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VARIATIONAL ALGORITHM FOR THE STABILITY OF THE FLOW OF MICROPOLAR FLUIDS WITH STRETCH

S. K. LAKSHMANA RAO and K. VENKATAPATHI RAJU
Department of Mathematics, Regional Engineering College, Warangal-506004, India

Abstract—The flow of incompressible microstretch fluids is governed by a coupled system of differential equations involving the velocity vector $\bar{q}$, microrotation vector $\bar{\nu}$ and the scalar ν representing the microstretch of the fluid element. The paper employs the energy method for obtaining criteria for the universal stability of microstretch fluid flows and provides an algorithm for the determination of the critical Reynolds number. It is seen that the Reynolds number can be defined in terms of the maximum speed of the primary flow.

1. INTRODUCTION

THE THEORY of simple microfluids developed by Eringen[1] takes note of the effects arising from the local structure of fluid elements and their intrinsic motions consequent to these effects. Apart from the usual translatory degrees-of-freedom reckoned by the velocity vector $\bar{q}$ the fluid element has additional degrees-of-freedom enabling it to undergo intrinsic rotation as well as deformation and the latter features are reckoned by the three gyration vector fields $\bar{\nu}_k$. Even the linear model of a simple microfluid poses difficulties in theoretical investigations as it involves twenty two material constants in its constitutive relations. A simplified version of the model leads to the class of micropolar fluids[2] in which the local motion of a fluid element is a rigid rotation. Another simple version is the case of a microstretch fluid[3, 4] in which the local motion of a fluid element involves both rotation and stretch. In this case the gyration tensor ν_{kl} and the tensor of first stress moment λ_{klm} are expressible in the form

$$\nu_{kl} = \nu\delta_{kl} + \epsilon_{klr}\nu_r \tag{1}$$

$$\lambda_{klp} = \lambda_k\delta_{lp} - \frac{1}{2}\epsilon_{lpr}m_{kr} \tag{2}$$

and the vector ν_r represents microrotation while the scalar denotes microstretch. The class of microstretch fluids is wider obviously than that of micropolar fluids and both the two fluid models depart from the classical Navier–Stokes theory in the two important features: the sustenance of couple stress and the non-symmetry of the stress tensor.

In this paper we examine the stability of microstretch fluid motions with the aim of obtaining a variational algorithm that enables the determination of the critical Reynolds number. The method employed is the energy criterion introduced earlier by Serrin[5] for the investigation of the stability of viscous fluid motions governed by the Navier–Stokes equations. This powerful method has been extended by Joseph[6] for the discussion of the stability of Boussinesq equations. Serrin's method has been employed by the authors[7, 8] to obtain criteria for the stability of micropolar as well as microstretch fluid flows and by Shahinpoor and Ahmadi[9] in the case of Cosserat fluid flows.

2. EQUATIONS GOVERNING THE FLOW OF INCOMPRESSIBLE MICROSTRETCH FLUID

We consider the motion of an incompressible microstretch fluid in an arbitrary time-dependent domain $R(t)$. The equations governing the flow are[3, 4]

$$\operatorname{div}\bar{q} = 0, \tag{3}$$

$$\frac{\partial j}{\partial t} + (\bar{q}\cdot\operatorname{grad})j - 2\nu j = 0, \tag{4}$$

$$\rho\left[\frac{\partial\bar{q}}{\partial t} - \bar{q}\times\operatorname{curl}\bar{q} + \operatorname{grad}\left(\frac{1}{2}q^2\right)\right] = \rho\bar{f} - \operatorname{grad}p + \lambda_0\operatorname{grad}\nu + k\operatorname{curl}\bar{\nu}$$
$$- (\mu + k)\operatorname{curl}\operatorname{curl}\bar{q} + (\lambda_1 + 2\mu + k)\operatorname{grad}(\operatorname{div}\bar{q}), \tag{5}$$

$$\rho j\left[\frac{\partial \bar{\nu}}{\partial t}+(\bar{q}\cdot\text{grad})\bar{\nu}\right]=\rho\bar{l}-2k\bar{\nu}+k\,\text{curl}\,\bar{q}$$

$$-\gamma\,\text{curl curl}\,\bar{\nu}+(\alpha+\beta+\gamma)\,\text{grad}\,(\text{div}\,\bar{\nu}), \tag{6}$$

$$\frac{1}{2}\rho j\left[\frac{\partial \nu}{\partial t}+(\bar{q}\cdot\text{grad})\nu\right]=\rho 1+\alpha_0\nabla^2\nu-(\eta_0-\lambda_0)\nu. \tag{7}$$

