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VISCOELASTIC ANALYSIS OF ADHESIVELY BONDED JOINTS

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Abstract—A direct formulation for the viscoelastic analysis of adhesively bonded joints using finite element method is presented. A six-noded quadratic isoparametric element is developed. Two variations for the adhesive layer element are presented. Hereditary integrals are used to represent the stress–strain relations. Relaxation modulus, either experimentally obtained or theoretically generated, can be directly used. Memory load storage/use is minimised using Prony series for the relaxation modulus.

A usual eight-noded quadratic isoparametric plane strain element is used to idealise the adherends. As the adhesive layer element is a curved one it can be effectively employed for the analysis of curved adhesive-bonded joints. It is seen from the results that the stress distributions in the joints are accurately determined using both the adhesive layer element formulations. As the element formulation 1 is stiff and formulation 2 is flexible, the average of these solutions using moderate mesh is an accurate solution for the given configuration.

1. INTRODUCTION

Adhesively bonded joints are extensively used in the manufacture of aerospace vehicles, fibre reinforced plastic components, wooden articles and many a general lightweight structural members. Stress analysis of these joints is a difficult problem owing to the high stress gradients in the adhesive layer and non-homogeneity as different materials are used for adherends and adhesive. The mathematical models are complicated, even for simplest possible configurations. Moreover, viscoelastic behaviour of adhesive is to be considered if it does not remain in its glassy state due to the thermal and loading environment. It is seen from the literature that a few typical configurations of the joint are analysed using both the analytical and the finite element methods. It is well known that the analytical solutions have the limitation of application to complex problems whereas the Finite Element Method (FEM) can be easily and effectively employed for the analysis of such problems.

1.1. *Adhesive-bonded joints*

Goland and Reissner [1], in their classical study of the adhesively bonded single lap joints, used two important physical idealisations of either beam-plate or elastic continuum in the formulation of the problem. In beam-plate theory the joint flexibility is assumed to be mainly due to the flexible adhesive layer. In this case the adherends are considered to be plates connected by an adhesive layer which is idealised into an elastic spring. As this formulation is simple and reliable, it is extended to various complex situations such as double lap, scarf and stepped joints and joints with non-identical, orthotropic and anisotropic adherends [2–6]. The procedure is also used to generate design data such as the effect of the adherend shapes and adhesive thickness on the strength of the joint [7–9]. Erdogan and Ratwani [10] have used a further simplified membrane idealisation for the analysis. In the elastic continuum idealisation [11–13], the adhesive layer is considered to be stiff and its presence is disregarded to simplify the problem. However, the analysis of joints using such an idealisation is tedious and the results obtained are not satisfactory. As the classical theory of Goland and Reissner [1] is concerned with the cases of flexible and stiff adhesive layers, it cannot be used for the joints with medium range flexibility. Recently Chen and Cheng [14] have presented a general approach applicable to all adhesive layer conditions and have applied it to a single lap bonded joint. This formulation is yet to be extended to various complex situations.

The Finite Element Method (FEM) is applied to a few typical adhesively bonded configurations in the literature [15–21]. In [19] the usual cubic plane strain triangular element is employed to idealise both the adhesive and adherends of the single plane lap joint. The thickness ratio of the adhesive layer and adherends chosen for the problem is quite high compared to practical situations. This solution has not produced zero shear stress at the ends of the adhesive layer, which is also the case with the classical solution of Goland and Reissner [1] and the FEM solutions of [15] and [16]. However, the FEM solutions reported in [17] and [18] do indicate the zero boundary shear stress in the adhesive layer. Barker and Hatt [20] have developed an adhesive layer linear element for the analysis of joints. In [21] a six-noded quadratic isoparametric adhesive layer element for elastic analysis is presented. The adhesive layer is assumed to be relatively thin and behaves elastically as a tension–compression and shear spring connecting the adherends.

1.2. *Viscoelasticity*

The viscoelastic analysis techniques are broadly classified into three basic approaches: (i) quasi-elastic solutions, (ii) integral transform techniques; and (iii) direct methods. Quasi-elastic solution uses elastic properties equivalent to the corresponding viscoelastic properties at the desired time. This approach essentially ignores the entire past history of loading and therefore yields a gross approximation to the true response. Integral transform technique [22] is based on the correspondence principle, in which the elastic solution is used to obtain the corresponding viscoelastic solution using the Laplace transform technique. This approach is exact for problems for which closed form solutions are possible and approximate Laplace transform inversion has to be employed for the problems with numerical elastic solutions [23]. The direct formulations are based on the finite element theory using either the differential form [24] or the integral form [25] of stress–strain relationships. Viscoelastic analysis of single lap adhesively bonded joints employing integral transform technique is reported in [19] and [26]. Delale and Erdogan [26] have developed closed form solution for the associated elastic problem and the corresponding viscoelastic solution is obtained using numerical evaluation of transform integrals. In [19] the FEM is used to solve the associated elastic problem using the usual cubic plane strain element. Viscoelastic solution is obtained by inverting the Laplace transforms approximately.

