} - I_{10}. \tag{30}$$

The surface integral $I_{10} = \oint \rho U(1/2\rho'^2)$ in (29) is positive or zero under each of the boundary conditions mentioned above and hence it follows that

$$\frac{\mathrm{d}J_1}{\mathrm{d}t} \leq -I_{11} - I_{12} - I_{13}. \tag{31}$$

To derive an estimate for dJ_1/dt, bounds for I_{11}, I_{12} and I_{13} are to be obtained and these are given below:

$$\begin{aligned}|I_{11}| &= \left|\int \rho\rho'\,\mathbf{q}'\,.\,\mathrm{grad}\,\rho^*\right| \\ &\leq \int |\rho\rho'\,\mathbf{q}'\,.\,\mathrm{grad}\,\rho^*| \\ &\leq n_1\int \rho'^2 + n_2\int \mathbf{q}'^2 \end{aligned} \tag{32}$$

where n_1 and n_2 are positive constants that can easily be evaluated.

$$\begin{aligned}|I_{12}| &\leq \left|\int \rho^2\rho'Ⓗ'\right| \\ &\leq \int \rho^2|\rho'Ⓗ'| \\ &\leq n_3\int \rho'^2 + \epsilon_1\int Ⓗ'^2 \end{aligned} \tag{33}$$

where ϵ_1 *is an arbitrary positive number* and n_3 is a positive number depending on the choice of ϵ_1.

$$\begin{aligned}|I_{13}| &= \left|\int \rho\rho'^2Ⓗ^*\right| \\ &\leq n_4\int \rho'^2. \end{aligned} \tag{34}$$

In deriving the bounds for I_{11}, I_{12}, I_{13} in the above, the basic tool employed is Cauchy's inequality

$$2ab \leq \alpha^2 a^2 + \frac{b^2}{\alpha^2} \tag{35}$$

where $\alpha(\neq 0)$ is *any real* constant. The constants n_1, n_2, n_3, n_4 in the above bounds depend on the functions pertaining to the primary and starred flows as well as their spatial derivatives and their precise evaluation is not required for proving the uniqueness theorem. From (32)–(34) it follows that

$$\frac{dJ_1}{dt} \leq (n_1+n_3+n_4)\int \rho'^2 + n_2\int (\mathbf{q}')^2 + \epsilon_1\int (Ⓗ')^2. \tag{36}$$

From the momentum balance (8) which is valid for both the primary and the starred flows over the interval $0 \leq t \leq \tau$, it follows that

$$\begin{aligned}\left(\rho\frac{d\mathbf{q}}{dt}\right)^* - \left(\rho\frac{d\mathbf{q}}{dt}\right) &= \rho'\mathbf{f} - \mathrm{grad}\,p' \\ &\quad + (\lambda_1 + 2\mu + k)\,\mathrm{grad}\,Ⓗ' + K\,\mathrm{curl}\,\boldsymbol{\nu}' - (\mu + k)\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{q}'\end{aligned} \tag{37}$$

and the l.h.s. of (37) equals

$$\rho\frac{d\mathbf{q}'}{dt} + \rho\mathbf{q}'\,.\,\mathrm{grad}\,\mathbf{q}^* + \rho'\left(\frac{d\mathbf{q}}{dt}\right)^*. \tag{38}$$

Let

$$J_2 = \int \frac{1}{2}\rho(\mathbf{q}')^2. \tag{39}$$

From the transport formula it follows that

$$\frac{dJ_2}{dt} = \int \frac{1}{2}\rho \frac{d}{dt}(\mathbf{q}')^2 - \frac{1}{2}\oint \rho U(\mathbf{q}')^2. \tag{40}$$

