Geometrically Nonlinear Torsion of an Aeolotropic Cylinder

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ABSTRACT: *The problem of finite torsion of a cylindrically aeolotropic hollow circular cylinder is considered. An approximate solution of the governing differential equation is obtained. Results are compared with those of earlier work.*

Introduction

Seth (1, 2) solved the problem of finite torsion of a solid circular cylinder when the material is either isotropic or transversely isotropic. Solution of the governing differential equation was obtained in the form of a convergent series. In this paper we apply the method of Seth for the torsion of a hollow cylinder when the material is cylindrically aeolotropic. As the governing differential equation attains complexity, its solution approaches that indicated by Mitra (3) up to a desired power of τ^2, where τ is the angle of twist.

Components of Displacement

Let the cylinder whose internal radius r_1, and external radius r_2 in the deformed state be subjected to a finite twist τ. From considerations of symmetry, we assume the displacement components are given as (1)

$$\begin{aligned} u &= x(1 - \beta \cos \tau z) - y\beta \sin \tau z \\ v &= y(1 - \beta \cos \tau z) + x\beta \sin \tau z \\ w &= \alpha z \end{aligned} \tag{1}$$

where β is a function of r $(= \sqrt{x^2 + y^2})$ and α is a constant to be determined.

In cylindrical coordinates these components are given as

$$\begin{aligned} u_r &= r(1 - \beta \cos \tau z) \\ u_\theta &= r\beta \sin \tau z \\ u_z &= \alpha z. \end{aligned} \tag{2}$$

Stresses, Strains and Stress-Strain Relations

In the theory of finite deformation strain components are described as employing the coordinates of the particle either in the strained state as independent variables or in the unstrained state as independent variables. The former is the Eulerian method and the latter is the Lagrangian method. For actual applications of the theory the importance of using the Eulerian method is stressed by Seth (1) and Murnagan (4). In terms of displacement vector u_i, the components of finite strain in the Eulerian coordinates are given as

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}. \tag{3}$$

In the Lagrangian coordinates the latter term in Eq. 3 is positive.

From Eqs. 2 and 3, we obtain

$$\begin{aligned} e_{rr} &= \tfrac{1}{2}[1 - (r\beta^1 + \beta)^2] \\ e_{\theta\theta} &= \tfrac{1}{2}(1 - \beta^2) \\ e_{zz} &= \alpha - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\tau^2 r^2 \beta^2 \\ e_{\theta z} &= \tau r \beta^2 \\ e_{\theta z} &= \tau r \beta^2 \\ e_{r\theta} &= e_{zr} = 0 \end{aligned} \tag{4}$$

where

$$\beta' = \frac{d\beta}{dr}.$$

When the material is cylindrically aeolotropic, the stress–strain relations are given as

$$\begin{aligned} \widehat{rr} &= c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} \\ \widehat{\theta\theta} &= c_{21}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz} \\ \widehat{zz} &= c_{31}e_{rr} + c_{32}e_{\theta\theta} + c_{33}e_{zz} \\ \widehat{z\theta} = c_{44}e_{\theta z}; \quad \widehat{rz} &= c_{55}e_{rz}; \quad \widehat{r\theta} = c_{66}e_{r\theta} \end{aligned} \tag{5}$$

where $c_{ij} = c_{ji}$.

Equations of Equilibrium

The only equation which is not identically satisfied is

$$\frac{\partial\widehat{rr}}{\partial r} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = 0 \tag{6}$$

which gives

$$c_{11}(r\beta' + \beta)(2\beta' + r\beta'') + c_{12}\beta\beta' + c_{13}\tau^2(r\beta^2 + r^2\beta\beta')$$
$$- \frac{1}{2r}\left[(c_{11} - c_{22})(1 - \beta^2) - (c_{11} - c_{12})r^2\left(\beta^{12} + \frac{2\beta\beta'}{r}\right)\right.$$
$$\left. + (c_{13} - c_{23})(2\alpha - \alpha^2 - \tau^2\beta^2)\right] = 0. \tag{7}$$