1.3. *Present work*

In the present paper a direct formulation for the viscoelastic analysis of adhesively bonded joints employing FEM is presented. The six-noded quadratic isoparametric adhesive layer element [21] is reinvestigated, modified removing the zero constraint of longitudinal normal stress, extended to linear viscoelastic analysis and is implemented in the special purpose program VANIS, Viscoelastic Analysis of Nearly Incompressible Solids [27]. Two formulations for the adhesive layer finite element are presented. In the first element formulation all three stress/strain components of the plane strain idealisation are used whereas in the second formulation the longitudinal normal stress is assumed to be zero [14, 21]. The quadratic eight-noded isoparametric plane strain element is used to idealise adherends. The results obtained using both the element formulations are discussed and it can be seen from the results that the element formulation 1 is stiff as it uses the straightforward displacement FEM. The element formulation 2, in which the zero stress constraint is introduced, exhibits flexible behaviour and gives lower stress values. The results obtained using both the element formulations agree well with the analytical solution [26], except for the lateral normal (peel) stress obtained using the element formulation 2, which deviates from the analytical solution. As one FEM solution is stiff and the other is flexible, the average of both the solutions using moderate mesh should be an accurate solution for the given problem, particularly for the peel stress.

2. VISCOELASTIC FORMULATION

In practical situations of adhesively bonded joints the adhesive layer is very thin compared to the adherends. The adherend thickness to bond length ratio is also small. The adhesively bonded joints may be considered as plates as the width of the joint is greater than the other dimensions. Therefore plane strain idealisation is used for the analysis of these joints.

In the formulation of element matrices for the adhesive layer, the shear and normal stresses are assumed to be constant in a lateral direction. The six-noded adhesive layer element and eight-noded element for adherends are shown in Fig. 1. The formulation for the eight-noded element is avail-

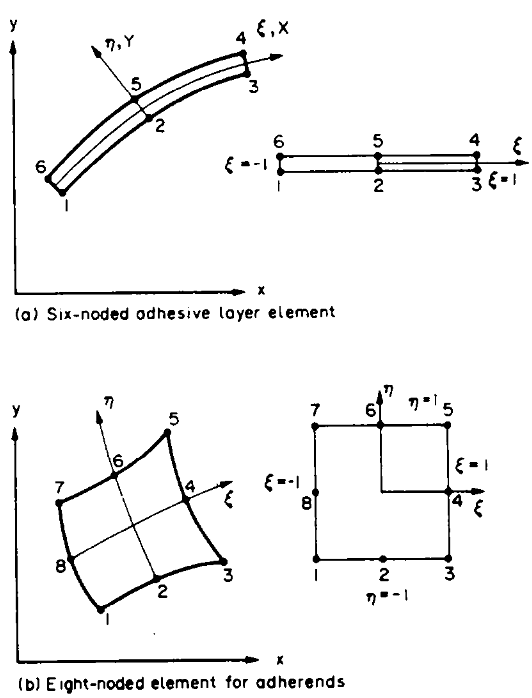

Fig. 1. Quadratic isoparametric elements.

able in [27]. The formulation for a six-noded element is discussed in the present section. As the present adhesive layer element is a curved isoparametric element it can be used for the analysis of curved adhesive layered joints. Two variations for the adhesive layer element are presented.

2.1. *Stress–strain relations*

Two types of stress–strain systems are used in formulating the two variations of the six-noded adhesive layer element. In the first formulation a usual two-dimensional plane strain system of stresses and strains is used and in the second one a modified plane strain system is used, which is obtained by introducing a zero longitudinal normal stress constraint into the usual 2-D plane strain system. The viscoelastic constitutive relation for plane strain analysis with constant bulk modulus K is given by the hereditary integral [27, 28] in the co-ordinate system $\bar{X}$ as

$$\sigma_{ij}(\bar{X},t) = 2\phi(0)\epsilon_{ij}(\bar{X},t) - \int_{0+}^{t} \frac{\partial\phi(t-t')}{\partial t'} \times \epsilon_{ij}(\bar{X},t')\mathrm{d}t' + \delta_{ij}[K - \tfrac{2}{3}\phi(0)]\epsilon_{mm}(\bar{X},t) + \delta_{ij}\frac{2}{3}\int_{0+}^{t} \frac{\partial\phi(t-t')}{\partial t'}\epsilon_{mm}(\bar{X},t')\mathrm{d}t', \quad (1)$$

where $\phi(t)$ is the shear relaxation modulus and is a function of time t.

Writing eqn (1) in matrix notation we have

$$\{\sigma(\bar{X},t)\} = [D]\{\epsilon(\bar{X},t)\} + [C]\mathscr{L}\{\epsilon(\bar{X},t)\}, \quad (2)$$

where

$$\mathscr{L}\{\epsilon(\bar{X},t)\} = \int_{0+}^{t} \frac{\partial\phi(t-t')}{\partial t'}\{\epsilon(\bar{X},t')\}\mathrm{d}t'. \quad (3)$$

$[D]$ is the usual elasticity matrix for plane strain analysis and $[C]$ the viscoelasticity matrix. The elements of matrix $[C]$ are

$$C_{ii} = -4/3, \quad i = 1, 2, \quad C_{33} = -1$$

$$C_{ij} = 2/3, \quad i,j = 1, 2, \quad i \neq j.$$

The stress and strain vectors for plane strain condition are

$$\langle\sigma(\bar{X},t)\rangle = \langle\sigma_X\sigma_Y\sigma_{XY}\rangle$$

$$\langle\epsilon(\bar{X},t)\rangle = \langle\epsilon_X\epsilon_Y\epsilon_{XY}\rangle.$$