From (37) and (38) it is seen that

$$\begin{aligned}\rho\mathbf{q}' \cdot \frac{d\mathbf{q}'}{dt} = &- \rho\mathbf{q}' \cdot (\mathbf{q}' \cdot \operatorname{grad} \mathbf{q}^*) + \rho'\mathbf{q}' \cdot \left(\mathbf{f} - \left(\frac{d\mathbf{q}}{dt}\right)^*\right)\\ &- \mathbf{q}' \cdot \operatorname{grad} p' + (\lambda_1 + 2\mu + k)\,\mathbf{q}' \cdot \operatorname{grad} Ⓗ' + k\,\mathbf{q}' \cdot \operatorname{curl} \boldsymbol{\nu} - (\mu + k)\mathbf{q}' \cdot \operatorname{curl}\operatorname{curl} \mathbf{q}'.\end{aligned} \tag{41}$$

From (40) and (41) it follows that

$$\begin{aligned}\frac{dJ_2}{dt} &= \int \rho\mathbf{q}' \cdot \frac{d\mathbf{q}'}{dt} - \frac{1}{2}\oint \rho U(\mathbf{q}')^2\\ &= - \int \rho\mathbf{q}' \cdot (\mathbf{q}' \cdot \operatorname{grad} \mathbf{q}^*) + \int \rho'\,\mathbf{q}' \cdot \left(\mathbf{f} - \frac{d\mathbf{q}}{dt}\right)^*\\ &\quad - \int \mathbf{q}' \cdot \operatorname{grad} p' + (\lambda_1 + 2\mu + k) \int \mathbf{q}' \cdot \operatorname{grad} Ⓗ'\\ &\quad + k \int \mathbf{q}' \cdot \operatorname{curl} \boldsymbol{\nu}' - (\mu + k) \int \mathbf{q}' \cdot \operatorname{curl}\operatorname{curl} \mathbf{q}' - \frac{1}{2}\oint \rho U(\mathbf{q}')^2 \end{aligned}\tag{42}$$

$$= - I_{21} + I_{22} - I_{23} + (\lambda_1 + 2\mu + k)I_{24} + kI_{25} - (\mu + k)I_{26} - I_{20} \tag{43}$$

and the surface integral I_{20} is zero in view of the implication $\mathbf{q}'|_{\partial R} = 0$ arising from the boundary condition on the velocity vector. The integrands in I_{21}, I_{23}, I_{24}, I_{25} and I_{26} can be expressed alternatively by using the identities

$$\begin{aligned}&\rho\mathbf{q}' \cdot [\mathbf{q}' \cdot (\operatorname{grad} \mathbf{q}'^*)] = \rho\mathbf{q}' \cdot D^* \cdot \mathbf{q}',\\ &\mathbf{q}' \cdot \operatorname{grad} p' = \operatorname{div}(p'\mathbf{q}') - p'Ⓗ',\\ &\mathbf{q}' \cdot \operatorname{grad} Ⓗ' = \operatorname{div}(Ⓗ'\mathbf{q}') - Ⓗ'^2\\ &\mathbf{q}' \cdot \operatorname{curl} \boldsymbol{\nu}' = \operatorname{div}(\boldsymbol{\nu}' \times \mathbf{q}') + \boldsymbol{\nu}' \cdot \operatorname{curl} \mathbf{q}',\\ &\mathbf{q}' \cdot \operatorname{curl}\operatorname{curl} \mathbf{q}' = (\operatorname{curl} \mathbf{q}')^2 - \operatorname{div}(q' \times \operatorname{curl} \mathbf{q}').\end{aligned} \tag{44}$$

