Computers & Structures Vol. 20, No. 5, pp. 817–825, 1985
Printed in Great Britain

0045-7949/85 $3.00 + .00
Pergamon Press Ltd.

VISCOELASTIC ANALYSIS OF NEARLY INCOMPRESSIBLE SOLIDS

S. YADAGIRI†
Kakatiya Institute of Technology and Science, Warangal-506 015, India

and

C. PAPI REDDY‡
Regional Engineering College, Warangal-506 004, India

Abstract—The finite element method is well established for the analysis of structures and other field problems. However, its straightforward application for the analysis of nearly incompressible solids yields erratic results. In the present work, an efficient special purpose code for the Viscoelastic Analysis of Nearly Incompressible Solids (VANIS) is developed using isoparametric elements with selective integration procedure, which is a third order Gauss rule for deviatoric response and second order Gauss rule for volumetric response. The software can be effectively employed for the structures with lower Poisson's ratios. VANIS is based on the direct formulation using linear uncoupled thermoviscoelastic theory for the thermorheologically simple materials. The element library consists of 8-noded plane strain, 8-noded axisymmetric solid and 20-noded three dimensional quadratic isoparametric elements. These elements meet all the possible structural idealisation requirements of the solid continua. Experimentally obtained rigidity modulus can be used directly or expressing it in Prony series. The software is tested on a number of problems and gives very accurate results for all the permissible values of the Poisson's ratio.

1. INTRODUCTION

Solid propellant rocket fuels are incompressible or nearly incompressible (Poisson's ratio approaching one half) viscoelastic solids. Also, the mechanical properties of these materials are highly temperature dependent. Such is the situation with the other polymeric solids and polymers. Metals in the plastic region and soils in the saturated condition also have higher Poisson's ratios. Application of the usual finite element method for the analysis of such solids yields severely oscillating stresses/strains across elements.

In the present work a simple and efficient special purpose code for the Viscoelastic Analysis of Nearly Incompressible Solids (VANIS) is developed using the plane strain, axisymmetric solid and 3-D qudratic isoparametric elements. Selective integration procedure is employed for computing the element stiffness matrices.

Thermal effects are incorporated through the use of the shifted time hypothesis. The viscoelastic formulation is presented in Section 2 and the performance of the software is discussed in Section 3. It is seen from the results that the software is a very useful code for the analysis of incompressible solid continua. Also, the software can be effectively used for the analysis of compressible structures through the use of a control parameter, which changes the computational flow from the selective integration to a third order Gauss rule for the computation of both the deviatoric and volumetric components of the stiffness matrix.

†Assistant Professor of Mechanical Engineering.
‡Professor of Mechanical Engineering.

1.1 *Incompressibility*

It is well known that the straightforward application of the displacement method to nearly incompressible structures yields erratic displacements and severely oscillating stresses about the exact solution and across the elements. This aspect has been studied for elastic materials and is well documented in literature[1–10]. The remedies suggested in literature to overcome the difficulties are the use of: (i) refined meshes, (ii) reduced Poisson's ratio, (iii) alternate formulations, such as the stress hybrid approach and the formulation based on Herrmann's (Semi-Reissner's) variational principle, and (iv) reduced integration for the troublesome portion of the strain energy. The proposition of mesh refinement[3] needs number of elements and yields doubtful results and therefore is not advisable. The results obtained using the reduced Poisson's ratio have to be extrapolated so as to obtain the results corresponding to the required Poisson's ratio[4, 5]. There are no systematic methods of extrapolation of results as of now. Previous experience or engineering intuition could provide hints to the juditious extrapolation. Improved results are obtained using Herrmann's variational formulation[1, 2] for incompressible structures. In this procedure simultaneous minimization of strain energy with respect to displacements and pressure variable is done. Stress hybrid formulation of Pian and Tong[6] also gave good results. Use of the reduced integration is versatile and is an economical approach[7–9]. Moreover, it is shown in literature [10] that Herrmann's formulation is identical to the displacement method with underintegration of certain troublesome portions of the strain energy.

1.2 *Viscoelasticity*

The viscoelastic analysis techniques may broadly be classified into three basic approaches, viz. (i) quasi-elastic solutions, (ii) integral transform techniques, and (iii) direct methods. Quasi-elastic solution uses elastic properties equivalent to the corresponding viscoelastic properties at the desired time and temperature. This approach essentially ignores the entire past history of loading and environment and therefore yields gross approximation to the true response. Integral transform technique[11] is based on the correspondence principle, in which using the elastic solution, the corresponding viscoelastic solution is obtained using the Laplace transform technique. This approach is exact for which closed form solutions are possible and approximate Laplace transform inversion has to be employed for the problems with the numerical elastic solutions[12]. Further, the transform technique is not directly applicable for the problems of non-homogeneous transient temperature distributions. To circumvent these problems, conditions of constant temperature over time increments are imposed and the correspondence principle is applied on an incremental basis[13]. The direct formulations are based on the finite element theory using either the differential form[14, 15] or the integral form[16, 17] of stress–strain relationships.

1.3 *The software 'VANIS'*

The special purpose code, VANIS is developed for the linear analysis of nearly incompressible viscoelastic solid continua. The software can be effectively employed for the structures with lower Poisson's ratio. It is based on the direct formulation using hereditary integral (linear) stress–strain relationships. Further, it is assumed in the formulation that: (i) reduced time hypothesis is valid (thermorheologically simple material), (ii) bulk modulus is constant with time, and (iii) the material is isotropic and homogeneous. The element library consists of 8-noded plane strain, 8-noded axisymmetric solid and 20-noded three dimensional solid quadratic isoparametric elements. The elements meet all the possible structural idealisation requirements of the solid continua.

