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# Mechanism Design by Chance Constrained Programming Techniques

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**Abstract**

The problem of optimum design of mechanisms for minimum mechanical and structural error is formulated as a stochastic programming problem. The nominal link lengths, the tolerances on link lengths and the clearances in joints are considered as design parameters. The constraints are also stated in probabilistic terms, i.e., each constraint is required to be satisfied with certain minimum specified probability. The techniques of chance constrained programming are applied to solve the optimization problem. The optimum design of a simple 4-bar function generating mechanism is considered to illustrate the efficiency of the proposed method.

## Introduction

CONSIDERABLE progress has been made in the field of optimum design of mechanisms using nonlinear programming techniques[1, 2]. In most of the reported literature, the minimization of structural error has been taken as the objective[3–6]. Due to the existence of play in the joints and tolerances on the link lengths, mechanical error of appreciable magnitude will also be introduced. Unless care is taken to assign proper clearances and tolerances to the members of linkage, the value of mechanical error may be much higher than that of the structural error. Some work has been done regarding the allocation of tolerances and clearances in mechanisms for minimum cost[7]. It can be seen from the available literature that no systematic effort has been made for the design of mechanisms for minimum structural and mechanical error.

It has been recognized that the very nature of clearances and tolerances makes the function generating process probabilistic[8]. Hence a more realistic procedure for the optimization of linkages would be to formulate the problem as a stochastic programming problem. A stochastic programming problem is an optimization problem in which some or all of the parameters are described by random variables rather than by deterministic quantities. The basic idea of all stochastic programming methods is to convert the probabilistic nature of the problem into an equivalent deterministic model. In this work, the idea of employing deterministic equivalence by applying the techniques of chance constrained programming due to Charnes and Cooper[9] is used. The design of a simple planar 4-bar function generating mechanism, for minimizing structural and mechanical errors, is considered for illustration. The nominal link lengths and tolerance widths of link lengths are considered as deterministic design variables (thus the actual link lengths would become probabilistic) and the clearances (locations of pin centers in their respective hinge joints) as probabilistic design variables. The behavioural constraints are also stated in probabilistic terms.

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**Deterministic formulation of the problem**

When a linkage with $n$ design parameters $x_1, x_2 \ldots, x_n$ transforming a motion defined by the input variable $\theta$ into another motion defined by the output variable $\emptyset$ is to be designed optimally, the objective function for minimization can be taken as

$$f(\vec{X}) = \sum_{i=1}^{q} [\phi_{a_i}(\vec{X}, \theta_i) - \phi_{r_i}]^2 \tag{1}$$

where the range of the input angle ($\Delta\theta$) is assumed to have been divided into $q - 1$ parts so that $\theta_1$ and $\theta_q$ indicate the starting and the final angular positions of the input link. Here $\phi_{a_i}$ and $\phi_{r_i}$ denote the actually generated (with structural and mechanical errors) and the required values of the output angle corresponding to the input angle $\theta_i$ respectively. The $q$ points considered in Eq. (1) can be treated as some sort of accuracy points since the error between the generated and the required values of the output angle is minimized at these points. Normally the link lengths $l_1, l_2 \ldots,$ are restricted to lie within certain limits as

$$l_j^{(l)} \leq l_j \leq l_j^{(u)} \tag{2}$$

where the superscripts $l$ and $u$ indicate the lower and upper bounds respectively. The transmission angle of the mechanisms at any accuracy point, $\gamma_i$, is restricted as

$$\gamma^{(l)} \leqslant \gamma_i \leqslant \gamma^{(u)} \tag{3}$$

For the four bar linkage shown in Fig. 1, the input–output relation can be expressed as

$$\phi_{a_i}(\vec{X}, \theta_i) = 2 \tan^{-1}\left(\frac{A_i \pm D_i}{B_i + C_i}\right) \tag{4}$$

where

$$A_i = \sin\theta_i, \tag{5}$$

$$B_i = \cos\theta_i - l_1/l_2, \tag{6}$$

$$C_i = \frac{l_1^2 + l_2^2 - l_3^2 + l_4^2}{2l_2l_4} - \frac{l_1}{l_4}\cos\theta_i, \tag{7}$$

$$D_i = (A_i^2 + B_i^2 - C_i^2)^{1/2}, \tag{8}$$

and the transmission angle is given by

$$\gamma_i = \cos^{-1}\left[-\left(\frac{l_2^2 - l_3^2 - l_4^2 + l_1^2 - 2l_1l_2\cos\theta_i}{2l_3l_4}\right)\right] \tag{9}$$