In the above, D^* denotes the rate of deformation matrix for the starred motion. In view of the boundary condition on the velocity vector, $\mathbf{q}'|_{\partial R} = 0$ as noted above each of the volume integrals

$$\int \operatorname{div}(p'\mathbf{q}'), \int \operatorname{div}(Ⓗ'\mathbf{q}'), \int \operatorname{div}(\boldsymbol{\nu}' \cdot \mathbf{q}'), \int \operatorname{div}(\mathbf{q}' \cdot \operatorname{curl} \mathbf{q}')$$

is seen to be zero on using the divergence theorem. Hence the following relation:

$$\begin{aligned}\frac{dJ_2}{dt} = &- \int \rho\,\mathbf{q}' \cdot D^* \cdot \mathbf{q}' + \int \rho'\,\mathbf{q}' \cdot \left(\mathbf{f} - \left(\frac{d\mathbf{q}}{dt}\right)^*\right)\\ &+ \int p'Ⓗ' - (\lambda_1 + 2\mu + k) \int Ⓗ'^2 + k \int \boldsymbol{\nu}' \cdot \operatorname{curl} \mathbf{q}' - (\mu + k) \int (\operatorname{curl} \mathbf{q}')^2.\end{aligned} \tag{45}$$

It is easy to see that

$$\begin{aligned}&|\rho\mathbf{q}' \cdot D^* \cdot \mathbf{q}'| \le n_5(\mathbf{q}')^2\\ &\left|\rho'\,\mathbf{q}' \cdot \left(\mathbf{f} - \left(\frac{d\mathbf{q}^*}{dt}\right)\right)\right| \le n_6\rho'^2 + n_7(\mathbf{q}')^2\\ &|p'Ⓗ'| \le n_8 p'^2 + \epsilon_2 Ⓗ'^2\end{aligned} \tag{46}$$

where ϵ_2 *is an arbitrary positive quantity* and n_5, n_6, n_7, n_8 are constants. The constant n_8

depends on the choice of ϵ_2. Also, in view of the equation of state (11),

$$p'^2 \le n_9 \rho'^2 + n_{10} \theta'^2. \tag{47}$$

From (45)–(47) it follows that

$$\frac{dJ_2}{dt} \le (n_6 + n_8 n_9) \int \rho'^2 + (n_5 + n_7) \int (\mathbf{q}')^2 + n_8 n_{10} \int \theta'^2 + \epsilon_2 \int Ⓗ'^2 - (\lambda_1 + 2\mu + k) \int Ⓗ'^2$$
$$+ k \int (\boldsymbol{\nu}' . \operatorname{curl} \mathbf{q}') - (\mu + k) \int (\operatorname{curl} \mathbf{q}')^2. \tag{48}$$

From the balance of first stress moments (9) which is valid for both the primary and the starred flows over the interval $0 \le t \le \tau$, it follows that

$$\left(\rho j \frac{d\boldsymbol{\nu}}{dt}\right)^* - \left(\rho j \frac{d\boldsymbol{\nu}}{dt}\right) = \rho' 1 + k \operatorname{curl} \mathbf{q}' - 2k\boldsymbol{\nu}' - \gamma \operatorname{curl} \operatorname{curl} \boldsymbol{\nu}' + (\alpha + \beta + \gamma) \operatorname{grad} (\operatorname{div} \boldsymbol{\nu}') \tag{49}$$

and the l.h.s. of (49) equals

$$\rho j \frac{d\boldsymbol{\nu}'}{dt} + \rho j (\mathbf{q} . \operatorname{grad}) \boldsymbol{\nu}^* + \rho' j \left(\frac{d\boldsymbol{\nu}}{dt}\right)^* \tag{50}$$

in arriving at which use has been made that the gyration parameter j is the same for the two flows under consideration. Let

$$J_3 = \int \frac{1}{2} \rho j (\boldsymbol{\nu}')^2. \tag{51}$$

From the transport formula (18), it follows that

$$\frac{dJ_3}{dt} = \int \rho \frac{d}{dt} \left(\frac{1}{2} j (\boldsymbol{\nu}')^2\right) - \oint \rho U \left(\frac{1}{2} j (\boldsymbol{\nu}')^2\right) \tag{52}$$