Adopting the method of Mitra (3) we assume that

$$\beta = B_0 + B\tau^2 + B_1\tau^4 + B_2\tau^6 + \cdots \tag{8}$$

where B_0, B, $B_1\cdots$ etc., are functions of r only. Then substituting this in Eq. 7 we must solve the problem of retaining the terms which contain up to the desired powers of τ^2. Here, we only take the first two terms. Hence,

$$\beta = B_0 + B\tau^2. \tag{9}$$

We see from Eq. 1 that

$$B_0 = 1 \tag{10}$$

and Eq. 7 gives

$$r^2B'' + 3rB' + c_0B_0 = c_1r^2 + c_2 \tag{11}$$

where

$$c_0 = \frac{c_{11} - c_{22}}{c_{11}}; \qquad c_1 = \frac{c_{23} - 3c_{13}}{c_{11}};$$

$$c_2 = \frac{(c_{13} - c_{23})(2\alpha - \alpha^2)}{2c_{11}\tau^2}. \tag{12}$$

Therefore, by integrating Eq. 11, we obtain

$$B = ar^{\alpha_1} + br^{\alpha_2} + \frac{c_1r^2}{8 + c_0} + \frac{c_2}{c_0} \tag{13}$$

where α_1 and α_2 are the roots of the equation

$$x^2 + 2x + c_0 = 0 \tag{14}$$

and a and b are the integration constants.

Boundary Conditions

a) The boundaries $r = r_1$ and $r = r_2$ should be free from tractions. This is satisfied if

$$\widehat{rr} = 0 \qquad \text{when} \quad r = r_1 \qquad \text{and} \quad r = r_2 \tag{15}$$

which gives

$$K_1ar^{\alpha_1} + K_2br^{\alpha_2} + K_3(1 - \alpha)^2 = K_3 + K_4r^2; \qquad r = r_1, r_2 \tag{16}$$

where

$$K_1 = r^2[c_{11}(1 + \alpha_1) + c_{12}]$$
$$K_2 = r^2[c_{21}(1 + \alpha_2) + c_{22}]$$
$$K_3 = \tfrac{1}{2}\left[\frac{(c_{11} + c_{12})(c_{23} - c_{13})}{c_{11} - c_{22}} - c_{13}\right] \tag{17}$$

and

$$K_4 = -\left[\frac{(3c_{11} - c_{12})c_1}{8 + c_0} + \frac{c_{13}}{2}\right]\tau^2.$$

b) The tractions on any cross section are statically equivalent to a single couple whose axis is the Z-axis. This is satisfied if

$$\int \widehat{zx}\, dx\, dy = 0 \qquad \int \widehat{yz}\, dx\, dy = 0 \tag{18}$$

$$\int x\widehat{zz}\, dx\, dy = 0 \qquad \int y\widehat{zz}\, dx\, dy = 0 \tag{19}$$

and

$$\int_{r_1}^{r_2} r\widehat{zz}\, dr = 0. \tag{20}$$

Since the Z-axis passes through the center of the circular cross section, we see that Eqs. 18 and 19 are identically satisfied and Eq. 20 gives

$$L_1 a + L_2 b + L_3(1 - \alpha)^2 = L_4 \tag{21}$$

where

$$L_1 = \tau^2[c_{31}(1 + \alpha_1) + c_{32}]\left[\frac{r_2^{\alpha_1+2} - r_1^{\alpha_1+2}}{\alpha_{1+2}}\right]$$

$$L_2 = \tau^2[c_{32}(1 + \alpha_2) + c_{32}]\left[\frac{r_2^{\alpha_2+2} - r_1^{\alpha_2+2}}{\alpha_{1+2}}\right]$$

$$L_3 = \frac{r_2^2 - r_1^2}{4}\left[c_{33} - \frac{(c_{31} + c_{32})(c_{13} - c_{23})}{c_{11} - c_{22}}\right]$$

$$L_4 = \left[c_{33} - \frac{c_{31}^2 - c_{32}^2}{c_{11} - c_{22}}\right]\frac{r_2^2 - r_1^2}{4} - \tau^2\,\frac{(r_2^4 - r_1^4)}{4}\left[c_1\,\frac{(3c_{11} + c_{12})}{8 + c_0} + \frac{c_{33}}{2}\right]. \tag{22}$$