In the above muster of equations ρ is the density of the fluid, j denotes the gyration parameter, p is an undetermined pressure, $\bar{f}$ and $\bar{l}$ are, respectively, the body force and body couple per unit mass and 1 in eqn (7) is one third of the trace of the first body moment per unit mass. The vectors $\bar{q}$ and $\bar{\nu}$ are the velocity and microtation vectors and the scalar ν denotes the microstretch of the fluid elements. The viscosity coefficients λ_1, μ, k, η_0 and λ_0 and the gyroviscosity coefficients α, β, γ and α_0 are constant and are subject to the following restrictions[3, 4].

$$3\lambda_1+2\mu+k\geq 0,\quad 2\mu+k\geq 0,\quad k\geq 0,\quad \eta_0-\lambda_0\geq 0,$$

$$(\eta_0-\lambda_0)(3\lambda_1+2\mu+k)\geq\frac{\lambda_0^2}{4},\quad 3\alpha+\beta+\gamma\geq 0,$$

$$\gamma\geq 0,\quad |\beta|\leq\gamma,\quad \alpha_0\geq 0. \tag{8}$$

The density ρ and the gyration parameter j are positive and the former is a constant.

Boundary conditions. We assume that on the boundary $\partial R(t)$ the field variables $\bar{q}$, $\bar{\nu}$, ν are prescribed. If $\bar{x}$ is a boundary point and t is the time and $\bar{U}(\bar{x}, t)$, $\bar{N}(\bar{x}, t)$ and $N_0(\bar{x}, t)$ are the velocity, microrotation and microstretch of the element at $\bar{x}$ and at time t, we have

$$\bar{q}(\bar{x}, t=\bar{U}(\bar{x}, t),\quad \bar{\nu}(\bar{x}, t=\bar{N}(\bar{x}, t),\quad \nu(\bar{x}, t)=N_0(\bar{x}, t). \tag{9}$$

These conditions reflect a sort of super adherence of the fluid to the solid boundary.

3. ENERGY EQUATION

The flow $(\bar{q}, \bar{\nu}, \nu)$ of an incompressible microstretch flow in the domain $R(t)$—referred to henceforth as the primary motion—is altered at some instant ($t=0$, say) to the starred flow $(\bar{q}^*, \bar{\nu}^*, \nu^*)$ and both the flows have the same density and gyration parameter j. The body force, body couple and the body moment trace are omitted. On the boundary $\partial R(t)$ both the flows conform to the adherence condition. The difference flow defined by $(\bar{u}=\bar{q}^*-\bar{q},\ \bar{\vartheta}=\bar{\nu}^*-\bar{\nu},\ \theta=\nu^*-\nu)$ satisfies the conditions

$$\bar{u}=\bar{0},\quad \bar{\vartheta}=\bar{0},\quad \theta=0\quad \text{on}\quad \partial R(t) \tag{9}$$

and the parameters ρ, j are the same for the primary, starred and the difference flows. To analyze the stability of the primary flow we may introduce the Liapunoff function T representing the kinetic energy of the difference flow $(\bar{u}, \bar{\vartheta}, \theta)$ and study the time-rate of variation of T and also its limit when $t\to\infty$. The Liapunoff function

$$T=T_1+T_2+T_3=\frac{1}{2}\int\rho(\bar{u})^2\,\mathrm{d}R+\frac{1}{2}\int\rho j(\bar{\vartheta})^2\,\mathrm{d}R+\frac{3}{4}\int\rho j\theta^2\,\mathrm{d}R \tag{10}$$

and each volume integral extends over the domain $R(t)$. The field quantities and the domain $R(t)$ are assumed smooth enough for the validity of the divergence theorem. In the sequel the volume infinitesimal $\mathrm{d}R$ in the volume integrals over the domain $R(t)$ is omitted. Since the primary as well as the starred flows satisfy the governing eqns (3)–(7) we see that