and this can also be written in the form

$$\frac{dJ_3}{dt} = \int \rho j \left(\boldsymbol{\nu}' . \frac{d\boldsymbol{\nu}'}{dt}\right) - \oint \frac{1}{2} \rho U j (\boldsymbol{\nu}')^2 \tag{53}$$

in view of the eqn (7). From (49), (50) and (53) it follows that

$$\begin{aligned} \frac{dJ_3}{Dt} &= \int \rho j \left(\boldsymbol{\nu}' . \frac{d\boldsymbol{\nu}'}{dt}\right) - \frac{1}{2} \oint \rho U j (\boldsymbol{\nu}')^2 \\ &= - \int \rho j \boldsymbol{\nu}' . [(\mathbf{q}' . \operatorname{grad}) \boldsymbol{\nu}^*] + \int \rho' \boldsymbol{\nu}' . \left(1 - j \left(\frac{d\boldsymbol{\nu}}{dt}\right)^*\right) \\ &\quad + k \int (\boldsymbol{\nu}' . \operatorname{curl} \mathbf{q}') - 2k \int (\boldsymbol{\nu}')^2 - \gamma \int (\boldsymbol{\nu}' . \operatorname{curl} \operatorname{curl} \boldsymbol{\nu}') \\ &\quad + (\alpha + \beta + \gamma) \int \boldsymbol{\nu}' . \operatorname{grad} (\operatorname{div} \boldsymbol{\nu}') - \frac{1}{2} \oint \rho U j (\boldsymbol{\nu}')^2 \\ &= - I_{31} + I_{32} + k I_{33} - 2k I_{34} - \gamma I_{35} + (\alpha + \beta + \gamma) I_{36} - I_{30} \end{aligned} \tag{54}$$

and the surface integral I_{30} vanishes in view of the boundary condition on the microrotation vector.

The integrands in I_{31}, I_{35} and I_{36} can be expressed alternatively by using the identities

$$\begin{aligned} \rho j \boldsymbol{\nu}' . [(\mathbf{q}' . \operatorname{grad}) \boldsymbol{\nu}^*] &= \rho j \mathbf{q}' . (\operatorname{grad} \boldsymbol{\nu}^*) . \boldsymbol{\nu}' \\ \boldsymbol{\nu}' . (\operatorname{curl} \operatorname{curl} \boldsymbol{\nu}') &= (\operatorname{curl} \boldsymbol{\nu}')^2 - \operatorname{div} (\boldsymbol{\nu}' \times \operatorname{curl} \boldsymbol{\nu}') \\ \boldsymbol{\nu}' . \operatorname{grad} (\operatorname{div} \boldsymbol{\nu}') &= \operatorname{div} ((\operatorname{div} \boldsymbol{\nu}') \boldsymbol{\nu}') - (\operatorname{div} \boldsymbol{\nu}')^2 \end{aligned} \tag{55}$$

where (grad ν^*) denotes the matrix gradient of the vector ν^*. It is easily recognised by the use of divergence theorem that the integrals $\int \operatorname{div}(\nu' \times \operatorname{curl} \nu')$, $\int \operatorname{div}((\operatorname{div} \nu')\nu')$ which arise in the course of simplification are zero. Hence the following relation:

$$\frac{dJ_3}{dt} = -\int \rho j \mathbf{q}' . (\operatorname{grad} \nu^*) . \nu' + \int \rho' \left(1 - j\left(\frac{d\nu}{dt}\right)^*\right) . \nu' + k \int \nu' . \operatorname{curl} \mathbf{q}'$$
$$- 2k \int (\nu')^2 - \gamma \int (\operatorname{curl} \nu')^2 - (\alpha + \beta + \gamma) \int (\operatorname{div} \nu'^2). \tag{56}$$

It is easy to see that

$$|\rho j \mathbf{q}' . (\operatorname{grad} \nu^*) . \nu'| \leq n_{11}(\mathbf{q}')^2 + n_{12}(\nu')^2 \qquad \left|\rho'\left(1 - j\left(\frac{d\nu}{dt}\right)^*\right) . \nu'\right| \leq n_{13}(\rho')^2 + n_{14}(\nu')^2. \tag{57}$$