Software uses a selective integration procedure, which is the third order Gauss rule for deviatoric response and the second order Gauss rule for volumetric response. Storage of large amount of memory load information is avoided using the recurrence relation developed expressing the relaxation modulus in Prony series. Only the two immediate previous load vectors are required to be stored in memory. Also, it must be noted that, in the present formulation, the experimentally obtained modulus can be used either directly or by expressing it in Prony series and therefore, the problem of looking for realistic material models is avoided, which has been always a problem in the integral transform techniques.

The software is tested on a number of problems. The results obtained agree well with the already published ones.

2. VISCOELASTIC FORMULATION

The element details of the three quadratic isoparametric finite elements available in "VANIS" are given in Fig. 1 and are also available in Ref. [9]. The present software uses the linear uncoupled thermoviscoelastic formulation[16, 17] and is based on the following assumptions

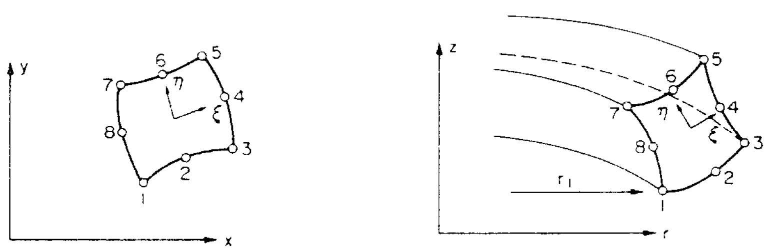
(i) the stress strain relation is a hereditary integral expression,

(ii) the bulk modulus is constant with time,

(iii) the reduced time hypothesis is valid,

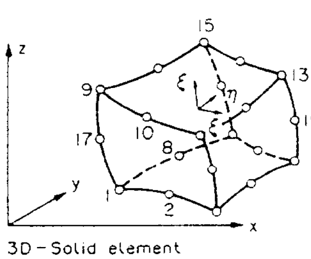
(iv) the material is isotropic and homogeneous.

The thermoviscoelastic formulation is briefly discussed in the present section.



(i) Plane strain element (ii) Axisymmetric sold element

$$N_i = \tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)\,\xi_i^2\eta_i^2 + \tfrac{1}{2}(1-\xi^2)(1+\eta\eta_i)(1-\xi_i^2) + \tfrac{1}{2}(1-\eta^2)(1+\xi\xi_i)(1-\eta_i^2),\quad i=1\ldots8.$$



$$N_i = \tfrac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)(\xi\xi_i+\eta\eta_i+\zeta\zeta_i-2)\,\xi_i^2\eta_i^2\zeta_i^2 + \tfrac{1}{4}(1-\xi^2)(1+\eta\eta_i)(1+\zeta\zeta_i)(1-\xi_i^2) + \tfrac{1}{4}(1-\eta^2)(1+\xi\xi_i)(1+\zeta\zeta_i)(1-\eta_i^2) + \tfrac{1}{4}(1-\zeta^2)(1+\xi\xi_i)(1+\eta\eta_i)(1-\zeta_i^2),\quad i=1\ldots20$$

3D – Solid element

Fig. 1. Element details.

2.1 *Stress–strain relation*

The thermoviscoelastic constitutive relation[18] with constant bulk modulus K and constant linear co-efficient of thermal expansion α is

$$\begin{aligned}\sigma_{ij}(\bar{X},t) &= 2G(0)\epsilon_{ij}(\bar{X},t)\\ &\quad -2\int_{0+}^{t}\frac{\partial G(\tau-\tau')}{\partial t'}\epsilon_{ij}(\bar{X},t')\,\mathrm{d}t'\\ &\quad +\delta_{ij}\left[K-\frac{2}{3}G(0)\right]\epsilon_{mm}(\bar{X},t)\\ &\quad +\delta_{ij}\frac{2}{3}\int_{0+}^{t}\frac{\partial G(\tau-\tau')}{\partial t'}\epsilon_{mm}(\bar{X},t')\,\mathrm{d}t'\\ &\quad -\delta_{ij}3\alpha K[T(\bar{X},t)-T_s(\bar{X},0)] \qquad (1)\end{aligned}$$

where $T(\bar{X},t)$ is the current temperature; $T_s(\bar{X},0)$ is the structural reference temperature; and $G(\tau)$ is the shear relaxation modulus given for an arbitrary "material" reference temperature, T_M. The shifted time τ is related to real time t through the relation

$$\tau=\tau(\bar{X},t)=\int_{0+}^{t}\frac{\mathrm{d}t'}{A_T[T(\bar{X},t')]} \qquad (2)$$

where A_T is shift function and is evaluated using the so called WLF equation[19] as

$$\log A_T = -\frac{C_1(T-T_M)}{C_2+(T-T_M)} = -h(T) \qquad (3)$$

i.e.

$$\frac{1}{A_T[T(\bar{X},t)]}=10^{h(T)} \qquad (4)$$

where C_1 and C_2 are material constants.