$$\text{div}\,\bar{u}=0, \tag{11}$$

$$(\bar{u}\cdot\text{grad})j-2\theta j=0, \tag{12}$$

$$\rho\left[\frac{\partial \bar{u}}{\partial t}+(\bar{q}^*\cdot\text{grad})\bar{u}+(\bar{u}\cdot\text{grad})\bar{q}\right]=-\text{grad}\,(p^*-p)+\lambda_0\,\text{grad}\,\theta$$
$$+k\,\text{curl}\,\bar{\vartheta}-(\mu+k)\,\text{curl curl}\,\bar{u}, \tag{13}$$

$$\rho j\left[\frac{\partial \bar{\vartheta}}{\partial t}+(\bar{q}^*\cdot\text{grad})\bar{\vartheta}+(\bar{u}\cdot\text{grad})\bar{\nu}\right]=-2k\bar{\vartheta}+k\,\text{curl}\,\bar{u}-\gamma\,\text{curl curl}\,\bar{\vartheta}$$
$$+(\alpha+\beta+\gamma)\,\text{grad}\,(\text{div}\,\bar{\vartheta}), \tag{14}$$

$$\frac{1}{2}\rho j\left[\frac{\partial \theta}{\partial t}+(\bar{q}^*\cdot\text{grad})\theta+(\bar{u}\cdot\text{grad})\nu\right]=\alpha_0\nabla^2\theta-(\eta_0-\lambda_0)\theta. \tag{15}$$

We can use the relations (11)–(15) and the boundary conditions (9) to evaluate the time-rate of change of each of the energy functionals T_1, T_2 and T_3 defined in (10). These are given below in a form suitable for later deduction of the variational algorithm for the critical Reynolds number

$$\frac{\mathrm{d}T_1}{\mathrm{d}t}=\int\rho\bar{u}\cdot(\text{grad}\,\bar{u})\cdot\bar{q}+k\int\bar{\vartheta}\cdot\text{curl}\,\bar{u}$$
$$-(\mu+k)\int(\text{curl}\,\bar{u})^2, \tag{16}$$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t}=\int\rho j\bar{u}\cdot(\text{grad}\,\bar{\vartheta})\cdot\bar{\nu}+\int\rho j\nu^*(\bar{\vartheta})^2$$
$$+2\int\rho j\theta\bar{\nu}\cdot\bar{\vartheta}+k\int\bar{\vartheta}\cdot\text{curl}\,\bar{u}$$
$$-2k\int(\bar{\vartheta})^2-\gamma\int(\text{curl}\,\bar{\vartheta})^2-(\alpha+\beta+\gamma)\int(\text{div}\,\bar{\vartheta})^2, \tag{17}$$

$$\frac{\mathrm{d}T_3}{\mathrm{d}t}=\frac{3}{2}\int\rho j(\bar{u}\cdot\text{grad}\,\theta)\nu+\frac{3}{2}\int\rho j\nu^*\theta^2$$
$$+3\int\rho j\nu\theta^2-3\alpha_0\int(\text{grad}\,\theta)^2-3(\eta_0-\lambda_0)\int\theta^2. \tag{18}$$

We define the quantities V, M_0 and M as the maximum modulus value of the speed, microrotation and microstretch over the domain $R(t)$ and over an arbitrary but fixed time interval $(0,\tau]$ so that

$$(V, M_0, M)=\max\left(|\bar{q}|,|\bar{\nu}|,|\nu|\right)\text{ on }R(t)\times(0,\tau] \tag{19}$$

and d = diameter of the domain $R(t)$, for $0\le t\le\tau$. We adopt the scheme of non-dimensionalization shown below in (20) and delete the primes over the non-dimensionalized quantities thereafter

$$\bar{u}=V(\bar{u}'),\quad \bar{q}=V(\bar{q}'),\quad \bar{\vartheta}=M_0(\bar{\vartheta}'),\quad \bar{\nu}=M_0(\bar{\nu}'),\quad \theta=M(\theta'),$$
$$\nu=M(\nu'),\quad \nu^*=M(\nu^{*\prime}),\quad t=\frac{d}{V}(t'),\quad \bar{x}=d(\bar{x}'),\quad j=d^2(j'). \tag{20}$$