In the second element formulation the longitudinal normal stress σ_X is assumed to be zero and the plane strain condition, i.e. $\epsilon_Z = 0.0$, is used. The stress–strain relations for the case are given as

$$\sigma_Y(X,t) = \psi(0)\epsilon_Y(X,t) - \int_{0+}^{t} \frac{\partial\psi(t-t')}{\partial t'}\epsilon_Y(X,t')\,\mathrm{d}t' \quad (4)$$

$$\sigma_{XY}(X,t) = \phi(0)\epsilon_{XY}(X,t) - \int_{0+}^{t} \frac{\partial\phi(t-t')}{\partial t'}\epsilon_{XY}(X,t')\,\mathrm{d}t', \quad (5)$$

where $\phi(t)$ is the shear relaxation modulus and $\psi(t)$ is an equivalent tension/compression modulus, which is derived from the rheological differential equations of the stress–strain system. The procedure for obtaining the $\psi(t)$ is discussed in the following steps.

The constitutive behaviour of the volumetric components of the stress and strain is given as

$$P''(\sigma_V) = R''(\epsilon_V), \quad (6)$$

and the relations of deviatoric components are

$$P'(\sigma_{Xd}) = R'(\epsilon_{Xd}) \quad (7)$$

$$P'(\sigma_{Yd}) = R'(\epsilon_{Yd}) \quad (8)$$

$$P(\sigma_{Zd}) = R'(\epsilon_{Zd}), \quad (9)$$

where P'', R'', P' and R' are appropriate differential operators.

The volumetric and deviatoric components for the problem are

$$\sigma_V = (\sigma_Y + \sigma_Z)/3; \quad \epsilon_V = (\epsilon_X + \epsilon_Y)/3 \quad (10)$$

$$\sigma_{Xd} = -(\sigma_Y + \sigma_Z)/3; \quad \epsilon_{Xd} = (2\epsilon_X - \epsilon_Y)/3 \quad (11)$$

$$\sigma_{Yd} = (2\sigma_Y - \sigma_Z)/3; \quad \epsilon_{Yd} = (-\epsilon_X + 2\epsilon_Y)/3 \quad (12)$$

$$\sigma_{Zd} = (-\sigma_Y + 2\sigma_Z)/3; \quad \epsilon_{Zd} = -(\epsilon_X + \epsilon_Y)/3. \quad (13)$$

Substituting the relations (10–13) into eqns (6–9) and then eliminating σ_Z and ϵ_X from the resulting equations we obtain the relation between σ_Y and ϵ_Y as given by eqn (14):

$$P'(2P''R' + R''P')\sigma_Y = R'(P''R' + 2R''P')\epsilon_Y. \quad (14)$$

The elastic behaviour is assumed for the volumetric components of stress and strain [28]. Therefore the operators P'' and R'' are given as $P'' = 1.0$ and $R'' = 3K$. Then eqn (14) is simplified as

$$P'(2R' + 3KP')\sigma_Y = R'(R' + 6KP')\epsilon_Y. \quad (15)$$

The required modulus $\psi(t)$ can be obtained for the given material by substituting its operators P' and R' into eqn (15) and by evaluating the stress response for Heavyside unit strain input. For elastic behaviour (instantaneous response) the operators P' and R' are 1.0 and $2\phi(0)$ respectively.

For the elastic case the eqn (15) is written as

$$\sigma_Y = \psi(0)\epsilon_Y, \tag{16}$$

where

$$\psi(0) = 4\phi(0)[3K + \phi(0)]/[3K + 4\phi(0)].$$

The eqns (4) and (5) are rewritten in matrix notation as

$$\{\sigma(X, t)\} = [D]\{\epsilon(X, t)\} + [C]\mathscr{J}\{\epsilon(X, t)\}, \tag{17}$$

where

$$\mathscr{J}\{\epsilon(X, t)\} = \int_{0^+}^{t} \begin{Bmatrix} \dfrac{\partial\psi(t-t')}{\partial t'}\epsilon_Y(t') \\ \dfrac{\partial\phi(t-t')}{\partial t'}\epsilon_{XY}(t') \end{Bmatrix} \mathrm{d}t'.$$

The elements of $[D]$ and $[C]$ matrices for this case are

$$d_{22} = \psi(0), \quad d_{33} = \phi(0), \quad C_{22} = C_{33} = -1.0.$$

The symbol $\mathscr{J}$ of eqn (17) is similar to the symbol $\mathscr{L}$ of eqn (2). However, proper care has to be taken to use appropriately the moduli $\psi(t)$ and $\phi(t)$ as seen in eqns (4) and (5).

2.2. *Element details*

As the adhesive layer is very thin the longitudinal, lateral normal stresses σ_ξ and σ_η and the shear stress $\sigma_{\xi\eta}$ are assumed to be constant in respect of the η co-ordinate (Fig. 1(a)). Therefore, the displacement field along η co-ordinate is linear. The geometry of the element is considered as a 2-D plane curve with thickness h_0. Only one non-dimensional curvilinear co-ordinate ξ is used to represent the element.