Writing eqn (1) in matrix notation, we have

$$\begin{aligned}\{\sigma(\bar{X},t)\} &= [D]\{\epsilon(\bar{X},t)\}+[C]\mathscr{L}\{\epsilon(\bar{X},t)\}\\ &\quad -3\alpha K[T(\bar{X},t)-T_s(\bar{X},0)]\{J\} \qquad (5)\end{aligned}$$

where

$$\mathscr{L}\{\epsilon(\bar{X},t)\}=\int_{0+}^{t}\frac{\partial G(\tau-\tau')}{\partial t'}\{\epsilon(\bar{X},t')\}\,\mathrm{d}t' \qquad (6)$$

$[D]$ is the usual elasticity matrix; and $[C]$ is the viscoelasticity matrix and is given in appendix for the three elements. $\{\sigma(\bar{X},t)\}$, $\{\epsilon(\bar{X},t)\}$ are the stress and strain vectors and are

for plane strain element

$$\langle\sigma(\bar{X},t)\rangle=\langle\sigma_x\sigma_y\sigma_{xy}\rangle$$
$$\langle\epsilon(\bar{X},t)\rangle=\langle\epsilon_x\epsilon_y\epsilon_{xy}\rangle$$

for axisymmetric solid element,

$$\langle\sigma(\bar{X},t)\rangle=\langle\sigma_x\sigma_y\sigma_\theta\sigma_{xy}\rangle$$
$$\langle\epsilon(\bar{X},t)\rangle=\langle\epsilon_x\epsilon_y\epsilon_\theta\epsilon_{xy}\rangle$$

and for 3-dimensional solid element,

$$\langle\sigma(\bar{X},t)\rangle=\langle\sigma_x\sigma_y\sigma_z\sigma_{xy}\sigma_{yz}\sigma_{zx}\rangle$$
$$\langle\epsilon(\bar{X},t)\rangle=\langle\epsilon_x\epsilon_y\epsilon_z\epsilon_{xy}\epsilon_{yz}\epsilon_{zx}\rangle.$$

It is seen from eqn (5) that the second component in the right hand side is due to the viscoelastic property of the material.

2.2 *Strain–displacement relation*

The geometry $\{\bar{X}\}$ and the displacement field $\{U\}$ in the isoparametric elements is given by

$$\begin{aligned}\{\bar{X}\} &= [N]\{x_i\}\\ \{U\} &= [N]\{q_i\}\end{aligned} \qquad (7)$$

where $[N]$ is matrix of shape functions and $\{x_i\}$ and $\{q_i\}$ are the nodal co-ordinates and nodal displacements.

Strain–displacement relation, after appropriate differentiation of shape functions is written in matrix notation as

$$\{\epsilon\}=[B]\{q\} \qquad (8)$$

the bar and suffix i of q are dropped for convenience. The shape functions are explicit functions of local co-ordinates. Strain quantities need differentiation of shape functions with respect to global co-ordinates and is done through a transformation[9] using Jacobian matrix $[\mathcal{J}]$.

2.3 *Elemental equations*

The elemental equations are derived using the principle of virtual displacements

$$\begin{aligned}\int_{t_1}^{t_2}\Bigg\{\int_v \delta\epsilon^T(\bar{X},t)\sigma(\bar{X},t)\,\mathrm{d}V-\int_v \delta U^T(\bar{X},t)F_v(\bar{X},t)\,\mathrm{d}v\\ -\int_s \delta U^T(\bar{X},t)F_s(\bar{X},t)\,\mathrm{d}s\Bigg\}\mathrm{d}t \qquad (9)\end{aligned}$$

where $F_v(\bar{X},t)$ is the prescribed body force vector; $F_s(\bar{X},t)$ is the prescribed surface tractions on the boundary s, and $U(\bar{X},t)$ are the virtual displacements. Substituting eqns (5), (7) and (8) into eqn (9) we get

$$\begin{aligned}\int_{t_1}^{t_2}\{\delta q\}^T\Bigg[\int_v B^TDB\,\mathrm{d}v\{q\}+\int_v BC\mathscr{L}\{\epsilon\}\,\mathrm{d}v\\ +\int_s N^T\{p\}\,\mathrm{d}s+\int_v N^T\{\varrho\}\,\mathrm{d}v\\ -\int_v 3\alpha K(T-T_s)B^T\,\mathrm{d}v\{J\}\Bigg]\mathrm{d}t \qquad (10)\end{aligned}$$

where $\{p\}$ is the surface traction vector per unit surface area; and $\{\varrho\}$ is the body force vector per unit volume. The elemental equations are obtained considering the stationarity of the potential energy (eqn 10) as

$$\begin{aligned}\int_v B^TDB\,\mathrm{d}v\{q\}+\int_v B^TC\mathscr{L}\{\epsilon\}\,\mathrm{d}v+\int_s N^T\{p\}\,\mathrm{d}s\\ +\int_v N^T\{\varrho\}\,\mathrm{d}v-3\alpha K(T-T_s)\int_v B^T\{J\}\,\mathrm{d}v=\{0\}. \qquad (11)\end{aligned}$$

It is seen from eqn (11) that the second term is the additional term and is due to viscoelastic property of

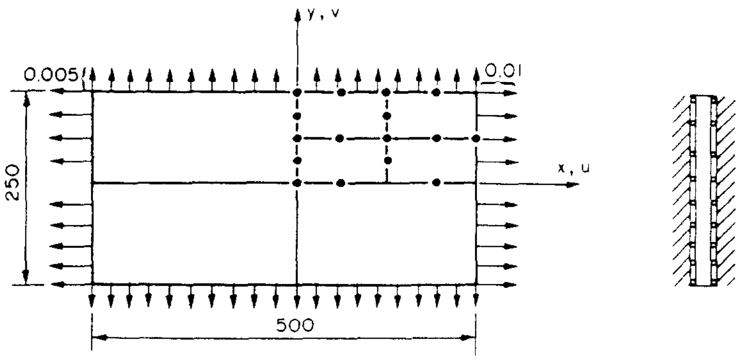


Fig. 2. Plane strain problem.