From (56) and (57) it follows that

$$\frac{dJ_3}{dt} \leq n_{13} \int (\rho')^2 + n_{11} \int (\mathbf{q}')^2 + (n_{12} + n_{14}) \int (\nu')^2$$
$$+ k \int (\nu' . \operatorname{curl} \mathbf{q}') - 2k \int (\nu')^2 - \gamma \int (\operatorname{curl} \nu')^2 - (\alpha + \beta + \gamma) \int (\operatorname{div} \nu')^2. \tag{58}$$

From the energy balance (14), which is valid for the primary as well as the starred flows over the interval $0 \leq t \leq \tau$, it follows that

$$\left(\rho c_v \frac{d\theta}{dt}\right)^* - \left(\rho c_v \frac{d\theta}{dt}\right) = \psi' - \operatorname{div} \mathbf{h}'. \tag{59}$$

The l.h.s. in the above equation is

$$\rho c_v \frac{d\theta'}{dt} + \rho c_v \mathbf{q}' . \operatorname{grad} \theta^* + \rho' c_v \left(\frac{d\theta}{dt}\right)^* \tag{60}$$

on recalling that the coefficient c_v has been assumed constant and is the same for both the flows. The quantity ψ introduced in (15) can also be written in the form

$$\psi = T : D + T : D^A + M : \operatorname{grad} \nu \tag{61}$$

in terms of the dyadics T, D, D^A, M and grad ν which represent respectively the force stress, the rate of deformation, the antisymmetric relative spin defined in the form $D^A = \epsilon_{ijm}(\omega_m - \nu_m)$, the couple stress and the matrix gradient of the microrotation vector. The force stress tensor T and the couple stress tensor M have been defined earlier in (1) and (2) and in dyadic notation, these assume the form

$$T = (-p + \lambda_1 Ⓗ)I + (2\mu + k)D - kIX\left(\frac{1}{2} \operatorname{curl} \mathbf{q} - \nu\right) \tag{62}$$

$$M = \alpha (\operatorname{div} \nu)I + \beta(\operatorname{grad} \nu)^T + \gamma(\operatorname{grad} \nu) - \sigma IX \operatorname{grad} \theta. \tag{63}$$

From (20) and (61) it is seen that

$$\psi' = T : D' + T' : D^* + T : (D^A)' + T' : (D^A)^* + M : \operatorname{grad} \nu' + M' : \operatorname{grad} \nu^*. \tag{64}$$

From (60) and (64) it follows that

$$\rho c_v \theta' \frac{d\theta'}{dt} = -\rho c_v \theta' \mathbf{q}' . (\operatorname{grad} \theta^*) - \rho' c_v \theta' \left(\frac{d\theta}{dt}\right)^* + \theta' \psi' - \theta' \operatorname{div} \mathbf{h}'. \tag{65}$$

Let

$$J_4 = \int \frac{1}{2} \rho c_v \theta'^2 \tag{66}$$

From the transport formula (18), it follows that

$$\frac{dJ_4}{dt} = \int \rho \frac{d}{dt}\left(\frac{1}{2} c_v \theta'^2\right) - \oint \frac{1}{2} \rho U (c_v \theta'^2) \tag{67}$$

and on using (65) it follows that

$$\frac{dJ_4}{dt} = -\int \rho c_v \theta' \mathbf{q}' . \operatorname{grad} \theta^* - \int \rho' c_v \theta' \left(\frac{d\theta}{dt}\right)^* + \int \theta' \psi' - \int \theta' \operatorname{div} \mathbf{h}' - \frac{1}{2} \oint \rho U (c_v \theta'^2). \tag{68}$$