The geometry of the element is given as

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \sum_{i=1}^{3} N_i(\xi)\begin{Bmatrix} x_i \\ y_i \end{Bmatrix}, \tag{18}$$

where $N_i(i = 1, 2, 3)$ are the shape functions and are given as function of $\xi(-1 \leqslant \xi \leqslant 1)$ as

$$N_1 = -\tfrac{1}{2}\xi(1-\xi), \quad N_2 = 1-\xi^2, \quad N_3 = \tfrac{1}{2}\xi(1+\xi).$$

The local top and bottom layer displacements (Fig. 1(a)) are given by eqns (19) and (20):

$$\begin{Bmatrix} u \\ v \end{Bmatrix}^{\text{top}} = \sum_{i=1}^{3} N_i(\xi)\begin{Bmatrix} u_{i+3} \\ v_{i+3} \end{Bmatrix} \tag{19}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix}^{\text{bottom}} = \sum_{i=1}^{3} N_i(\xi)\begin{Bmatrix} u_i \\ v_i \end{Bmatrix}. \tag{20}$$

The local normal and shear strains are given by eqns (21), (22) and (23):

$$\epsilon_\xi = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(u^{\text{top}} + u^{\text{bottom}}) \tag{21}$$

$$\epsilon_\eta = (v^{\text{top}} - v^{\text{bottom}})/h_0 \tag{22}$$

$$\epsilon_{\xi\eta} = (u^{\text{top}} - u^{\text{bottom}})/h_0, \tag{23}$$

where s is the actual co-ordinate measured along the element and is given as

$$(\mathrm{d}s/\mathrm{d}\xi)^2 = (\mathrm{d}x/\mathrm{d}\xi)^2 + (\mathrm{d}y/\mathrm{d}\xi)^2. \tag{24}$$

Equations (18)–(20) and (24) are substituted into eqns (21)–(23) and the strain matrix B as given by eqn (25) is generated.

$$\{\epsilon\} = [B]\{U\}, \tag{25}$$

where $\{\epsilon\}$ is the vector of strains given by eqns (21)–(23) and $\{U\}$ is the vector of nodal displacements in local co-ordinate directions.

The elements of matrix $[B]$ are as follows:

$$b_{1,1} = b_{1,11} = \tfrac{1}{2}(\xi - 0.5)\frac{\mathrm{d}\xi}{\mathrm{d}s}$$

$$b_{1,3} = b_{1,9} = -\xi\frac{\mathrm{d}\xi}{\mathrm{d}s}$$

$$b_{1,5} = b_{1,7} = \tfrac{1}{2}(\xi + 0.5)\frac{\mathrm{d}\xi}{\mathrm{d}s}$$

$$b_{2,2} = b_{3,1} = -b_{2,12} = -b_{3,11} = -N_1/h_0$$

$$b_{2,4} = b_{3,3} = -b_{2,10} = -b_{3,9} = -N_2/h_0$$

$$b_{2,6} = b_{3,5} = -b_{2,8} = -b_{3,7} = -N_3/h_0.$$

The remaining elements of matrix $[B]$ are zero elements.

2.3. *Elemental equations*

The elemental equations are derived in [27] considering the stationarity of potential energy as

$$\{[K_1] + \tfrac{1}{2}[\phi(0) - \phi(t_k - t_{k-1})][K_2]\}\{q(t_k)\} = \{P(t_k)\} + \{M(t_k)\}. \tag{26}$$

The trapezoidal rule is used to evaluate the integrals in the time domain. t_k is the time step at which solution is sought.

The matrices $[K_1]$ and $[K_2]$ of eqn (26) for the adhesive layer element are

$$[K_1] = h_0 \int_{-1}^{1} [Q]^T[B]^T[D][B][Q]\left(\frac{ds}{d\xi}\right)d\xi \quad (27)$$

$$[K_2] = h_0 \int_{-1}^{1} [Q]^T[B]^T[C][B][Q]\left(\frac{ds}{d\xi}\right)d\xi, \quad (28)$$

where $[Q]$ is the transformation matrix relating the local and global displacements $\{U\}$ and $\{q\}$ respectively and $[Q]$ is given as

$$[Q] = \begin{bmatrix} \bar{Q} & & & & & \\ & \bar{Q} & & 0 & & \\ & & \bar{Q} & & & \\ & & & \bar{Q} & & \\ & 0 & & & \bar{Q} & \\ & & & & & \bar{Q} \end{bmatrix},$$

where

$$[\bar{Q}] = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

with

$$\theta = \tan^{-1}\left\{\left(\frac{dy}{d\xi}\right)\Big/\left(\frac{dx}{d\xi}\right)\right\}.$$

$\{P(t_k)\}$ and $\{M(t_k)\}$ of eqn (26) are the applied and memory load vectors respectively. The memory load vector $\{M(t_k)\}$ is detailed as:

$$\{M(t_k)\} = -[K_2]\Big(\tfrac{1}{2}[\phi(0) - \phi(t_k - t_{k-1})]\{q(t_{k-1})\} + \sum_{j=1}^{k-2} [\phi(t_k - t_{j+1}) - \phi(t_k - t_j)]\{q^*(t_j)\}\Big), \quad (29)$$

where

$$\{q^*(t_j)\} = 0.5[\{q(t_j)\} + \{q(t_{j+1})\}].$$

Nodal displacements $\{q(t_k)\}$ at kth time are obtained by solving eqn (26).