the material. Using the trapezoidal rule for time domain and using eqn (8), this term is rewritten for kth time step as

$$\int_v B^T C \mathscr{L}\{\epsilon\}\,\mathrm{d}v = \frac{1}{2}[G(0) - G(\tau_k - \tau_{k-1})][K_2]\{q(t_k)\}$$

$$+\frac{1}{2}[G(0) - G(\tau_k - \tau_{k-1})][K_2]\{q(t_{k-1})\}$$

$$+[K_2]\left(\sum_{j=1}^{k-2}[G(\tau_k - \tau_{j+1}) - G(\tau_k - \tau_j)]\{q^*(t_j)\}\right) \quad (12)$$

where

$$\{q^*(t_j)\} = 0.5[\{q(t_j)\} + \{q(t_{j+1})\}]$$

$$[K_2] = \int_v B^T C B\,\mathrm{d}v$$

substituting eqn (12) into eqn (11) we get the elemental equilibrium equation as:

$$\left[[K_1] + \frac{1}{2}[G(0) - G(\tau_k - \tau_{k-1})][K_2]\right]\{q(t_k)\}$$

$$= \{P(t_k)\} + \{H(t_k)\} + \{M(t_k)\} \quad (13)$$

where

$$[K_1] = \int_v B^T D B\,\mathrm{d}v$$

$$\{P(t_k)\} = \int_s N^T\{p\}\,\mathrm{d}s + \int_v N^T\{\varrho\}\,\mathrm{d}v$$

$$\{H(t_k)\} = -3\alpha K(T - T_s)\int_v B^T\{J\}\,\mathrm{d}v$$

$$\{M(t_k)\} = -[K_2]\left(\frac{1}{2}[G(0) - G(\tau_k - \tau_{k-1})]\{q(t_{k-1})\}\right.$$

$$\left. - \sum_{j=1}^{k-2}[G(\tau_k - \tau_{j+1}) - G(\tau_k - \tau_j)]\{q^*(t_j)\}\right). \quad (14)$$

Nodal displacements $\{q(t_k)\}$ at kth time step are obtained by solving the eqn (13).

2.4 *Stresses and strains*

After obtaining $\{q(t_k)\}$, the strains at the required location are obtained using eqn (8).

The stresses are then computed using the strains evaluated as

$$\{\sigma(t_k)\} = [S_1]\{\epsilon(t_k)\} + \{S_2\} - 3\alpha K(T_k - T_S)\{J\} \quad (15)$$

where

$$[S_1] = [D] + \frac{1}{2}[G(0) - G(\tau_k - \tau_{k-1})][C]$$

$$\{S_2\} = [C][B]\left(\sum_{j=1}^{k-2}[G(\tau_k - \tau_{j+1}) - G(\tau_k - \tau_j)]\{q^*(t_j)\}\right.$$

$$\left. + \frac{1}{2}[G(0) - G(\tau_k - \tau_{k-1})]\{q(t_{k-1})\}\right) \quad (16)$$

T_k is the temperature of the structure at the kth time step and T_s is the structural reference temperature.

2.5 *Memory load considerations*

The memory load $\{M(t_k)\}$ is the summation of $(k-2)$ load vectors to be stored in memory or backup storage. The large amount of information thus to be stored can easily exceed the available core

Table 1. Strains in plane strain problem

t	ϵ_x		ϵ_y	$v(t)$
hrs.	Exact	FEM	FEM	
0.0	0.0125341	0.0125340	− 0.0124659	0.4995456
1.0	0.0306363	0.0306385	− 0.0305347	0.4998915
2.0	0.0482035	0.0481386	− 0.0480704	0.4999687
3.0	0.0652515	0.0651570	− 0.0650888	0.4999859
4.0	0.0817957	0.0816735	− 0.0816052	0.4999898
5.0	0.0978509	0.0977026	− 0.0976344	0.4999906
10.0	0.1712484	—	—	—

memory of a digital computer or computation will become costly for large t, when the backup storage is used, as the large amount of data has to be read into core and transferred back to peripherals several times. Storing of such a large amount of data is eliminated using a recurrence relation obtained by expressing the relaxation modulus in Prony series. Only set of quantities from the two previous time steps have to be retained.

The relaxation modulus in Prony series is written as

$$G(\tau) = A_0 + \sum_{i=1}^{n} A_i e^{-\tau/\beta_i} \tag{17}$$

where A_0, A_i and β_i are the material constants. Equation (17) amounts to a representation of the relaxation modulus by either a generalised Maxwell or a generalised Kelvin model[19].

The expression for the memory load $\{M(t_k)\}$ (eqn 14) is rewritten, using eqn (17) as

$$\{M(t_k)\} = -[K_2]\left(\sum_{i=1}^{n} A_i\{\mu_{i,k}\} + \frac{1}{2}\left[G(0) - A_0 - \sum_{i=1}^{n} A_i e^{-(\tau_k - \tau_{k-1})/\beta_i}\right]\{q(t_{k-1})\}\right) \tag{18}$$

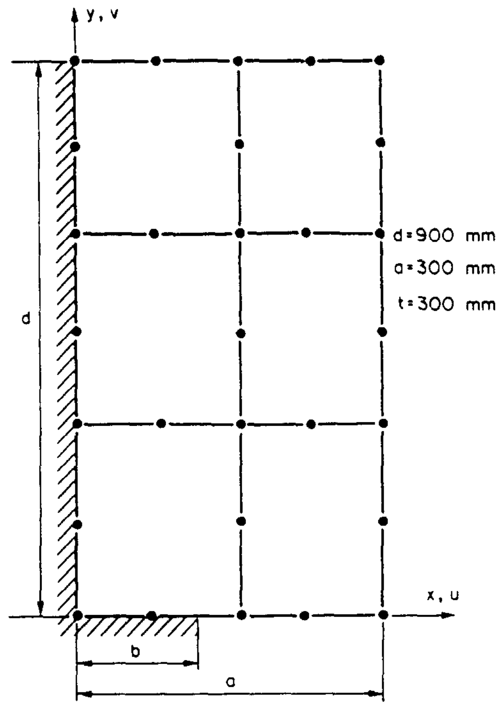


Fig. 3. Solid mass slump problem.