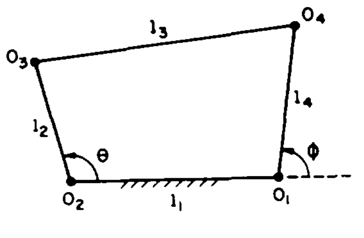



**Figure 1.** A planar 4-bar linkage.

for $i = 1, 2, \ldots, q$. In order to avoid imaginary values for $D_i$ in eqn (8), one has to impose the constraints

$$A_i^2 + B_i^2 - C_i^2 \geq 0, \quad i = 1, 2, \ldots, q. \tag{10}$$

The design vector $\vec{X}$ may represent the parameters like $l_1$, $l_2$, $l_3$ and $l_4$. Thus the above mechanism optimization problem, according to the deterministic design philosophy, can be stated in the form of a standard nonlinear programming problem as

$$\left.\begin{array}{l}\text{Find } \vec{X} \text{ to minimize } f(\vec{X}) \\ \text{subject to } g_j(\vec{X}) \geq 0,\ j = 1, 2, \ldots, m\end{array}\right\} \tag{11}$$

where

$$\vec{X} = \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix}$$

is as $n$-dimensional vector of design variables and $m$ is the number of constraints.

**Stochastic formulation of the problem**

When some of the parameters involved in the objective function and constraints vary about their mean values, the problem has to be formulated as a stochastic programming problem, which can be stated in standard form as:

$$\text{Find } \vec{X} \text{ to minimize } f(\vec{Y}) \tag{12}$$

subject to

$$P[g_j(\vec{Y}) \geq 0] \geq p_j,\ j = 1, 2, \ldots, m \tag{13}$$

where $\vec{X}$ is the vector of $n$ design variables (some or all of them may be random) and $\vec{Y}$ is the vector of $N$ random variables (may contain some or all of the design variables $x_i$), the symbol $P[\ldots]$ indicates the probability of occurrence of the event $[\ldots]$, and eqns (13) denote that the probability of realizing $g_j(\vec{Y})$ greater than zero must be greater than or equal to the specified probability $p_j$. The problem stated in eqns (12) and (13) can be converted into an equivalent deterministic programming problem by applying the chance constrained programming techniques as follows.

**Objective function**

If $F$ represents the objective function in terms of the random variables $y_i$, $i = 1, 2, \ldots, N$, $F(\vec{Y})$ can be expanded about the mean values of $y_i$, $\mathbf{y}_i$, as

$$F(\vec{Y}) = F(\vec{\mathbf{Y}}) + \sum_{i=1}^{N} \left.\frac{\partial F}{\partial y_i}\right|_{\vec{\mathbf{Y}}} \cdot (y_i - \mathbf{y}_i) + \text{higher order derivative terms} \tag{14}$$

If the standard deviations of $y_i$ are small, $F(\vec{Y})$ can be approximated by the first two terms of eqn (14):

$$F(\vec{Y}) = F(\vec{\mathbf{Y}}) + \sum_{i=1}^{N} \left.\frac{\partial F}{\partial y_i}\right|_{\vec{\mathbf{Y}}} \cdot \mathbf{y}_i + \sum_{i=1}^{N} (\partial F/\partial y_i)\Big|_{\vec{\mathbf{Y}}} \cdot y_i \equiv \psi(\vec{Y}) \tag{15}$$

If all $y_i$'s are assumed to follow normal distribution, $\psi$ also follows normal distribution. The mean and the variance of $\psi$ are respectively given by

$$\boldsymbol{\psi} = \psi(\vec{\mathbf{Y}}) = F(\vec{\mathbf{Y}}) \tag{16}$$

and

$$\mathrm{Var}\,[\psi] = \sigma_\psi^{\,2} = \sum_{i=1}^{N} [(\partial F/\partial y_i)|_{\bar{Y}}]^2 \cdot \sigma_{y_i}^2 \tag{17}$$

since all $y_i$'s are independent. Notice that $\sigma_{()}$ has been used to denote the standard deviation of ( ). For the purpose of optimization, a new objective function $F_1(\vec{Y})$ is constructed as

$$F_1(\bar{\vec{Y}}) = a_1 \cdot \psi(\vec{Y}) + a_2 \cdot \sigma_\psi^{\,2}(\vec{Y}) \tag{18}$$

where $a_1 \geq 0$ and $a_2 \geq 0$, and their numerical values indicate the relative importance of $\psi$ and $\sigma_\psi^{\,2}$ for minimization. Another way of dealing with the standard deviation of $\psi$ is to minimize $F_2 \equiv \psi$ subject to $\sigma_\psi \leq a_3 \cdot \psi$ where $a_3$ is a constant, along with the other constraints.

### Constraints

If some parameters are random in nature, the constraints will also be probabilistic and one would like to have the probability that a given constraint is satisfied to be greater than a certain value. This is precisely what is stated in eqn (13) also.