The time-rates of change of the energy functionals are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int(\bar{u})^2/2=\int\bar{u}\cdot(\text{grad}\,\bar{u})\cdot\bar{q}+\frac{kM_0}{\rho V^2}\int\bar{\vartheta}\cdot\text{curl}\,\bar{u}-\frac{\mu+k}{\rho Vd}\int(\text{curl}\,\bar{u})^2, \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}j(\bar{\vartheta})^2=\int j\bar{u}\cdot(\text{grad}\,\bar{\vartheta})\cdot\bar{\nu}+\frac{Md}{V}\int j\nu^*(\bar{\vartheta})^2$$
$$+\frac{2Md}{V}\int j\theta\bar{\nu}\cdot\bar{\vartheta}+\frac{k}{\rho M_0 d^2}\int\bar{\vartheta}\cdot\text{curl}\,\bar{u}-\frac{2k}{\rho Vd}\int(\bar{\vartheta})^2$$
$$-\frac{\gamma}{\rho Vd^3}\int(\text{curl}\,\bar{\vartheta})^2-\frac{(\alpha+\beta+\gamma)}{\rho Vd^3}\int(\text{div}\,\bar{\vartheta})^2, \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{3^j}{4}\theta^2 = \frac{3}{2}\int j[\bar{u}\cdot(\mathrm{grad}\ \theta)]\nu + \frac{3Md}{2V}\int^j \nu^*\theta^2$$

$$+\frac{3Md}{V}\int^j \nu\theta^2 - \frac{3\alpha_0}{\rho V d^3}\int (\mathrm{grad}\ \theta)^2 - \frac{3(\eta_0-\lambda_0)}{\rho V d}\int \theta^2. \tag{23}$$

The stability criteria can be obtained by seeking conditions that ensure the decay of the Liapunoff energy function T to zero as time $t\to\infty$. Since the paper aims to develop an algorithm for the determination of the critical parameter that ensures stability, we have to select the parameter whose role is to be accentuated in comparison with that of others. Let us define the parameters

$$n_1 = \frac{\mu+k}{\rho V d} = Re^{-1},\quad n_2 = \frac{k}{\mu+k},\quad n_3 = \frac{\gamma}{(\mu+k)d^2},$$

$$n_4 = \frac{\alpha+\beta+\gamma}{(\mu+k)d^2},\quad n_5 = \frac{3\alpha_0}{(\mu+k)d^2},\quad n_6 = \frac{3(\eta_0-\lambda_0)}{(\mu+k)},\quad n_7 = \frac{M_0 d}{V},\quad n_8 = \frac{Md}{V}. \tag{24}$$

Each of the above eight parameters is positive and we have $0<n_2<2$. The number $Re = n_1^{-1}$ is the Reynolds number of the primary flow and the algorithm to be obtained below is derived by accentuating the role of this parameter in the stability criteria. The time rate of energy eqns (21)–(23) can now be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (\bar{u})^2/2 = \int \bar{u}\cdot(\mathrm{grad}\ \bar{u})\cdot\bar{q} + n_1 n_2 n_7\int \bar{\vartheta}\cdot\mathrm{curl}\ \bar{u} - n_1\int(\mathrm{curl}\ \bar{u})^2, \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}j(\bar{\vartheta})^2 = \int j\bar{u}.(\mathrm{grad}\ \bar{\vartheta})\cdot\bar{\nu} + n_8\int j\nu^*(\bar{\vartheta})^2$$

$$+2n_8\int j\theta\bar{\nu}\cdot\bar{\vartheta} + \frac{n_1 n_2}{n_7}\int \bar{\vartheta}\cdot\mathrm{curl}\ \bar{u}$$

$$-2n_1 n_2\int(\bar{\vartheta})^2 - n_1 n_3\int(\mathrm{curl}\ \bar{\vartheta})^2$$

$$-n_1 n_4\int(\mathrm{div}\ \bar{\vartheta})^2, \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{3}{4}j\theta^2 = \frac{3}{2}\int j[\bar{u}\cdot(\mathrm{grad}\ \theta)]\nu + \frac{3}{2}n_8\int j\nu^*\theta^2$$

$$+3n_8\int j\nu\theta^2 - n_1 n_5\int(\mathrm{grad}\ \theta)^2 - n_1 n_6\int\theta^2. \tag{27}$$