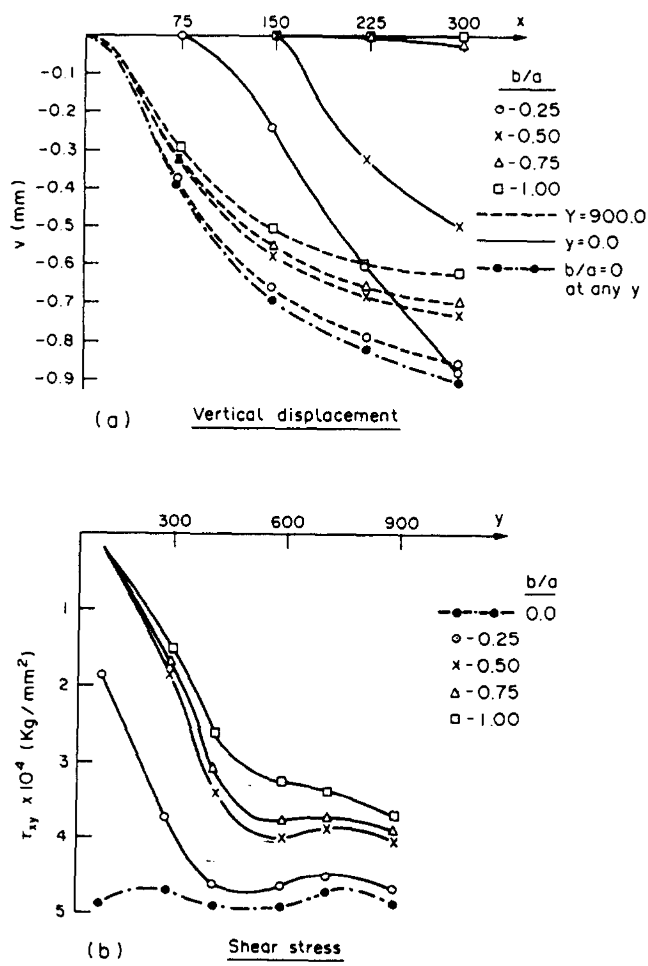


Fig. 4. Displacements and stresses in solid mass stump problem at $t = 0.0$.

where $\{\mu_{i,k}\}$ is,

$$\{\mu_{i,k}\} = \{\mu_{i,k-1}\} + e^{-(\tau_k - \tau_{k-1})\beta_i}[1.0 - e^{-(\tau_{k-1} - \tau_{k-2})\beta_i}]\{q(t_{k-2})\}. \quad (19)$$

In eqn (19) the $\{\mu_{i,1}\}$, $\{\mu_{i,2}\}$ are null vectors and

$$\{\mu_{i,3}\} = e^{-\tau_3\beta_i}(e^{-\tau_2\beta_i} - 1)\{q(0)\}.$$

Thus, the summation over the time realm is replaced by a summation over Prony series plus a recurrence relation. In certain cases however, eqn (17) can be used wherein the solution is required for less number of time steps.

2.6 *Incompressibility considerations*

The formulation is made applicable for nearly incompressible structures using a selective integration procedure, which is exact for the shear component and approximate for the bulk component of the elastic stiffness matrix.

The elastic stiffness matrix $[K_l]$ of eqn (13) is written into two (shear and bulk) components[9] as

$$[K_l] = [K_l^e] + [K_l^s] \quad (20)$$

for incompressible structures ($\nu \rightarrow 0.5$) the elements of $[K_l^e]$ tend to infinity and govern complete equilibrium. The contribution of the shear component $[K_l^s]$ becomes insignificant.

The software "VANIS" employs a selective integration procedure, which is third order Gauss rule for $[K_l^s]$ and second order Gauss rule for $[K_l^e]$, introducing numerical singularity into $[K_l^e]$. This approach gives very accurate stresses/strains at second order Gauss points[7, 9, 20].

3. NUMERICAL DISCUSSION

The code "VANIS" is tested on several problems. A few typical problems are reported in the present paper to demonstrate the behaviour of the software. It gives very accurate results for all the permissible values of Poisson's ratio.

The bulk modulus K is taken to be 110 Kg/mm^2 for all the problems. Kilogram, millimetre and hour are the units used for the force, length and time respectively. One hour time is divided into ten time steps for the analysis of the problems.

3.1 *Plane strain problem*

The geometrical and loading details are given in Fig. 2. In view of the symmetry, only quarter of the