The memory load $\{M(t_k)\}$ is the summation of $(k-2)$ load vectors to be stored in memory or backup storage. The large amount of information thus to be stored can easily exceed the available core memory of a digital computer, or computation becomes costly for large t, when backup storage is used, as the large amount of data has to be read into core and transferred back to peripherals several times. Storing of such a large amount of data is eliminated using a recurrence relation obtained by expressing the relaxation modulus in Prony series [27], where only the set of quantities from the two previous time steps have to be retained.

2.4. *Stresses and strains*

After obtaining $\{q(t_k)\}$, the strains at the required location are computed using eqn (30) as

$$\{\epsilon(t_k)\} = [B][Q]\{q(t_k)\}. \quad (30)$$

The stresses are then computed as

$$\{\sigma(t_k)\} = [S_1]\{\epsilon(t_k)\} + \{S_2\}, \quad (31)$$

where

$$[S_1] = [D] + \tfrac{1}{2}[\phi(0) - \phi(t_k - t_{k-1})][C] \quad (32)$$

$$\{S_2\} = [C][B]\Big(\tfrac{1}{2}[\phi(0) - \phi(t_k - t_{k-1})] \times \{q(t_{k-1})\} + \sum_{j=1}^{k-2} [\phi(t_k - t_{j+1}) - \phi(t_k - t_j)]\{q^*(t_j)\}\Big). \quad (33)$$

The element matrices given by eqns (27) and (28) are evaluated using 2 × 2 Gauss quadrature.

2.5. *Element formulaton* 1

In the element formulation 1, the required element matrices $[K_1]$ and $[K_2]$ are generated using the element details given in Sec. 2.2 and the constitutive relation given by eqn (2). The strains and stresses are computed using eqns (30) and (31).

2.6. *Element formulation* 2

In the second formulation, as σ_X is taken to be zero, the strain ϵ_X is not included in generating matrix $[B]$ (eqn (29)) as the contribution to the potential energy due to these components is zero. Therefore, the first row of the matrix $[B]$ is made a null row in generating the matrices $[K_1]$ and $[K_2]$. In this case the constitutive relation given by eqn (17) is used. Proper care has to be taken of it using $\psi(t)$ and $\phi(t)$ for the appropriate components of stress–strain in the eqns (26), (29) and (30)–(33).

3. NUMERICAL DISCUSSION

The procedure developed in the present work for the viscoelastic analysis of adhesively bonded joints is applied to a single lap joint, with two identical adherends. The adhesive layer is assumed to obey the three parameter viscoelastic solid behaviour. The analytical solutions for the problem for three load cases, viz. membrane, bending and transverse shear, are given in [26]. The geometrical and load details and three parameter visco-elastic solid model are explained in Fig. 2. The finite element idealisation for

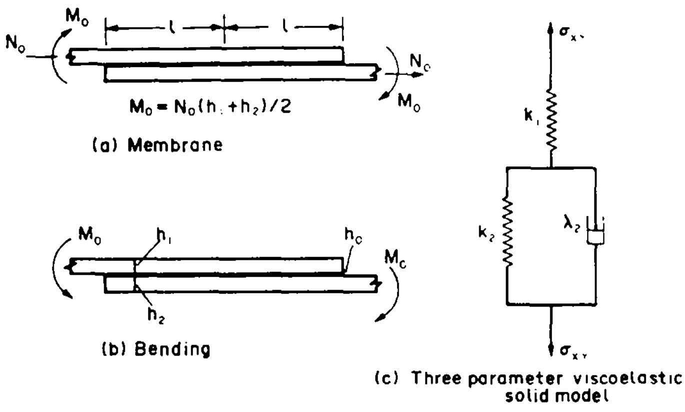

Fig. 2. Joint loading and viscoelastic model.

the joint is shown in Fig. 3. The number of nodes and elements are 394 and 116 respectively.

The data used for aluminum adherends are

$$E = 7030.768 \text{ kg/mm}^2$$

$$\nu = 0.3$$

$$h_1 = h_2 = 2.286 \text{ mm}$$

$$l = 12.7 \text{ mm (Fig. 2).}$$

For the particular epoxy used as the adhesive the properties at $t = 0.0$ hr are

$$K = 344.286 \text{ kg/mm}^2, \quad \phi(0) = 156.435 \text{ kg/mm}^2.$$

The equilibrium shear modulus $\phi(\infty)$ and the retardation time t_0 are taken as

$$\phi(\infty) = \phi(0)/3, \quad t_0 = 4.0 \text{ hr}.$$

The thickness of the adhesive layer h_0 is 0.1016 mm.

The governing differential equation for the viscoelastic model (Fig. 2) is given by eqn (34):

$$\left(1 + P_1 \frac{\partial}{\partial t}\right)\sigma_{XY} = \left(r_0 + r_1 \frac{\partial}{\partial t}\right)\epsilon_{XY}, \tag{34}$$