The constraint inequality (13) can be written as

$$\int_0^\infty f_{g_j}(g_j) \cdot \mathrm{d}g_j \geq p_j \tag{19}$$

where $f_{g_j}(g_j)$ is the probability density function of the random variable $g_j$ (a function of several random variables is also a random variable) whose range is assumed to be 0 to $\infty$. The constraint function $g_j(\vec{Y})$ can be expanded around, say, the vector of mean values of the random variables $\bar{\vec{Y}}$ as

$$g_j(\vec{Y}) \simeq g_j(\bar{\vec{Y}}) + \sum_{i=1}^{N} [(\partial g_j/\partial y_i)|_{\bar{Y}}](y_i - \bar{y}_i) \tag{20}$$

From this equation, the mean value, $\bar{g}_j$, and the standard deviation, $\sigma_{g_j}$, of $g_j$ can be obtained as

$$\bar{g}_j = g_j(\bar{\vec{Y}}) \tag{21}$$

and

$$\sigma_{g_j} = \left\{\sum_{i=1}^{N} [(\partial g_j/\partial x_i)|_{\bar{Y}}]^2 \cdot \sigma_{y_i}^{\,2}\right\}^{1/2} \tag{22}$$

With the transformation of variable

$$\theta = \frac{g_j - \bar{g}_j}{\sigma g_j}, \tag{23}$$

and noting that

$$\int_{-\infty}^{t} \frac{1}{\sqrt{(2\pi)}} e^{-t^2/2} \cdot \mathrm{d}t = 1, \tag{24}$$

eqn (19) can be expressed as

$$\int_{-(\bar{g}_j/\sigma_{g_j})}^{\infty} \frac{1}{\sqrt{(2\pi)}} e^{-\theta^2/2} \cdot \mathrm{d}\theta \geq \int_{-\epsilon_j(p_j)}^{\infty} \frac{1}{\sqrt{(2\pi)}} e^{-t^2/2} \cdot \mathrm{d}t \tag{25}$$

where $\epsilon_j(p_j)$ depends upon the specified probability level $p_j$. Thus

$$-g_j/\sigma g_j \leq -\epsilon_j(p_j) \tag{26}$$

i.e.,

$$-g_j + \epsilon_j(p_j) \cdot \sigma g_j \leq 0 \tag{27}$$

Equation (27) can be rewritten as

$$G_j(\vec{Y}) = g_j(\vec{Y}) - \epsilon_j(p_j)\left\{\sum_{i=1}^{N} [\partial g_j/\partial y_i)|_{\bar{Y}}]^2 \cdot \sigma y_i^2\right\}^{1/2} \geq 0 \tag{28}$$

Thus the optimization problem in eqns (12) and (13) can be stated in its equivalent deterministic form as:

minimize $F_1$ given by eqn (18) subject to $m$
constraints of the type shown in eqn (28).

The expressions of $\psi(\vec{Y})$, $\sigma_\psi(\vec{Y})$ and $g_j(\vec{Y})$ can be derived from the known equations of $F$ and $g_j$ by using the standard techniques of probability theory.

**Stochastic formulation of the 4-bar linkage problem**

In the present work, the nominal link lengths $l_2$, $l_3$ and $l_4$, the starting angular position of the input link, $\theta_1$, the tolerance widths on the link lengths, $t_i$, $i = 1, \ldots, 4$ and the clearances in joints $r_{12}$, $r_{23}$, $r_{34}$ and $r_{41}$ (strictly speaking, the locations of pin centers in respective hinge joints) are considered as the design variables $x_1, x_2, \ldots, x_{12}$ respectively (Figs. 1 and 2). The nominal link length $l_1$ is taken as unity. It is to be noted that the first eight design variables are deterministic (the link lengths with tolerances would, however, be probabilistic) and the remaining four are probabilistic.

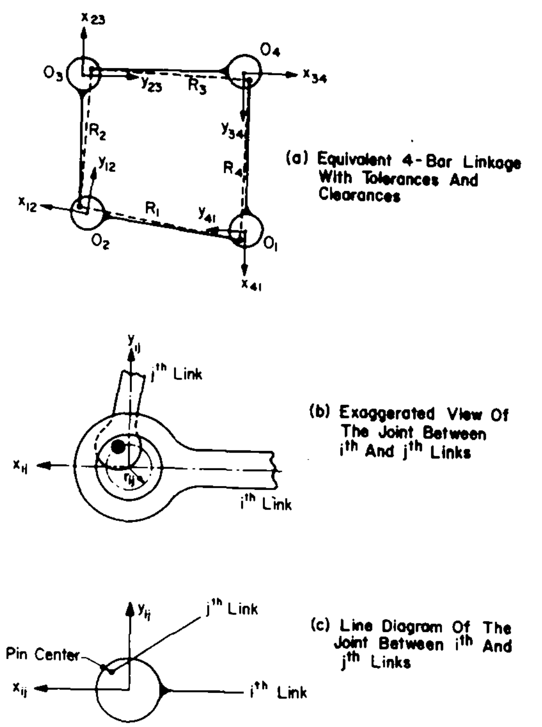



**Figure 2.** A 4-bar linkage with tolerances and clearances.

The link lengths with tolerances and the clearances in the joints are treated as random variables and are collectively represented by the vector $\vec{Y}$. The vector of mean values of these random variables is given by