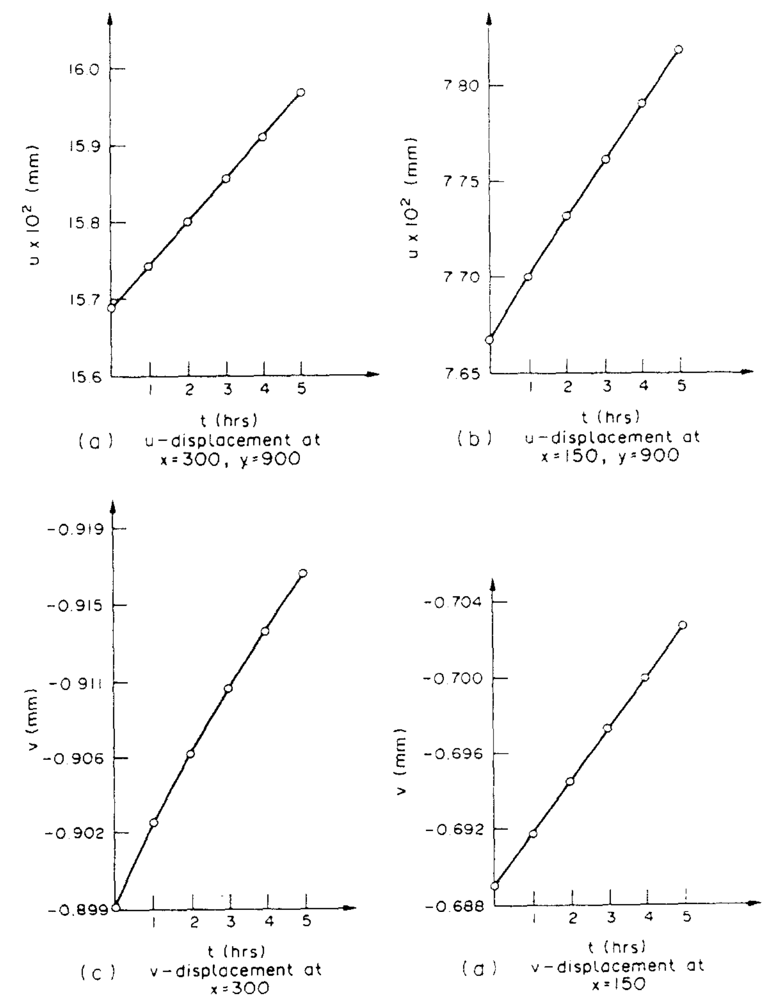


Fig. 5. Displacement with time.

structure is considered for the analysis and is divided into four plane strain finite elements. The applied stresses $\sigma_x(t)$ and $\sigma_y(t)$ are 0.01 Kg/mm^2 and 0.005 Kg/mm^2 respectively for all the positive values of t.

The modulus of rigidity is taken to be

$$G(t) = 0.002 + 0.098\, e^{-1.5t}.$$

It is a constant stress problem. Strains vary with time (creep) but are constant with geometry. The closed form solution for the problem is

$$u(x, t) = 1.25341 \times 10^{-2}[49.866960 - 48.866959 \times e^{-0.3t} - 10^{-6}\, e^{-1.499955t}]x. \quad (21)$$

The results obtained using the present code along with the closed form (exact) solution are given in Table 1. It is seen from the Table 1, that the present software gives very accurate results for nearly incompressible structures ($v(t) > 0.49955$). It is also seen from the Table 1 that the Poisson's ratio changes with time (as rigidity modulus is function of time and the bulk modulus is constant).

3.2 *Solid mass slump problem*

The propellant slump is a serious problem in solid propellant engineering. The material being viscoelastic in nature, the propellant grains stored for long time, undergo dimensional deviations due to their own weight. Normally the grains are supported by a casing. It is expected that the slumping can be minimised, by supporting the grain at the bottom. This problem is studied through a simple example of rectangular prism structure. Details of the structure and fem idealisation are given in Fig. 3. This problem is analysed using both the plane strain and 3-D finite elements.

The material properties used are:

$$G(t) = 0.022 + 0.03\, e^{-0.25t} + 0.048\, e^{-0.5t}$$
$$(\text{density}) = 1.8 \times 10^{-6}\ \text{Kg/mm}^3.$$

The results are given in Figs. 4 and 5. Both the plane strain and 3-D idealisations gave identical results. It is seen from these figures that both the displacements and stresses have come down with the increased b/a (bottom support). Also, it is seen from Fig. 5 that the displacements increase with time (of course for all the support conditions). It is found that stress variation with time is very small.

3.3 *A typical rocket grain*

The geometrical details of the grain, considered for the analysis are given in Fig. 6. It is a long grain encased in a rigid sheath. It is analysed for a thermal shrinkage of 30°C (from 60 to 30°C). In view of the symmetry only quarter of the grain is considered for the analysis. Plane strain and 3-D idealisations are used and is divided into 24 elements. The active degrees of freedom are 144 and 338 for plane strain and 3-D idealisations respectively. Both the idealisations gave identical results.

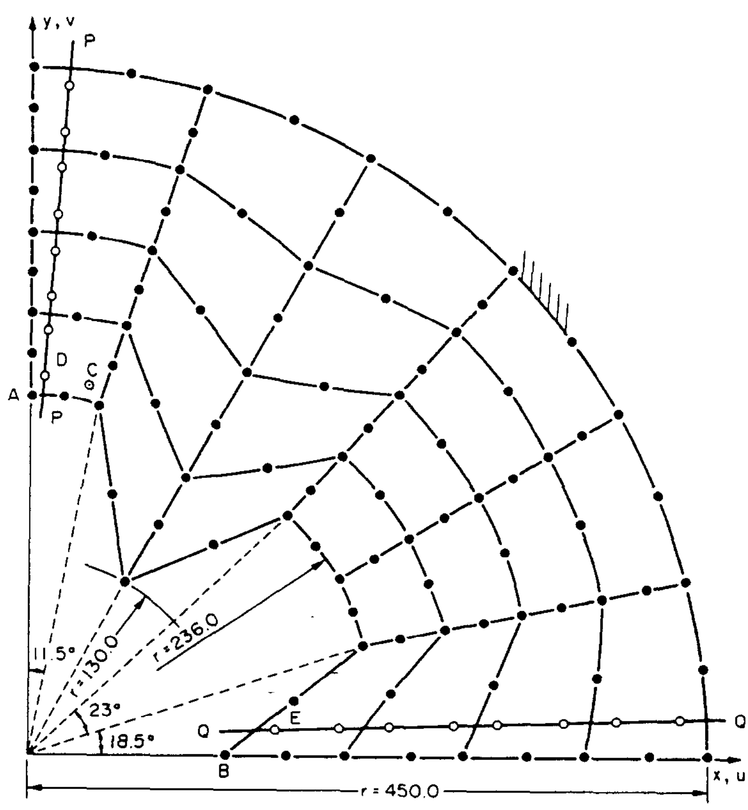


Fig. 6. Grain configuration (quarter section).