We now define a two-parametered energy functional

$$T = T_1 + \lambda_2 T_2 + \lambda_3 T_3 = \int (\bar{u})^2/2$$

$$+\lambda_2\int \frac{1}{2}j(\bar{\vartheta})^2 + \lambda_3\int\frac{3}{4}j\theta^2 \tag{28}$$

where the two parameters λ_2 and λ_3 are positive and observe that to achieve the universal stability of the primary flow it is essential that

$$\frac{\mathrm{d}}{\mathrm{d}t}T<0 \tag{29}$$

for all $t>0$ and every positive pair (λ_2, λ_3). From (25)–(28) we see that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = I_1(\bar{u},\bar{\vartheta}) + \lambda_2 I_2(\bar{u},\bar{\vartheta},\theta) + \lambda_3 I_3(\bar{u},\theta) - n_1 D(\bar{u},\bar{\vartheta},\theta), \tag{30}$$

and the functionals I_1, I_2, I_3 and D are given by

$$I_1 = I_1(\bar{u}, \bar{\vartheta}) = \int \bar{u} \cdot (\text{grad } \bar{u}) \cdot \bar{q} + n_1 n_2 n_7 \int \bar{\vartheta} \cdot \text{curl } \bar{u}, \tag{31}$$

$$I_2 = I_2(\bar{u}, \bar{\vartheta}, \theta) = \int j\bar{u} \cdot (\text{grad } \bar{\vartheta}) \cdot \bar{\nu} + n_8 \int j\nu^*(\bar{\vartheta})^2$$
$$+ 2n_8 \int j\theta\bar{\nu} \cdot \bar{\vartheta} + \frac{n_1 n_2}{n_7} \int \bar{\vartheta} \cdot \text{curl } \bar{u}, \tag{32}$$

$$I_3(\bar{u}, \theta) = \frac{3}{2} \int j[\bar{u} \cdot (\text{grad } \theta)]\nu + \frac{3}{2} n_8 \int j\nu^* \theta^2 + 3n_8 \int j\nu\theta^2, \tag{33}$$

$$D(\bar{u}, \bar{\vartheta}, \theta) = \int (\text{curl } \bar{u})^2 + \lambda_2 (2n_2 \int (\bar{\vartheta})^2$$
$$+ n_3 \int (\text{curl } \bar{\vartheta})^2 + n_4 \int (\text{div } \bar{\vartheta})^2) + \lambda_3 (n_5 \int (\text{grad } \theta)^2 + n_6 \int \theta^2). \tag{34}$$

The condition (29) is true if and only if we have

$$I_1 + \lambda_2 I_2 + \lambda_3 I_3 - n_1 D(\bar{u}, \bar{\vartheta}, \theta) < 0. \tag{35}$$

We can connect the problem of finding the critical value of Reynolds number Re for universal stability of the primary flow to the variational problem of finding the maximum of

$$I(\bar{u}, \bar{\vartheta}, \theta) = I_1(\bar{u}, \bar{\vartheta}) + \lambda_2 I_2(\bar{u}, \bar{\vartheta}, \theta) + \lambda_3 I_3(\bar{u}, \theta) \tag{36}$$

among the class of the functions $\bar{u}$, $\bar{\vartheta}$, θ which satisfy (i) the constraint

$$\text{div } \bar{u} = 0 \quad \text{in} \quad R(t) \times (0, \tau] \tag{37}$$

(ii) the normalization condition

$$D(\bar{u}, \bar{\vartheta}, \theta) = 1 \tag{38}$$

and (iii) the boundary conditions

$$\bar{u} = \bar{0}, \quad \bar{\vartheta} = \bar{0}, \quad \theta = 0 \quad \text{on} \quad \beta\partial R(t) \tag{39}$$

for prescribed values of the parameters n_2, n_3, n_4, n_5, n_6, n_7 and n_8 as well as λ_2 and λ_3. We consider the variational problem

$$\delta\left(\int f(\bar{u}, \bar{\vartheta}, \theta)\right) = 0 \tag{40}$$

where

$$f(\bar{u}, \bar{\vartheta}, \theta) = \bar{u} \cdot (\text{grad } \bar{u}) \cdot \bar{q} + n_1 n_2 n_7 \bar{\vartheta} \cdot \text{curl } \bar{u}$$
$$+ \lambda_2 \Big(j\bar{u} \cdot (\text{grad } \bar{\vartheta}) \cdot \bar{\nu} + n_8 j\nu^* (\bar{\vartheta})^2$$
$$+ 2n_8 j\theta\bar{\nu} \cdot \bar{\vartheta} + \frac{n_1 n_2}{n_7} \bar{\vartheta} \cdot \text{curl } \bar{u} \Big)$$
$$+ \lambda_3 \Big(\frac{3}{2} j[\bar{u} \cdot (\text{grad } \theta)]\nu + \frac{3}{2} n_8 j\nu^* \theta^2$$
$$+ 3n_8 j\nu\theta^2 \Big) - P \text{ div } \bar{u} - \frac{1}{R} D(\bar{u}, \bar{\vartheta}, \theta). \tag{41}$$