From (62)–(64), it is possible to see that

$$\begin{aligned}\psi' = & (+p + \lambda_1 Ⓗ)Ⓗ' + (-p' + \lambda_1 Ⓗ')Ⓗ^* + (2\mu + k) D : D' + (2\mu + k) D' : D^* \\ & + 2k(\boldsymbol{\omega} - \boldsymbol{\nu}) . (\boldsymbol{\omega}' - \boldsymbol{\nu}') + 2k(\boldsymbol{\omega}' - \boldsymbol{\nu}') . (\boldsymbol{\omega}^* - \boldsymbol{\nu}^*) + \alpha(\operatorname{div} \boldsymbol{\nu})(\operatorname{div} \boldsymbol{\nu}') + \beta(\operatorname{grad} \boldsymbol{\nu})^T : (\operatorname{grad} \boldsymbol{\nu}') \\ & + \gamma(\operatorname{grad} \boldsymbol{\nu}) : (\operatorname{grad} \boldsymbol{\nu}') + \sigma(\operatorname{grad} \theta) . (\operatorname{curl} \boldsymbol{\nu}') + \alpha(\operatorname{div} \boldsymbol{\nu}')(\operatorname{div} \boldsymbol{\nu}^*) + \beta(\operatorname{grad} \boldsymbol{\nu}')^T : (\operatorname{grad} \boldsymbol{\nu}^*) \\ & + \gamma(\operatorname{grad} \boldsymbol{\nu}') : (\operatorname{grad} \boldsymbol{\nu}^*) + \sigma(\operatorname{grad} \theta') . (\operatorname{curl} \boldsymbol{\nu}^*). \end{aligned} \tag{69}$$

Using Cauchy's inequality one can deduce that

$$\begin{aligned} |\rho c_v \theta' \mathbf{q}' . \operatorname{grad} \theta^*| &\le n_{15} (\mathbf{q}')^2 + n_{16} \theta'^2, \\ \left| \rho' c_v \theta' \left(\frac{d\theta}{dt}\right)^* \right| &\le n_{17} \rho'^2 + n_{18} \theta'^2. \end{aligned} \tag{70}$$

To estimate the term $\theta'\psi'$, the Cauchy's inequality can be applied to each of the individual terms obtained by product of θ' with ψ' given in (69) and the individual bounds for the various terms in $\theta'\psi'$ can be pooled. It is seen easily that

$$\begin{aligned} |\theta'\psi'| \le & n_{19} \rho'^2 + n_{20} \theta'^2 + \epsilon_3 Ⓗ'^2 + \epsilon_4 (\boldsymbol{\omega}' - \boldsymbol{\nu}')^2 + \epsilon_5 (\operatorname{div} \boldsymbol{\nu}')^2 \\ & + \epsilon_6 (\operatorname{grad} \boldsymbol{\nu}')^2 + \epsilon_7 (\operatorname{curl} \boldsymbol{\nu}')^2 + \epsilon_8 (\operatorname{grad} \theta')^2 + \epsilon_9 D' : D' \end{aligned} \tag{71}$$

where the ϵ's are arbitrary positive numbers and the n's are constants that depend on the choice of the ϵ's.

The term $\int \theta' \operatorname{div} \mathbf{h}'$ can be expressed in the form

$$\begin{aligned} \int \theta' \operatorname{div} \mathbf{h}' &= \int \operatorname{div} (\theta' \mathbf{h}') - \int \mathbf{h}' . \operatorname{grad} \theta' \\ &= \oint \theta' \mathbf{h}' . \mathbf{n} - \int \mathbf{h}' . \operatorname{grad} \theta' \end{aligned} \tag{72}$$

on using the divergence theorem. The surface integral $\oint \theta' \mathbf{h}' . \mathbf{n}$ is nonnegative under every form of the boundary conditions considered. From (3) it follows that

$$\mathbf{h}' . \operatorname{grad} \theta' = -\chi_1 (\operatorname{grad} \theta')^2 + \chi_2 (\operatorname{curl} \boldsymbol{\nu}') . (\operatorname{grad} \theta'). \tag{73}$$