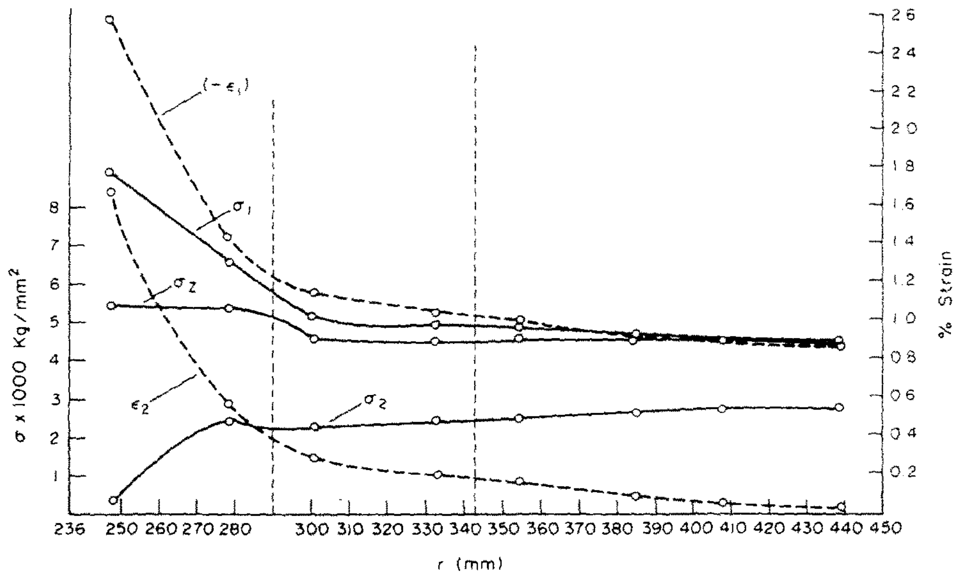


Fig. 7. Stresses and strains along P–P ($t = 2.0$ hr).

The following material properties are used for the analysis.

$$G(t) = 0.022 + 0.03\,e^{-0.5t} + 0.048\,e^{-t}$$
$$\alpha = 0.00011\ \text{mm/mm}^{\circ}\text{C}$$
$$C_1 = 8.0,\ C_2 = 150.0.$$

The stresses and strains along P–P and Q–Q (Fig. 6) are given in Figs. 7 and 8 at $t = 2.0$. P–P and Q–Q are the lines joining 2×2 Gaussian points and are close to y and x axes respectively. The displacements, stresses and strains at points A, B, C, D and E are given in Table 2 for various values of t. It is seen from the results that C and D are the stress concentration regions and the region E is compressively stressed.

Also, it is observed from the results that σ_1 and σ_2 represent (closely) hoop and radial stresses respectively. It is seen from Figs. 7 and 8 that the principal stress σ_2 (radial) approaches zero at the inner surface which is the condition that has to be satisfied at the free boundary. In Fig. 8 oscillation of σ_2 about zero is seen in the free boundary region owing to its small numerical values. Also it is seen, for the larger values of $r(r = \sqrt{x^2 + y^2})$, σ_1 is of the same order either along P–P or Q–Q indicating that the geometrical irregularities affect the stress distribution locally. The situation is the same with σ_2.

The stresses build up with time (Table 2) until $t = 2.0$ and then relax, whereas the strains keep on building up leading to the strain failure for such materials and loading conditions.

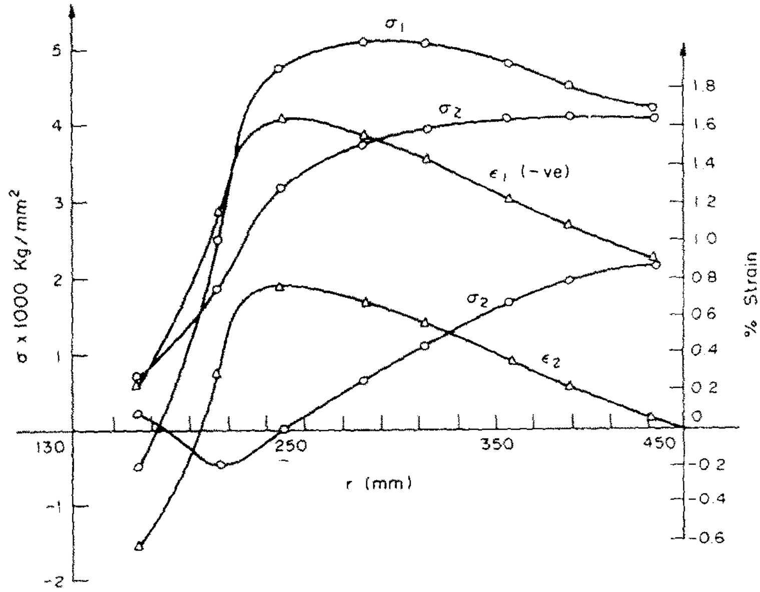


Fig. 8. Stresses and strains along Q–Q ($t = 2.0$).