The parameters $P = P(x, y, z, t)$ and $1/R = (1/R_{\lambda_2,\lambda_3})$ are the Lagrange parameters in the extremum problem. The Euler–Lagrange variational equations corresponding to (40) are found to be

$$\operatorname{div}(\bar{u}\bar{q}) + \frac{2}{R}\operatorname{curl}\operatorname{curl}\bar{u} - (\operatorname{grad}\bar{u})\cdot\bar{q} - \lambda_2 j(\operatorname{grad}\bar{\vartheta})\cdot\bar{\nu}$$

$$-\frac{3}{2}\lambda_3 j\nu\operatorname{grad}\theta - \left(n_1n_2n_7 + \frac{\lambda_2n_1n_2}{n_7}\right)\operatorname{curl}\bar{\vartheta} - \operatorname{grad}P = \bar{0}, \tag{42}$$

$$\operatorname{div}(\lambda_2 j\bar{u}\bar{\nu}) - \frac{2\lambda_2n_4}{R}\operatorname{grad}(\operatorname{div}\bar{\vartheta}) + \frac{2\lambda_2n_3}{R}\operatorname{curl}\operatorname{curl}\bar{\vartheta}$$

$$-\left(n_1n_2n_7 + \frac{\lambda_2n_1n_2}{n_7}\right)\operatorname{curl}\bar{u} - \left(2\lambda_2n_8j\nu^* - \frac{4\lambda_2n_2}{R}\right)\bar{\vartheta}$$

$$-2\lambda_2n_8j\theta\bar{\nu} = \bar{0}, \tag{43}$$

$$\frac{3}{2}\lambda_3\operatorname{div}(j\nu\bar{u}) - \frac{2\lambda_3n_5}{R}\nabla^2\theta - 2\lambda_2n_8j\bar{\nu}\cdot\bar{\vartheta}$$

$$-3\lambda_3n_8j(2\nu+\nu^*)\theta + \frac{2\lambda_3n_6}{R}\theta = 0. \tag{44}$$

Scalar multiplication of (42) with $\bar{u}$ and integration of the result over the domain $R(t)$ yields the result

$$-2\int\bar{u}\cdot(\operatorname{grad}\bar{u})\cdot\bar{q} - \left(n_1n_2n_7 + \frac{\lambda_2n_1n_2}{n_7}\right)\int\bar{\vartheta}\cdot\operatorname{curl}\bar{u}$$

$$-\lambda_2\int j\bar{u}\cdot(\operatorname{grad}\bar{\vartheta})\cdot\bar{\nu} - \frac{3}{2}\lambda_3\int j\nu\bar{u}\cdot\operatorname{grad}\theta$$

$$+\frac{2}{R}\int(\operatorname{curl}\bar{u})^2 = 0. \tag{45}$$

From (43) we can derive the result below after scalar multiplication by $\bar{\vartheta}$ and integration of the product over the domain $R(t)$.

$$-\left(n_1n_2n_7 + \frac{\lambda_2n_1n_2}{n_7}\right)\int\bar{\vartheta}\cdot\operatorname{curl}\bar{u} - \lambda_2\int j\bar{u}\cdot(\operatorname{grad}\bar{\vartheta})\cdot\bar{\nu}$$

$$-2\lambda_2n_8\int j\nu^*(\bar{\vartheta})^2 - 2\lambda_2n_8\int j\theta\bar{\nu}\cdot\bar{\vartheta}$$

$$+\frac{2}{R}\left(2\lambda_2n_2\int(\bar{\vartheta})^2 + \lambda_2n_3\int(\operatorname{curl}\bar{\vartheta})^2\right.$$

$$\left.+\lambda_2n_4\int(\operatorname{div}\bar{\vartheta})^2\right) = 0. \tag{46}$$

Multiplication of (44) by θ and integration of the product over the domain $R(t)$ yields

$$-2\lambda_2n_8\int j\theta\bar{\nu}\cdot\bar{\vartheta} - \frac{3}{2}\lambda_3\int j\nu\bar{u}\cdot\operatorname{grad}\theta$$

$$-3\lambda_3n_8\int j\nu^*\theta^2 - 6\lambda_3n_8\int j\nu\theta^2$$

$$+\frac{2}{R}\left(\lambda_3n_5\int(\operatorname{grad}\theta)^2 + \lambda_3n_6\int\theta^2\right) = 0. \tag{47}$$