$$\vec{\mathbf{Y}} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The expressions of $\psi(\vec{\mathbf{Y}})$ and $\sigma_\psi(\vec{\mathbf{Y}})$ can be obtained as (from Appendix A):

$$\psi(\vec{\mathbf{Y}}) \simeq \sum_{i=1}^{q} [\phi_{a_i}(\vec{X}, \theta_i)|_{\vec{Y}} - \phi_{r_i}]^2$$

= sum of the squares of error at the mean values of the random variables
= sum of the squares of differences between the curves $A_1B_1$ and $A_2B_2$ in Fig. 3
= structural error (29)

and

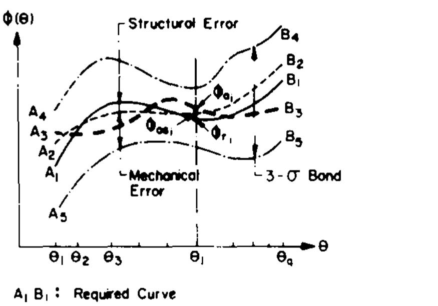
$$\sigma_\psi^2(\vec{\mathbf{Y}}) \simeq \sum_{i=1}^{q} \left\{ \sum_{j=1}^{8} \left( \frac{\partial \phi_{a_i}}{\partial y_j} \bigg|_{\vec{Y}} \right)^2 \cdot \sigma y_j^2 \right\}$$

= sum of the squares of standard deviation of the total error
= sum of the squares of difference between the curves $A_4B_4$ and $A_2B_2$
= mechanical error (30)

where $\sigma_{y_j}$ indicates the standard deviation of $y_j$, $j = 1, \ldots, 8$. If the tolerances and clearances are assumed to correspond to 3-sigma band[10], $\sigma_{y_j} = (t_j/3)$ if $y_j$ denotes link lengths with tolerances and $\sigma_{y_j} = (r_{jk}/3)$ if $y_j$ denotes clearance. Since tolerance widths and clearances are taken as design variables $x_i$, $i = 5, \ldots, 12$, eqn (30) can be rewritten as

$$\sigma_\psi^2(\vec{\mathbf{Y}}) \simeq \sum_{i=1}^{q} \sum_{j=1}^{8} \left( \frac{\partial \phi_{a_i}}{\partial y_j} \bigg|_{\vec{Y}} \right)^2 \cdot \frac{x_{j+4}^2}{9} \tag{31}$$

In the stochastic formulation, the constraints given by eqn (2) for $j = 2, 3, 4$ remain deterministic as $l_j$ are assumed to be the nominal link lengths. However, the constraints of eqns




$A_1B_1$: Required Curve
$A_2B_2$: Generated Curve Without Tolerances And Clearances (i.e. When $x_i = 0$, $i = 5, 6, \ldots 12$)
$A_3B_3$: Actually Generated Curve With Tolerances And Clearances
$A_4B_4$ & $A_5B_5$: Define The Band Width In Which The Curve $A_3B_3$ Falls 99.73% Of The Time

**Figure 3.** Required and generated outputs.

(3) and (10) can be expressed probabilistically as

$$P[\gamma_i - \gamma^{(l)} \geq 0] \geq p_i, \quad i = 1, 2, \ldots, q \tag{32}$$

$$P[\gamma^{(u)} - \gamma_i \geq 0] \geq p_i, \quad i = 1, 2, \ldots, q \tag{33}$$

$$P[A_i^2 + B_i^2 - C_i^2 \geq 0] \geq p_i, \quad i = 1, 2, \ldots, q \tag{34}$$

where

$$\gamma_i = \cos^{-1}\left[-\left(\frac{L_2^2 - L_3^2 - L_4^2 + L_1^2 - 2L_1L_2\cos\theta_i}{2L_3L_4}\right)\right] \tag{35}$$