In view of the identity

$$\operatorname{div}(\boldsymbol{\nu}' \times \operatorname{grad} \theta') = (\operatorname{grad} \theta') . (\operatorname{curl} \boldsymbol{\nu}') - \boldsymbol{\nu}' . \operatorname{curl}(\operatorname{grad} \theta') \tag{74}$$

and the divergence theorem, it follows that

$$\int (\text{curl}\, \boldsymbol{\nu}') \,.\, (\text{grad}\, \theta') = \oint (\boldsymbol{\nu}' \times \text{grad}\, \theta') \,.\, \mathbf{n} \tag{75}$$

and this last integral vanishes in view of the boundary condition on the microrotation. Thus one has

$$\int \mathbf{h}' \,.\, \text{grad}\, \theta' = -\chi_1 \int (\text{grad}\, \theta')^2. \tag{76}$$

From (72) and (76) it follows that

$$\int \theta' \,\text{div}\, \mathbf{h}' \geq \chi_1 \int (\text{grad}\, \theta')^2. \tag{77}$$

The surface integral $1/2 \oint \rho U (c_v \theta'^2)$ in eqn (68) is zero in view of the boundary conditions on the temperature θ. Thus it follows from (68), (70), (71) and (77) that

$$\begin{aligned} \frac{dJ_4}{dt} \leq (n_{17} + n_{19}) \int \rho'^2 + n_{15} \int (\mathbf{q}')^2 + (n_{16} + n_{18} + n_{20}) \int (\theta')^2 + \epsilon_3 \int Ⓗ'^2 \\ + \epsilon_4 \int (\boldsymbol{\omega}' - \boldsymbol{\nu}')^2 + \epsilon_5 \int (\text{div}\, \boldsymbol{\nu}')^2 + \epsilon_6 \int (\text{grad}\, \boldsymbol{\nu}')^2 + \epsilon_7 \int (\text{curl}\, \boldsymbol{\nu}')^2 \\ + \epsilon_8 \int (\text{grad}\, \theta')^2 + \epsilon_9 \int D' : D' - \chi_1 \int (\text{grad}\, \theta')^2. \end{aligned} \tag{78}$$

It can be shown that if $\mathbf{a}$ is a continuously differentiable vector field over the region $R(t) \times [0, \tau]$ with vanishing boundary values,

$$\int (\text{grad}\, \mathbf{a})^2 = \int (\text{div}\, \mathbf{a})^2 + \int (\text{curl}\, \mathbf{a})^2. \tag{79}$$

From the inequalities (36), (48), (58), (78) and the identity (79), one can see that

$$\begin{aligned} \frac{d}{dt}(J_1 + J_2 + J_3 + J_4) \leq {} & [(n_1 + n_3 + n_4) + (n_6 + n_8 n_9) + n_{13} + (n_{17} + n_{19})] \int \rho'^2 \\ & + (n_2 + n_5 + n_7 + n_{11} + n_{15}) \int (\mathbf{q}')^2 + (n_{12} + n_{14}) \int (\boldsymbol{\nu}')^2 \\ & + (n_8 n_{10} + n_{16} + n_{18} + n_{20}) \int (\theta')^2 - 2k \int (\boldsymbol{\omega}' - \boldsymbol{\nu}')^2 - (\lambda_1 + 2\mu + k) \int Ⓗ'^2 \\ & - \left(\mu + \frac{k}{2}\right) \int (\text{curl}\, \mathbf{q}')^2 - \gamma \int (\text{curl}\, \boldsymbol{\nu}')^2 - (\alpha + \beta + \gamma) \int (\text{div}\, \boldsymbol{\nu}')^2 \\ & + (\epsilon_1 + \epsilon_2 + \epsilon_3) \int Ⓗ'^2 + \epsilon_4 \int (\boldsymbol{\omega}' - \boldsymbol{\nu}')^2 + \epsilon_5 \int (\text{div}\, \boldsymbol{\nu}')^2 \\ & + \epsilon_6 \left[\int (\text{div}\, \boldsymbol{\nu}')^2 + \int (\text{curl}\, \boldsymbol{\nu}')^2 \right] + \epsilon_7 \int (\text{curl}\, \boldsymbol{\nu}')^2 + \epsilon_8 \int (\text{grad}\, \theta')^2 \\ & + \epsilon_9 \int D' : D' - \chi_1 \int (\text{grad}\, \theta')^2. \end{aligned} \tag{80}$$