where

$$p_1 = \lambda_2/(k_1 + k_2)$$

$$r_0 = k_1 k_2/(k_1 + k_2)$$

$$r_1 = \lambda_2 k_2/(k_1 + k_2).$$

The relaxation shear modulus for eqn (34) is

$$\phi(t) = r_0 + \left(\frac{r_1}{p_1} - r_0\right)e^{-t/p_1}. \tag{35}$$

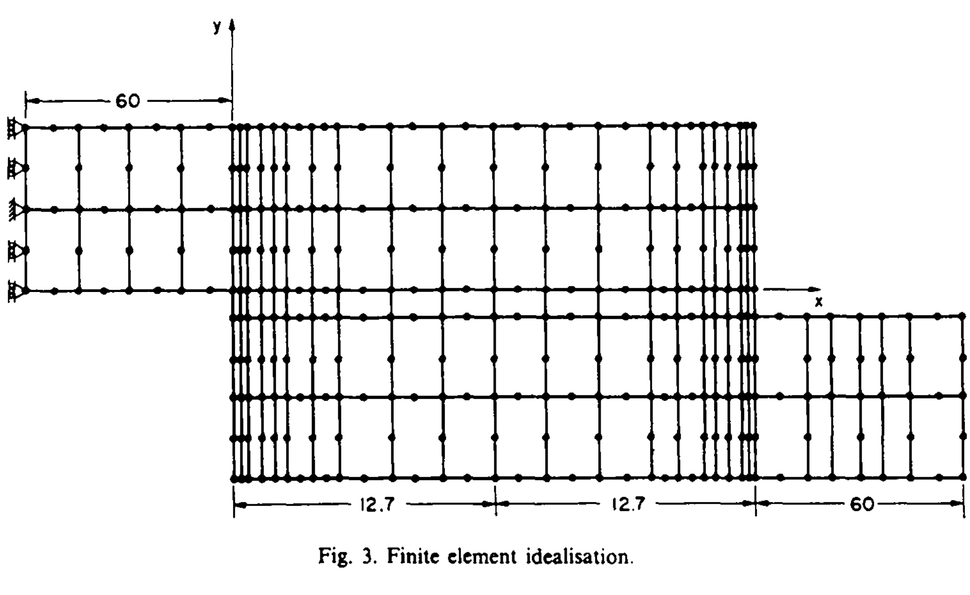

Fig. 3. Finite element idealisation.

Using the data given in [26], the relaxation modulus for the material is found to be

$$\phi(t) = 52.144896 + 104.29\,e^{-t/\tau_1}, \quad (36)$$

where $\tau_1 = 4/3$.

In addition to shear relaxation modulus, we need the relaxation modulus $\psi(t)$ for the element formulation 2. The governing differential equation which gives the modulus $\psi(t)$ is obtained by substituting the operators

$$P' = \left(1.0 + P_1\frac{\partial}{\partial t}\right) \quad \text{and} \quad R' = \left(r_0 + r_1\frac{\partial}{\partial t}\right)$$

into eqn (15), and is given by eqn (37):

$$\left\{(3K + 2r_0) + (2r_1 + 6Kp_1 + 2r_0p_1)\frac{\partial}{\partial t} + p_1(2r_1 + 3Kp_1)\frac{\partial^2}{\partial t^2}\right\}\sigma_Y = \left\{r_0(6K + r_0) + (6Kp_1r_0 + 6Kr_1 + 2r_0r_1)\frac{\partial}{\partial t} + (6Kp_1r_1 + r_1^2)\frac{\partial^2}{\partial t^2}\right\}\epsilon_Y. \quad (37)$$

Equation (37), together with data from [26], yields a relaxation modulus as

$$\psi(t) = 182.29637 + 104.28966\,e^{-t/\tau_1} + 162.09848\,e^{-t/\tau_2}, \quad (38)$$

where $\tau_1 = 4/3$ and $\tau_2 = 1.7813716$.

The results for the two load cases, viz. membrane and bending, are discussed in the following. For the case of membrane loading, a pressure of 1.0 kg/mm^2 is applied on the edges of adherends, which is equivalent to a longitudinal load of 2.286 kg/mm and a bending moment of 2.613 kg [26]. For the bending analysis, a moment of 1.0 kg is applied on the edges of adherends. Simply supported boundary conditions are used on the left edge of the joint and the loads are applied on the right edge. The stresses in adherends and adhesive layer are computed up to 4.0 hr, which is the retardation time for the material. Using the equilibrium moduli, equilibrium solutions are also obtained.

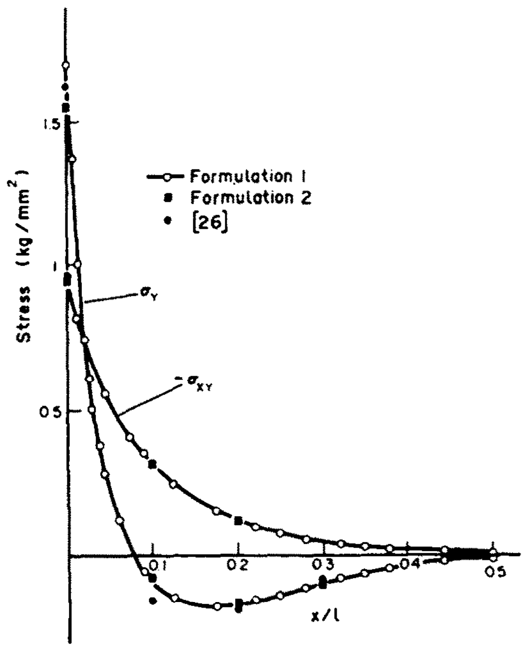

Fig. 4. Peel and shear stresses in adhesive layer (loading—membrane; $t = 0.0$ hr).