with $L_i$ given by eqn ($A_2$) and $A_i$, $B_i$ and $C_i$ by eqns. ($A_4$), ($A_5$) and ($A_6$), respectively, in appendix A. By comparing eqns (32) to (34) with eqn (13), $g_i$ can be identified to be $\gamma_i - \gamma^{(l)}$ in eqn (32), $\gamma^{(u)} - \gamma_i$ in eqn (33) and $A_i^2 + B_i^2 - C_i^2$ in eqn (34). Thus $g_i = g_i(\vec{\mathbf{Y}})$ and $(\partial g_i/\partial y_j)|_{\vec{Y}}$ can be evaluated readily at the vector of mean values of the random variables,

$$\vec{\mathbf{Y}} \equiv \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

**Solution procedure**

The equivalent deterministic optimization problem that resulted from the application of chance constrained programming techniques has been solved using nonlinear programming techniques. The interior penalty function method, coupled with a variable metric method of unconstrained minimization and cubic interpolation method of one-dimensional search, is used to solve the constrained problem[11]. The composite objective function ($\beta$) to be minimized in this method is given by

$$\beta(\vec{\mathbf{X}}, s_k) = F_1(\vec{X}, \vec{\mathbf{Y}}) + s_k \sum_{j=1}^{m} \frac{1}{G_j(\vec{X}, \vec{\mathbf{Y}})} \tag{36}$$

where $F_1$ is given by eqn (18) and $G_j$ by eqn (28), and $s_k$ is called the penalty parameter. The minimization of $\beta$ starts from an interior feasible point, $\vec{X}_0$. A number of minimizations with successively reduced values of the penalty parameter $s_k$ leads to the desired solution. In the present work, the initial value of $s_k$ is chosen such that

$$s_1 \sum_{j=1}^{m} \frac{1}{G_j(\vec{X}_0, \vec{\mathbf{Y}})} \simeq 0.5\, \mathrm{F}_1(\vec{X}_0, \vec{\mathbf{Y}}) \tag{37}$$

at the starting design vector $\vec{X}_0$. The subsequent values of $s_k$ are found as $s_{k+1} = 0.1\ s_k$, $k = 1, 2, \ldots$. A proper choice of the termination criterion is very important. For the results given in this paper, the algorithm is terminated whenever the decrease in the value of the $\beta$ function in two successive iterations is less than a predetermined small number. This varies from 0.000001 to 0.001 in the present work.

**Example problems**

To illustrate the application of chance constrained programming techniques to mechanism design, the design of a planar 4-bar linkage, for generating function $y = \sin x$ in the range $0° \leq x \leq 90°$, is considered. The ranges of input and output angles are taken as $\Delta\theta = \Delta\phi = 90°$

and the nominal link length $l_1$ is taken as unity. 11 accuracy points ($q = 11$) are considered in the range of the independent variable for the minimization of $F_1$. The transmission angle at any of the accuracy points is required to lie between 30° and 150° with a probability of $p_j = 0.9973$. The same probability value is taken in the constraint of eqn (34). The minimum and maximum permissible nominal lengths of any link are taken as 0.0 and 10.0 respectively. To reduce the scale disparities between the two terms of eqn (18), eqn (18) is modified as

$$F_1(\vec{Y}) = a_1 \cdot w_1 \cdot \psi(\vec{Y}) + a_2 \cdot w_2 \cdot \sigma_\psi^2(\vec{Y}) \tag{38}$$

where the weights $w_1$ and $w_2$ are selected so that

$$w_1 \cdot \psi(\vec{Y}) = w_2 \cdot \sigma_\psi^2(\vec{Y})$$

at the starting point $\vec{X}_0$. The optimization results obtained by giving different values to the weights $a_1$ and $a_2$ are given below.

*Case (i)* When $a_1 = a_2 = 1$:

In this case, the optimum design vector, given in the third column of Table 1, corresponds to $\psi(\vec{Y}) = 0.1352$ and $\sigma_\psi^2(\vec{Y}) = 0.0736 \times 10^{-6}$. The required and mean generated angular positions of the output link, the structural error, the mean value of transmission angle and the 3-$\sigma$ value of the mechanical error at various positions of the input link corresponding to the initial and optimum designs are shown in Table 2. It can be seen that the mean transmission angle at the first three accuracy points is very near to its lower bound of 30° at the optimum point. The maximum value of structural error has been increased from 12.19° to 13.05° (although the total structural error was reduced) and the 3-$\sigma$ value of mechanical error decreased from 3.29° to 1.46°. The progress of optimization is shown in Fig. 4 as a plot between the values of the objective function and the number of one-dimensional minimization steps. The computer time taken on IBM 7044 computer to obtain the optimum point (65 optimization steps) is about 15 min.

*Case (ii)* When $a_1 = 1$ and $a_2 = 10^5$:

In this case the minimization of mechanical error is assumed to be $10^5$ times more important than that of structural error. The required and the mean generated angular positions of the output link, the mean transmission angle and the 3-$\sigma$ values of the mechanical error at various positions of the input link are shown in Fig. 5. The optimum design vector is shown in Table 1.