Solving Eqs. 16 and 21 for a, b, and α, we obtain

$$\begin{aligned} Da &= K_2L_3[K_4r_2^{\alpha_2}(r_1^2 - r_2^2) - (K_3 + K_4r_2^2)(r_1^{\alpha_2} - r_2^{\alpha_2})] \\ &\qquad + K_3[K_2L_4(r_1^{\alpha_2} - r_2^{\alpha_2}) - K_4L_2(r_1^2 - r_2^2)] \qquad (23) \\ Db &= K_1L_3[(K_3 + K_4r_2^2)(r_1^{\alpha_1} - r_2^{\alpha_1}) - K_4r_2^{\alpha_1}(r_1^2 - r_2^2)] \\ &\qquad + K_3[L_1K_4(r_1^2 - r_2^2) - L_4K_1(r_1^{\alpha_1} - r_2^{\alpha_1})] \qquad (24) \\ D(1 - \alpha)^2 &= L_1K_2[(K_3 + K_4)r_1^{\alpha_2} - (K_3 + K_4r_1^2)r_2^{\alpha_2}] \\ &\qquad + L_2K_1[(K_3 + K_4r_1^2)r_2^{\alpha_1} - (K_3 + K_4r_2^2)r_1^{\alpha_1}] \\ &\qquad + L_4K_1K_2[r_1^{\alpha_1}r_2^{\alpha_2} - r_1^{\alpha_2}r_2^{\alpha_1}] \qquad (25) \end{aligned}$$

where

$$D = K_1K_2L_3[r_2^{\alpha_2}(r_1^{\alpha_1} - r_2^{\alpha_1}) - r_2^{\alpha_1}(r_1^{\alpha_2} - r_2^{\alpha_2})] + K_3[L_1K_2(r_1^{\alpha_2} - r_2^{\alpha_2}) - L_2K_1(r_1^{\alpha_1} - r_2^{\alpha_1})]. \quad (26)$$

Components of the Stress and the Torsional Couple

The components of stress are given by

$$\widehat{rr} = -K_1ar^{\alpha_1} - K_2br^{\alpha_2} - K_3(1-\alpha)^2 + K_4\tau^2r^2 + K_5 \quad (27)$$

$$\widehat{\theta\theta} = -a\tau^2r_1^{\alpha_1}[c_{21}(1+\alpha_1) + c_{22}] - br^{\alpha_2}\tau^2[c_{21}(1+\alpha_2) + c_{22}] - \tau^2r^2\left[\frac{(3c_{21}+c_{22})}{8+c_0} + \frac{c_{23}}{2}\right] - \left[\frac{c_{23}}{2} - \frac{(c_{12}+c_{22})(c_{22}-3c_{12})}{c_{11}-c_{22}}\right](1-\alpha)^2 + \frac{c_{23}}{2} - \frac{(c_{12}+c_{22})(c_{23}-3c_{13})}{c_{11}-c_{22}} \quad (28)$$

$$\widehat{zz} = -ar^{\alpha_1}\tau^2[c_{31}(1+\alpha_1) + c_{32}] - b\tau^2r^{\alpha_2}[c_{32}(1+\alpha_2) + c_{33}] - \left[\frac{(3c_{31}-c_{32})c_1}{8+c_0} + \frac{c_{23}}{2}\right]\tau^2r^2 - \left[\frac{c_{33}}{2} - \frac{c_{23}^2 - c_{13}^2}{c_{11}-c_{22}}\right](1-\alpha)^2 + \frac{c_{33}}{2} - \frac{c_{23}^2 - c_{13}^2}{c_{11}-c_{22}} \quad (29)$$

$$\widehat{\theta z} = c_{44}\tau^2\left[1 + 2\tau^2(ar^{\alpha_1} + br^{\alpha_2}) + \frac{c_1r^2}{8+c_0} + \frac{c_2}{c_0}\right] \qquad \widehat{r\theta} = \widehat{rz} = 0. \quad (30)$$

The torsional couple is given by

$$N = \int_0^{2\pi}\int_{r_1}^{r_2} \widehat{z\theta}r^2\,dr\,d\theta = 2\pi c_{44}\tau\int_{r_1}^{r_2}\beta^2r^3\,dr$$

$$= 2\pi c_{44}\tau\left[\left(1 + \frac{2\tau^2c_2}{c_0}\right)\frac{r_2^4 - r_1^4}{4} + \frac{2\tau^2c_1}{8+c_0}\,\frac{r_2^6 - r_1^6}{6} + 2a\tau^2(r_2^{\alpha_1+2} - r_1^{\alpha_1+2}) + 2b\tau^2\left(\frac{r_2^{\alpha_2+4} - r_1^{\alpha_2+4}}{\alpha_{2+4}}\right)\right]. \quad (31)$$

Observe that only when the first power of τ is retained, we get

$$N = 2\pi c_{44}\tau\left[\frac{r_2^4 - r_1^4}{4}\right].$$

This tallies with the results of Murnagan (4), Riz (5) and Rivlin (6).

References

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