The shear and normal stresses in the adhesive layer at time $t = 0.0$ hr in the end regions ($0.0 \leq x/l \leq 0.2$) are given in Tables 1 and 2, and are plotted in Figs 4 and 5 up to $x/l = 0.5$. Results obtained using both

Table 1. Shear stress ($-\sigma_{XY}$) kg/mm^2 in adhesive layer with time

Time (hr)	Formulation	Membrane loading (x/l) 0.0	0.1	0.2	Bending loading 2.0	1.9	1.8
0.0	1	0.9630	0.3195	0.1253	0.2819	0.0910	0.0355
	2	0.9510	0.3210	0.1261	0.2777	0.0915	0.0357
	*	0.9704	0.3301	0.1123	0.2675	0.0914	0.0312
1.0	1	0.7866	0.3189	0.1430	0.2296	0.0911	0.0407
	2	0.7745	0.3203	0.1440	0.2252	0.0916	0.0410
	*	0.7788	0.3310	0.1366	0.2191	0.0919	0.0379
2.0	1	0.6902	0.3148	0.1529	0.2012	0.0901	0.0436
	2	0.6783	0.3162	0.1539	0.1968	0.0905	0.0439
	*	0.6908	0.3269	0.1499	0.1919	0.0908	0.0416
4.0	1	0.6060	0.3057	0.1610	0.1763	0.0875	0.0460
	2	0.5948	0.3068	0.1620	0.1722	0.0879	0.0463
	*	0.6059	0.3163	0.1604	0.1683	0.0878	0.0445
∞	1	0.5599	0.2913	0.1617	0.1629	0.0834	0.0462
	2	0.5501	0.2922	0.1625	0.1592	0.0837	0.0465
	*	0.5602	0.3006	0.1613	—	—	—

* Reference [26] values.

Table 2. Peel Stress σ_Y kg/mm² in adhesive layer with time

Time (hr)	Formulation	Membrane loading (x/l) 0.0	0.1	0.2	Bending loading 2.0	1.9	1.8
0.0	1	1.6981	−0.0894	−0.1740	0.6213	−0.0304	−0.0669
	2	1.5502	−0.0706	−0.1678	0.5681	−0.0247	−0.0644
	*	1.6231	−0.1588	−0.1894	0.5917	−0.0578	−0.0691
1.0	1	1.5466	−0.0745	−0.1720	0.5693	−0.0250	−0.0661
	2	1.2890	−0.0397	−0.1593	0.4765	−0.0134	−0.0610
	*	1.5115	−0.1359	−0.1886	0.5538	−0.0498	−0.0691
2.0	1	1.4698	−0.0662	−0.1707	0.5430	−0.0220	−0.0655
	2	1.1372	−0.0194	−0.1524	0.4228	−0.0060	−0.0583
	*	1.4573	−0.1242	−0.1881	0.5340	−0.0455	−0.0690
4.0	1	1.4101	−0.0592	−0.1695	0.5226	−0.0195	−0.0650
	2	1.0043	0.0014	−0.1432	0.3752	0.0017	−0.0548
	*	1.4170	−0.1150	−0.1872	0.5193	−0.0422	−0.0686
∞	1	1.3860	−0.0558	−0.1689	0.5146	−0.0184	−0.0648
	2	0.9409	0.0136	−0.1332	0.3518	0.0062	−0.0510
	*	1.4042	−0.1118	−0.1867	—	—	—

* Reference [26] values.

the element formulations along with the analytical solution [26] are indicated. The shear stress distribution in the adhesive layer at various times in the end region is given in Figs 6 and 7 for membrane and bending loads respectively using the element formulation 1. The normal and shear stresses at $x/l = 0.0$ are given with respect to time in Figs 8 and 9, employing both the element formulations.

It is seen from the results that the stresses are highly localised at the ends of the adhesive layer and depict severe stress concentrations in the end regions. In the middle region of the joint they are practically non-existent.

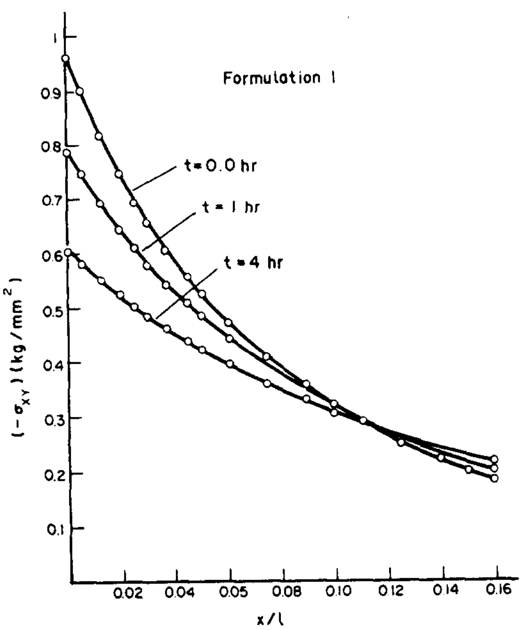

Fig. 6. Shear stress in adhesive layer (membrane loading).