**Table 1.** Optimization results

| Quantity | | Initial value | Optimum value: Case (i) | Optimum value: Case (ii) | Optimum value: Case (iii) |
|---|---|---|---|---|---|
| Link lengths | $x_1$ | 1.90 | 3.9449 | 2.2955 | 1.9665 |
| | $x_2$ | 2.70 | 4.3398 | 3.1487 | 2.4897 |
| | $x_3$ | 0.85 | 4.9697 | 1.0807 | 0.6614 |
| Starting position of the input link: (radians) | $x_4$ | 2.0283 | 0.2550 | 1.7293 | 2.0193 |
| Tolerances: | $x_5$ | 0.0002 | 0.0004235 | 0.0001001 | 0.0004830 |
| | $x_6$ | 0.0002 | 0.0027189 | 0.0001011 | 0.0005001 |
| | $x_7$ | 0.0002 | 0.0003635 | 0.0001174 | 0.0001874 |
| | $x_8$ | 0.0002 | 0.0004439 | 0.0001076 | 0.0006389 |
| Clearances: | $x_9$ | 0.0002 | 0.0004273 | 0.0001001 | 0.0001486 |
| | $x_{10}$ | 0.0002 | 0.002621 | 0.0001008 | 0.0004762 |
| | $x_{11}$ | 0.0002 | 0.0003576 | 0.0001005 | 0.0001572 |
| | $x_{12}$ | 0.0002 | 0.0004428 | 0.0001076 | 0.0006943 |
| Structural error $\psi(\vec{Y})$ | | 0.2463 | 0.1352 | 0.0280 | 0.0011 |
| Mechanical error: $\sigma_\psi^2(\vec{Y})$ | | $0.3796 \times 10^{-6}$ | $0.0736 \times 10^{-6}$ | $0.0672 \times 10^{-6}$ | $0.1123 \times 10^{-6}$ |

**Table 2.** Comparison of structural and mechanical errors at initial and optimum designs in case (i)

| Accuracy point $i$ | Initial design: Angular position of the output link (degrees) | | Initial design | | | Optimum design: Angular position of the output link (degrees) | | Optimum design | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Required | mean generated | Structural error (degrees) | Mean trans-mission angle (degrees) | 3-$\sigma$ value of mechanical error (degrees) | Required | Mean generated | Structural error (degrees) | Mean trans-mission angle (degrees) | 3 − $\sigma$ value of mechanical error (degrees) |
| 1. | 43.4510 | 43.4510 | 0.0000 | 67.9370 | 3.2946 | −40.6707 | −40.6707 | 0.0000 | 36.6559 | 1.4264 |
| 2. | 57.5301 | 56.5167 | −1.0134 | 74.6963 | 3.1956 | −26.5916 | −28.8213 | −2.2297 | 37.5483 | 1.4196 |
| 3. | 71.2625 | 68.3578 | −2.9047 | 80.5610 | 3.1601 | −12.8592 | −17.2558 | −4.3966 | 38.8019 | 1.4444 |
| 4. | 84.3101 | 79.0859 | −4.2242 | 85.5106 | 3.1630 | 0.1884 | −6.0413 | −5.8529 | 40.3591 | 1.4618 |
| 5. | 96.3516 | 88.7510 | −7.6006 | 89.4913 | 3.1851 | 12.2300 | 4.7914 | −7.4386 | 42.1563 | 1.4565 |
| 6. | 107.0905 | 97.3676 | −9.7229 | 92.4349 | 3.2104 | 22.9689 | 15.2389 | 7.7300 | 44.1289 | 1.4306 |
| 7. | 116.2624 | 104.9340 | −11.3284 | 94.2753 | 3.2260 | 32.1408 | 25.3148 | −6.8260 | 46.2150 | 1.3944 |
| 8. | 123.6415 | 111.4473 | −12.1942 | 94.9637 | 3.2232 | 39.5199 | 35.0414 | −4.4785 | 48.3567 | 1.3609 |
| 9. | 129.0460 | 116.9146 | −12.1314 | 94.4803 | 3.1989 | 44.9243 | 44.4445 | 0.4798 | 50.5012 | 1.3422 |
| 10. | 132.3429 | 121.3578 | −10.9851 | 92.8391 | 3.1563 | 48.2212 | 53.5500 | 5.3288 | 52.6007 | 1.3469 |
| 11. | 133.4509 | 124.8113 | −8.6396 | 90.0845 | 3.1045 | 49.3293 | 62.3818 | 13.0525 | 54.6121 | 1.3792 |