The sum of eqns (45)–(47) is recognized to be

$$I_1(\bar{u}, \bar{\vartheta}) + \lambda_2 I_2(\bar{u}, \bar{\vartheta}, \theta) + \lambda_3 I_3(\bar{u}, \theta) - \frac{1}{R} D(\bar{u}, \bar{\vartheta}, \theta) = 0 \tag{48}$$

whence, it follows that any solution $(\bar{u}, \bar{\vartheta}, \theta)$ of the Euler–Lagrange variational eqns (42)–(44) for the variational problem (36)–(40) is such that

$$I(\bar{u}, \bar{\vartheta}, \theta) = \frac{1}{R} = \frac{1}{R_{\lambda_2,\lambda_3}}. \tag{49}$$

It is known from the calculus of variations (Courant and Hilbert[10]) that the variational problem

$$\text{maximum}\,\{I(\bar{u}, \bar{\vartheta}, \theta)\} \tag{50}$$

$$\operatorname{div} \bar{u} = 0 \quad \text{in} \quad R(t) \times (0, \tau] \tag{51}$$

$$D(\bar{u}, \bar{\vartheta}, \theta) = 1 \tag{52}$$

$$\bar{u}|_{\partial R(t)} = \bar{0}, \quad \bar{\vartheta}|_{\partial R(t)} = \bar{0}, \quad \theta|_{\partial R(t)} = 0 \tag{53}$$

possesses maximizing solution functions $(\tilde{u}, \tilde{\vartheta}, \tilde{\theta})$. These functions are also eigenfunctions of the variational eqns (42)–(44) (Courant and Hilbert[11]) corresponding to the eigenvalue $(1/R_{\lambda_2,\lambda_3})$; further it is seen that

$$\frac{1}{\tilde{R}_{\lambda_2,\lambda_3}} = I(\tilde{u}, \tilde{\vartheta}, \tilde{\theta}) = \text{maximum}\,\{I(\bar{u}, \bar{\vartheta}, \theta)\}. \tag{54}$$

From (49) we know that for any solution $(\bar{u}, \bar{\vartheta}, \theta)$ of the E–L variational eqns (42)–(44) we have

$$I(\bar{u}, \bar{\vartheta}, \theta) = \frac{1}{R_{\lambda_2,\lambda_3}} \tag{55}$$

and comparison of (54) and (55) shows that

$$\tilde{R}_{\lambda_2,\lambda_3} \le R_{\lambda_2,\lambda_3}. \tag{56}$$

The variational problems (50)–(53) generates a complete set of eigenfunctions $(\bar{u}_i, \bar{\vartheta}_i, \theta_i)$ and a corresponding set of eigenvalues $(R^{(i)}_{\lambda_2,\lambda_3})^{-1}$[10]. This set of functions has the orthogonal property

$$\begin{aligned}
I(\bar{u}_i, \bar{\vartheta}_i, \theta_i;\quad \bar{u}_k, \bar{\vartheta}_k, \theta_k) = {}&\\
&\int \bar{u}_i \cdot (\operatorname{grad} \bar{u}_k) \cdot \bar{q} + \int \bar{u}_k \cdot (\operatorname{grad} \bar{u}_i) \cdot \bar{q}\\
&+ n_1 n_2 n_7 \int (\bar{\vartheta}_i \cdot \operatorname{curl} \bar{u}_k + \bar{\vartheta}_k \cdot \operatorname{curl} \bar{u}_i)\\
&+ \lambda_2 \left\{ \frac{n_1 n_2}{n_7} \int (\bar{\vartheta}_i \cdot \operatorname{curl} \bar{u}_k + \bar{\vartheta}_k \cdot \operatorname{curl} \bar{u}_i) \right.\\
&+ \int j\bar{u}_i \cdot (\operatorname{grad} \bar{\vartheta}_k) \cdot \bar{\nu} + \int j\bar{u}_k \cdot (\operatorname{grad} \bar{\vartheta}_i) \cdot \bar{\nu}\\
&+ 2n_8 \int j\nu^* \bar{\vartheta}_i \cdot \bar{\vartheta}_k + 2n_8 \int j\bar{\nu} \cdot (\theta_i \bar{\vartheta}_k + \theta_k \bar{\vartheta}_i)\\
&+ \lambda_3 \left\{ \frac{3}{2} \int j\nu \operatorname{div} (\theta_i \bar{u}_k + \theta_k \bar{u}_i) \right.\\
&\left.\left. + 3n_8 \int j\nu^* \theta_i \theta_k + 6n_8 \int j\nu \theta_i \theta_k \right\} \right\} = 0 \qquad (i \ne k)
\end{aligned} \tag{57}$$