The integral $\int D' : D'$ in the above can be written in a more convenient alternative form on recognising the identity

$$(\text{grad}\, \mathbf{q})^2 = D : D + \frac{1}{2} (\text{curl}\, \mathbf{q})^2. \tag{81}$$

Since the density ρ, the gyration parameter j and the specific heat c_v are all strictly positive in the closure of the flow region, it is possible to write

$$\left.\begin{aligned} &\int \rho'^2 \le n_{21}J_1; \quad \int (\mathbf{q}')^2 \le n_{22}J_2 \\ &\int (\boldsymbol{\nu}')^2 \le n_{23}J_3; \quad \int \theta'^2 \le n_{24}J_4 \end{aligned}\right\} \tag{82}$$

where n_{21}, n_{22}, n_{23}, n_{24} are constants that depend on ρ, j and c_v. Defining

$$J = J_1 + J_2 + J_3 + J_4 \tag{83}$$

one can deduce from (80)–(82) that

$$\begin{aligned} \frac{dJ}{dt} \le mJ &- [(\lambda_1 + 2\mu + k) - (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_9)] \int Ⓗ'^2 \\ &- \left[\left(\mu + \frac{k}{2}\right) - \frac{\epsilon_9}{2}\right] \int (\text{curl } \mathbf{q}')^2 - [2k - \epsilon_4] \int (\boldsymbol{\omega}' - \boldsymbol{\nu}')^2 - [(\alpha + \beta + \gamma) - (\epsilon_5 + \epsilon_6)] \int (\text{div } \boldsymbol{\nu}')^2 \\ &- [\gamma - (\epsilon_6 + \epsilon_7)] \int (\text{curl } \boldsymbol{\nu}')^2 - [\chi_1 - \epsilon_8) \int (\text{grad } \theta')^2 \end{aligned} \tag{84}$$

where

$$\begin{aligned} m = \max\{&(n_1 + n_3 + n_4 + n_6 + n_8 n_9 + n_{13} + n_{17} + n_{19}) n_{21}; (n_2 + n_5 + n_5 + n_{11} + n_{15}) n_{22}; \\ &(n_{12} + n_{14}) n_{23}; (n_8 n_{10} + n_{16} + n_{18} + n_{20}) n_{24}\}. \end{aligned} \tag{85}$$

Since the ϵ's are arbitrary it is possible to ensure that each of the six coefficients in parentheses ( ) multiplying the integrals on the r.h.s. of (84) is positive. It therefore follows that

$$\frac{dJ}{dt} \ge mJ, \quad 0 < t \le \tau. \tag{86}$$

Integrating this differential inequality one obtains

$$J \le J_0 e^{mt} \quad 0 \le t \le \tau \tag{87}$$

where J_0 is the value of J at $t = 0$. The integrals J_1, J_2, J_3, J_4 are zero when $t = 0$ and it, therefore, follows from (87) that J remains zero throughout the whole time interval $[0, \tau]$. In view of (83) it follows that each of the measures J_1, J_2, J_3, J_4 remains zero throughout the time interval $[0, t]$. Hence the conclusion that the primary and the starred flows are identical and one has the proof of uniqueness theorem.

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