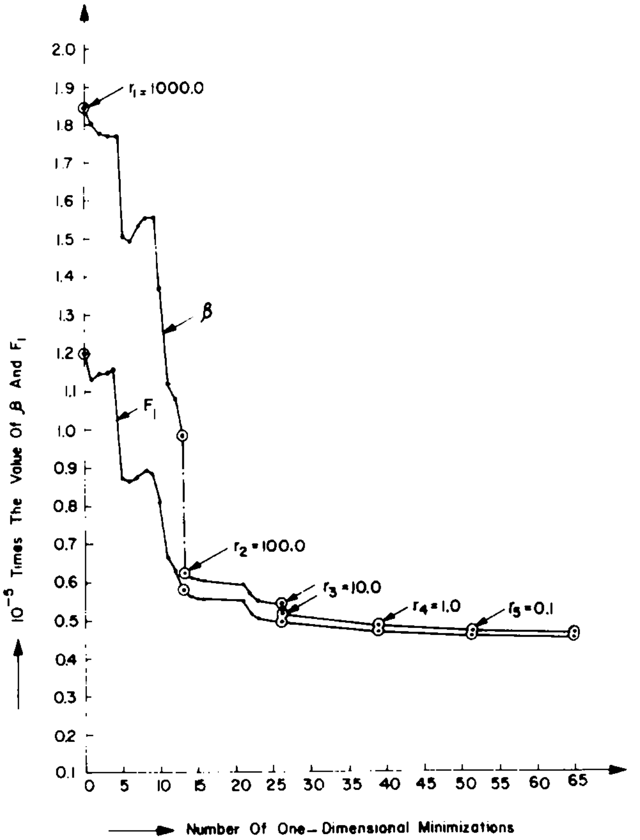



**Figure 4.** Progress of optimization process in case (i).

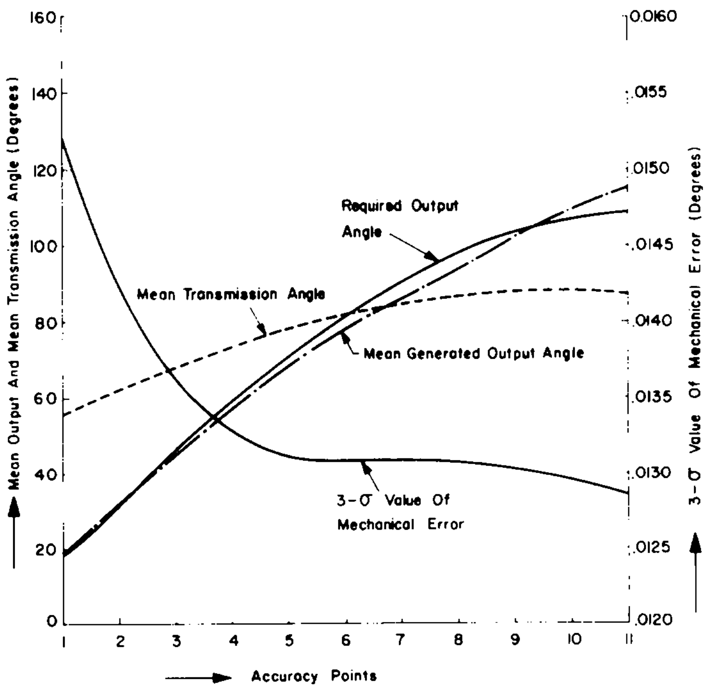



**Figure 5.** Characteristics of the optimum design in case (ii).

It can be seen that all the tolerances and clearances assumed their minimum permissible values at the optimum point. The maximum structural error has been reduced from 12.19° to 4.69° and the maximum 3-$\sigma$ value of the mechanical error from 3.29° to 0.015°.

*Case (iii)* When $a_1 = 10^5$ and $a_2 = 1$:

Here the minimization of structural error is considered to be $10^5$ times more important compared to that of mechanical error. The optimization results are shown in the last column of Table 1. It can be seen that the mean transmission angle at accuracy points 8 and 9 approached its upper bound value. The maximum structural error reduced from 12.19° to 1.00° and the 3-$\sigma$ value of mechanical error from 3.29° to 2.12°. As expected, the structural error is very low and the mechanical error is very large at the optimum design.

## Conclusion

The application of chance constrained programming techniques to the optimum design of mechanisms is presented. Although the design of a simple 4-bar function generating mechanism is considered for illustration, the method presented is quite general and is applicable for the design of any mechanism. Since the probabilistic nature of manufacturing tolerances and clearances is a practical reality, the present method is expected to be a more realistic approach for the optimum design of function generating mechanisms.