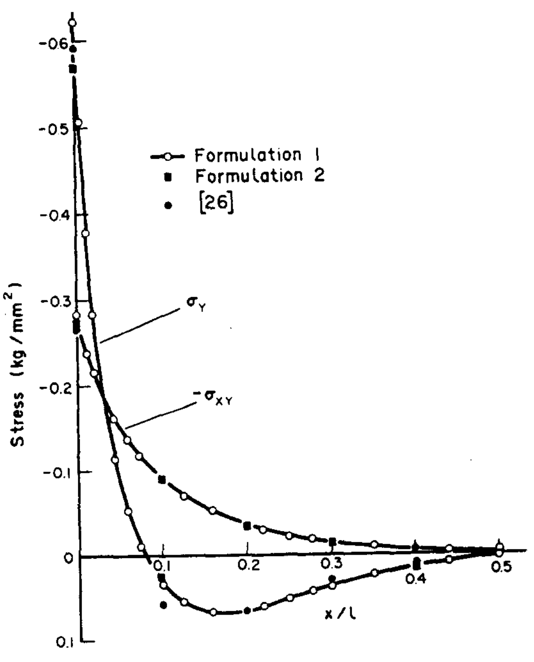

Fig. 5. Peel and shear stresses in adhesive layer ($t = 0.0$ hr; $M = 1.0$ kg).

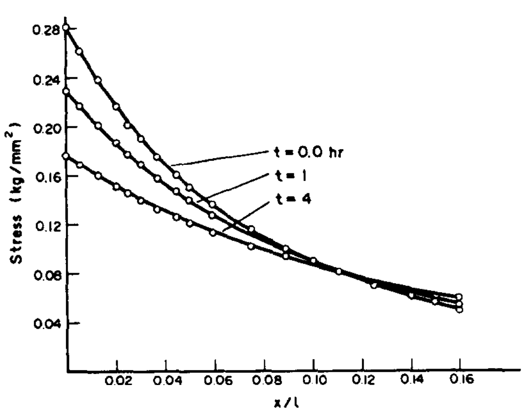

Fig. 7. Shear stress in adhesive layer (bending loading).

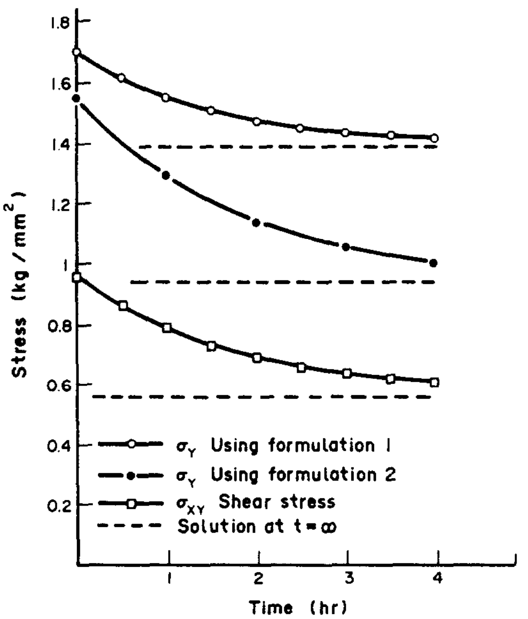

Fig. 8. Peel and shear stresses in adhesive layer at $x/l = 0.0$ for membrane loading.

The peel stress distribution changes its sign for both the load cases when plotted along the adhesive layer, whereas the shear stress distribution does not. The relaxation of stresses is quite considerable at retardation time; about 17% and 37% in peel and shear stresses respectively at $x/l = 0.0$. However, it is to be noted that the relaxation depends upon the type of adhesive used. The formulation 1 exhibits stiff behaviour as it uses straightforward displacement technique whereas the formulation 2 is flexible as it is equivalent to a constraint approach.

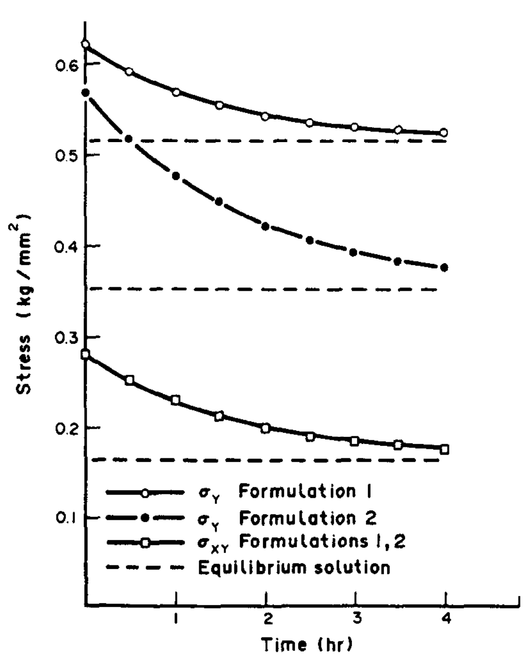

Fig. 9. Stresses in adhesive layer at $x/l = 0.0$ for bending loading.

4. CONCLUSIONS

The procedure developed for the viscoelastic analysis of adhesively bonded joints gives very accurate results and as it is based on the finite element method using curved isoparametric elements it can be easily and effectively employed for the analysis of complex configurations. Two variations of adhesive layer element are presented. As the element formulation 1 is stiff and the formulation 2 is flexible, the average of these solutions using moderate mesh is an accurate solution for the given problem, particularly for peel stress. A typical single lap bonded joint is analysed using the procedure developed. It is observed from the results that there are severe stress concentrations in an adhesively bonded joint, which need to be properly taken care of in the design of such joints. The storage of a large amount of memory load information is minimised using Prony series for relaxation moduli and therefore the computational effort and cost are reduced.

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