Table 2. Stresses and displacements in rocket grain

Time Hrs.	Displacements (mm) at A	B	$\sigma_1 \times 1000$ kg/mm^2 at			% strains at C	C	D
	v	u	C	D	E	ϵ_1	ϵ_2	ϵ_1
1.0	1.925	2.503	5.655	5.339	−0.305	−1.514	0.877	−1.881
2.0	2.630	3.416	9.386	8.862	−0.526	−2.073	1.194	−2.575
3.0	2.894	3.762	8.420	7.949	−0.452	−2.285	1.315	−2.840
4.0	2.995	3.901	6.205	5.858	−0.302	−2.369	1.364	−2.948
5.0	3.034	3.960	4.558	4.304	−0.193	−2.405	1.386	−2.994

4. CONCLUSIONS

The code "VANIS" presented in the paper is based on the linear uncoupled thermoviscoelastic theory. The software is made useful for nearly incompressible structures through the use of a selective integration procedure. It gives very accurate results for all the permissible values of Poisson's ratio. As it uses the displacement formulation it is versatile and economical.

Practical analysis of rocket motor type structures needs incorporation of the stiffness of the casing and the bonding adhesive layer characteristics. These aspects are being incorporated by implementing two more elements, viscoelastic adhesive layer element[21, 22] and a shell element[23] with isotropic/orthotropic properties in the code.

REFERENCES

1. L. R. Herrmann, Elasticity equations for incompressible and nearly incompressible materials by a variational theorem. *AIAA J.* **3**, 1896–1900 (1965).
2. L. R. Herrmann, Interpretation of finite element procedure as stress error minimization procedure. *J. Engng Mech., ASCE* **98**, 1330–1336 (1972).
3. I. M. Smith, Incremental numerical solution of a simple deformation problem in soil mechanics. *Geotechnique* **20**, 357–372 (1970).
4. Y. Huang, Finite element analysis of non-linear soil media. *Application of Finite Elements in Civil Engineering*, pp. 663–690. Vanderbilt University (1969).
5. B. M. Irons, Round-off criteria in direct stiffness solutions. *AIAA J.* **6**, 1308–1312 (1968).
6. T. H. H. Pain and Pin Tong, Basis of finite element methods for solid continua. *Int. J. Num. Meth. Engng* **1**, 3–28 (1969).
7. D. J. Naylor, Stresses in nearly incompressible materials by the finite elements with application to the calculation of excess pore pressures. *Int. J. Num. Meth. Engng* **8**, 443–460 (1974).
8. I. Fried, Finite element analysis of incompressible material by residual energy balancing. *Int. J. Solids Structures* **10**, 993–1002 (1974).
9. O. C. Zienkiewicz, *The Finite Element Method*. McGraw-Hill, New York (1977).
10. T. J. R. Hughes, Equivalence of the finite elements for nearly incompressible elasticity. *J. Appl. Mech.* **44**, 181–183 (1977).
11. E. H. Lee, Stress analysis of viscoelastic bodies. *Quart. Appl. Math.* **13**, 183–190 (1955).
12. R. A. Schapary, Approximate methods of transform inversion for viscoelastic stress analysis. *Proc. 4th U.S. Nat. Cong. Appl. Mech.* **2**, 1075–1085 (1962).
13. H. H. Hilton and R. G. Russel, An extension of Alfrey's analogy to thermal stress problems in temperature dependent linear viscoelastic media. *J. Appl. Phys. Sci.* **9**, 152–164 (1961).
14. O. C. Zienkiewicz, M. Watson and I. P. King, A numerical method of viscoelastic analysis. *Int. J. Mech. Sci.* **10**, 807–827 (1968).
15. W. C. Carpenter, Viscoelastic stress analysis. *Int. J. Num. Meth. Engng* **4**, 357–366 (1972).
16. J. L. White, Finite elements in linear viscoelasticity. *Proc. 2nd Conf., Matrix, Meth. Struct. Mech.* **AFFDLTR-150**, 489–516 (1968).
17. S. Yadagiri, Three dimensional thermoviscoelastic analysis of rocket grains using 3-D quadratic isoparametric solid finite element. *Proc. Workshop on Structural Integrity of Large Solid Propellant Grains*. SHAR Centre, ISRO, India, Paper -2- (1982).
18. M. Muki and E. Sternberg, On the transient thermal stresses in viscoelastic materials with temperature dependent properties. *J. Appl. Mech.* **28**, 193–207 (1961).
19. M. L. Williams, Structural analysis of viscoelastic materials. *AIAA J.* **5**, 785–808 (1964).
20. S. Yadagiri, On the numerical stability of the finite element method. *J. Aero. Sec. India* **34**, 19–27 (1982).
21. B. Nageswara Rao, Y. V. K. Sadasiva Rao, and S. Yadagiri, Analysis of composite bonded joints. *Fibre Sci. Tech.* **17**, 77–90 (1982).
22. S. Yadagiri, Viscoelastic analysis of adhesively bonded joints. To be published.
23. SAP2 User's manual, 16-node isoparametric thick shell element.

APPENDIX

Elements of viscoelastic matrix [C]

The elements of viscoelastic matrix [C] are

$$C_{ii} = -4/3 \qquad i = 1, m$$
$$C_{ij} = +2/3 \qquad i, j = 1, m, i \neq j$$
$$C_{ii} = -1 \qquad i = m + 1, n$$

$m = 2$, $n = 3$ for plane strain element
$m = 3$, $n = 4$ for axisymmetric solid element.
$m = 3$, $n = 6$ for 3-D element.

All other elements of [C] matrix are zeros.