## References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## Appendix A

*Derivation of the expressions for $\psi(\vec{Y})$ and $\sigma_\psi(\vec{Y})$*

The link lengths of the equivalent 4-bar mechanism with a consideration of tolerances on link lengths and clearances in joints (shown by dotted lines in Fig. 2c) can be expressed as

$$L_i^2 = (l_i \pm t_i + x_{ij})^2 + y_{ij}^2 \tag{A1}$$

where $t_i$ is the tolerance width on link $i$ and $x_{ij}$ and $y_{ij}$ are the coordinates of the axis of the pin between the links $i$ and $j$ as shown in Fig. 2a. By assuming that $(l_i \pm t_i + x_{ij}) \gg y_{ij}$, eqn (A1) can be reduced to

$$L_i = l_i \pm t_i + x_{ij} \tag{A2}$$

By denoting $y_j = l_j \pm t_j$, $j = 1, \ldots, 4$ and $y_j = x_{j-4,k}$, $J = 5, \ldots, 8$; $k = j - 3$ for $j = 5, 6, 7$ and $k = 1$ for $j - 8$, eqn (4) can be expressed as

$$\phi_{a_i}(\vec{Y}, \theta_i) = 2 \tan^{-1}\left[\frac{A_i(\vec{Y}) \pm D(\vec{Y})}{B_i(\vec{Y}) + C(\vec{Y})}\right] \tag{A3}$$

with

$$A_i(\vec{Y}) = \sin\theta_i \tag{A4}$$

$$B_i(\vec{Y}) = \cos\theta_i - L_2/L_3 \tag{A5}$$

$$C_i(\vec{Y}) = \left(\frac{L_1^2 + L_2^2 - L_3^2 + L_4^2}{2L_2L_4}\right) - \frac{L_1}{L_4}\cos\theta_i \tag{A6}$$

$$D_i(\vec{Y}) = [A_i^2(\vec{Y}) + B_i^2(\vec{Y}) - C_i^2(\vec{Y})]^{1/2} \tag{A7}$$

*Derivation of $\psi(\bar{\mathbf{Y}})$*

If the link lengths with tolerance and clearances are assumed to be random variables, the generated output angle at the mean values of the random variables ($\bar{\mathbf{Y}}$) will be the curve $A_2B_2$ in Fig. 3. Thus the function $\psi(\bar{\mathbf{Y}})$ in eqn (16) can be expressed as

$$\psi(\bar{\mathbf{Y}}) \simeq \sum_{i=1}^{q} [\phi_{a_i}(\bar{\mathbf{Y}}, \theta_i) - \phi_{r_i}]^2 \tag{A8}$$

where $\phi_{a_i}(\vec{Y}, \theta_i)$ can be obtained by setting $\vec{Y} = \bar{\mathbf{Y}}$ in eqn (A3). Thus $\psi(\bar{\mathbf{Y}})$ can be seen to be the sum of the squares of structural error at the accuracy points $\theta_1, \theta_2, \ldots, \theta_q$.

*Derivation of $\sigma_\psi(\bar{\mathbf{Y}})$*

Since $\phi_{r_i}$ is a constant at any $\theta_i$, the differentiation of the error term $(\phi_{a_i} - \phi_{r_i})$ gives $(\partial\phi_{a_i}/\partial y_j)$. Thus eqn (17) can be expressed as

$$\sigma_\psi^2(\bar{\mathbf{Y}}) = \sum_{i=1}^{q} \left\{ \sum_{j=1}^{8} \left(\frac{\partial\phi_{a_i}}{\partial y_j}\right)\Bigg|_{\bar{Y}}^2 \cdot \sigma_{y_j}^2 \right\} \tag{A9}$$

where

$$\frac{\partial\phi_{a_i}}{\partial y_j} = \frac{2\left[(B_i + C_i)\left(\dfrac{\partial A_i}{\partial y_j} \pm \dfrac{\partial D_i}{\partial y_j}\right) - (A_i \pm D_i)\left(\dfrac{\partial B_i}{\partial y_j} + \dfrac{\partial C_i}{\partial y_j}\right)\right]}{(A_i \pm D_i)^2 + (B_i + C_i)^2} \tag{A10}$$

and the partial derivatives of $A_i$, $B_i$, $C_i$ and $D_i$ can be obtained from eqns (A2) and (A4) to (A7).

Etude de mécanismes par la technique du programme aux contraintes aléatoires

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Résumé - Dans cet article on formule le problème de la synthèse optimale des mécanismes aux erreurs minimales à l'aide d'un programme stochastique. On considère comme paramètres les longueurs nominales des membres et les jeux dans les articulations. Les contraintes sont spécifiées pour satisfaire le critère d'une certaine probabilitée minimale. La technique de programmation aléatoire a été développée pour résoudre le problème de l'optimisation. On présente l'exemple d'un mécanisme à quatre barres générateur de fonction pour illustrer l'efficacité de la méthode.