$$D(\bar{u}_i, \bar{\vartheta}_i, \theta_i;\quad \bar{u}_k, \bar{\vartheta}_k, \theta_k) = \int (\text{curl } \bar{u}_i) \cdot (\text{curl } \bar{u}_k) + \lambda_2 \left\{ 2n_2 \int \bar{\vartheta}_i \cdot \bar{\vartheta}_k + n_3 \int (\text{curl } \bar{\vartheta}_i) \cdot (\text{curl } \bar{\vartheta}_k) + n_4 \int (\text{div } \bar{\vartheta}_i) \text{ div } \bar{\vartheta}_k) \right\} + \lambda_3 \left\{ n_5 \int (\text{grad } \theta_i) \cdot (\text{grad } \theta_k) + n_6 \int \theta_i \theta_k \right\} = 0 \qquad (i \neq k). \tag{58}$$

For any admissible function $(\bar{u}, \bar{\vartheta}, \theta)$ of the maximization problems (36)–(39) we have

$$I(\bar{u}, \bar{\vartheta}, \theta) \leq \underset{(i)}{l.u.b.} \left(\frac{1}{R^{(i)}_{\lambda_2,\lambda_3}} \right). \tag{59}$$

The left side expression in (59) can be made arbitrarily close to its maximum value by a suitable combination of the complete and orthonormal set of eigenfunctions $(\bar{u}_i, \bar{\vartheta}_i, \theta_i)$ and we have the result

$$\frac{1}{\tilde{R}_{\lambda_2,\lambda_3}} = \underset{(i)}{l.u.b.} \left(\frac{1}{R^{(i)}_{\lambda_2,\lambda_3}} \right). \tag{60}$$

Theorem. Let $(\bar{u}, \bar{\vartheta}, \theta)$ be any solution of the variational problems (36)–(39) for fixed values of $n_2, n_3, n_4, n_5, n_6, n_7$ and n_8, and positive values of λ_2 and λ_3, and let

$$\max I(\bar{u}, \bar{\vartheta}, \theta) = \frac{1}{\tilde{R}_{\lambda_3,\lambda_3}}. \tag{61}$$

The eigenvalue problem defined by (42)–(44) has a least eigenvalue and the primary flow is stable if its Reynolds number Re is less than the least eigenvalue $\tilde{R}_{\lambda_2,\lambda_3}$. Further, given a complete set of eigenfunctions $(\bar{u}_i, \bar{\vartheta}_i, \theta_i)$ corresponding to the eigenvalues $R^{(i)}_{\lambda_2,\lambda_3}$, we have

$$\tilde{R}_{\lambda_2,\lambda_3} = \underset{(i)}{g.l.b.} \{R^{(i)}_{\lambda_2,\lambda_3}\}. \tag{62}$$

The above assertion follows on noting that for a suitably normalized solution $(\bar{u}, \bar{\vartheta}, \theta)$ we have from (21)–(23), (28) and (30)

$$\frac{\mathrm{d}}{\mathrm{d}t} T = I(\bar{u}, \bar{\vartheta}, \theta) - n_1 \tag{63}$$

and for any admissible solution $(\bar{u}, \bar{\vartheta}, \theta)$, eqn (54)

$$\frac{\mathrm{d}T}{\mathrm{d}t} \leq \frac{1}{\tilde{R}_{\lambda_2,\lambda_3}} - n_1 \tag{64}$$

Since $n_1^{-1} = Re$ is the Reynolds number of the primary flow, the result follows in view of (29).

The least eigenvalue $\tilde{R}_{\lambda_2,\lambda_3}$ involves the two parameters λ_2 and λ_3 both positive and arbitrary otherwise. One may seek the maximum of this eigenvalue over the first quadrant of the (λ_2, λ_3)—plane and this gives a sharp estimate of the critical value of the Reynolds number of the primary flow. Flows with Reynolds number Re less than this critical value are stable.

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