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THE RECTILINEAR OSCILLATIONS OF AN ELLIPTIC CYLINDER IN INCOMPRESSIBLE MICROPOLAR FLUID

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Abstract—The paper is a study of the flow of incompressible micropolar fluid arising from the harmonic oscillation of an elliptic cylinder parallel to either of the principal axes of its cross section. The stream function as well as the velocity and microrotation vectors are obtained and expressed in infinite series form involving Mathieu functions and functions allied to them. The drag on the cylinder is evaluated and the effect of variations of the polarity and frequency parameters on the drag as revealed by numerical studies is shown through figures.

1. INTRODUCTION

Micropolar fluids [1] are a subclass of simple microfluids [2] and the theory of either of them takes account of the microscopic effects arising from the local structure and micromotion of the fluid elements. In both theories, there is the sustenance of couple stress which is a distinguishing feature of these fluids as contrasted with nonpolar fluids whose motion is governed by the Navier–Stokes equations. Another significant departure of the polar fluid theory is the non-symmetry of the surface stress tensor. The field equations of micropolar fluid flow are representable in terms of the fluid velocity vector and the microrotation vector and the constitutive equations of the theory involve first order spatial gradients on the above two field vectors.

The study of harmonic oscillations of a body in a fluid domain has fascinated many over the years and the oscillations of symmetric bodies like the circular cylinder, sphere and spheroid as well as the rectilinear oscillations of an elliptic cylinder in classical viscous fluids have been analytically examined by Kanwal [3,4]. The harmonic oscillation of a sphere and a circular cylinder has been discussed for viscoelastic fluids of Oldroyd's three constant model (fluid of B-type) by Frater [5,6] and numerical information is provided to reveal the effects of the viscoelastic and frequency parameters on the drag experienced by the bodies. The harmonic oscillations of symmetric bodies like a circular cylinder, a sphere and a spheroid in micropolar fluids have already been examined in [7–9] and numerical information extracted to show the thrust of the micropolarity parameter as well as the frequency parameter on the drag/couple experienced by the bodies.

The present study is centred round the flow of incompressible micropolar fluid arising from the harmonic oscillation of an elliptic cylinder oscillating rectilinearly along its major or minor axis. The velocity and microrotation are evaluated in analytical form and the drag on the cylinder is computed. The result of numerical study has been presented in the form of figures showing the critical variations of the drag parameters for various frequency levels.

2. DIFFERENTIAL EQUATIONS OF THE PROBLEM

The field equations of incompressible micropolar fluid flow are given by the collection of equations

$$\operatorname{div}\bar{\mathbf{q}} = 0 \tag{2.1}$$

$$\rho\frac{d\bar{\mathbf{q}}}{dt} = \rho\bar{\mathbf{l}} - \operatorname{grad} p + k\operatorname{curl}\bar{v} - (\mu + k)\operatorname{curl}\operatorname{curl}\bar{\mathbf{q}} \tag{2.2}$$

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$$\rho j\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \rho\bar{l} - 2k\bar{v} + k\,\mathrm{curl}\,\bar{q} - \gamma\,\mathrm{curl}\,\mathrm{curl}\,\bar{v}$$

$$+ (\alpha + \beta + \gamma)\mathrm{grad}\,\mathrm{div}\,\bar{v} \tag{2.3}$$

In these equations, $\bar{\mathbf{q}}$ and $\bar{\mathbf{v}}$ are the velocity and microrotation vectors, j is the gyration parameter, (μ, k) are the viscosity coefficients while (α, β, γ) are the gyroviscosity coefficients. The symbols ρ, p, $\bar{\mathbf{f}}$, $\bar{\mathbf{l}}$ denote per unit mass and body couple per unit mass respectively. The force stress tensor t_{ij} and the couple stress tensor m_{ij} are given by [1]

$$t_{ij} = -p\delta_{ij} + (2\mu + k)e_{ij} + k_{ijr}(\omega_r - v_r) \tag{2.4}$$

$$m_{ij} = \alpha(\mathrm{div}\,\bar{\mathbf{v}})\delta_{ij} + \beta\gamma_{i,j} + \gamma v_{j,i} \tag{2.5}$$

where e_{ij} is the rate of strain tensor and $\omega_k = \frac{1}{2}\varepsilon_{kij}v_{j,i}$ is the spin tensor. The viscosity coefficients (μ, k) and the gyroviscosity coefficients (α, β, γ) conform to the following inequalities:

$$k \geqslant 0; \qquad 2\mu + k \geqslant 0;$$

$$\alpha + \beta + \gamma \geqslant 0; \qquad 3\alpha + \beta + \gamma \geqslant 0; \qquad |\beta| \leqslant \gamma \tag{2.6}$$

An elliptic cylinder oscillates harmonically and rectilinearly with velocity, $U\exp(i\sigma t)$ along its major or minor axis. The cross-section ellipse has the semifocal distance c and the quantity $U/(c\sigma)$ is assumed small and the inertial and gyroinertial terms in the equations of motion are linearized and the body force and body couple terms are deleted.

Let $\bar{\mathbf{e}}_\alpha$, $\bar{\mathbf{e}}_\beta$, $\bar{\mathbf{e}}_z$ be the base vectors of an elliptic coordinate system with the line element

$$\mathrm{d}s^2 = h_\alpha^2\mathrm{d}\alpha^2 + h_\beta^2\mathrm{d}\beta^2 + \mathrm{d}z^2 \tag{2.7}$$

The velocity and microrotation vectors appropriate to the problem are

$$\bar{\mathbf{q}} = u(\alpha, \beta, t)\bar{\mathbf{e}}_\alpha + v(\alpha, \beta, t)\bar{\mathbf{e}}_\beta \tag{2.8}$$

$$\bar{\mathbf{v}} = C(\alpha, \beta, t)\bar{\mathbf{e}}_z \tag{2.9}$$

and in terms of the stream function $\psi(\alpha, \beta, t)$ we can write

$$h_\beta u = -\frac{\partial\psi}{\partial\beta}, \qquad h_\alpha v = \frac{\partial\psi}{\partial\alpha} \tag{2.10}$$

The linearized versions of the equations of motion are

$$\rho h\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial\alpha} + k\frac{\partial C}{\partial\beta} - (\mu + k)\frac{\partial}{\partial\beta}(\nabla_1^2\psi) \tag{2.11}$$

$$\rho h\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial\beta} - k\frac{\partial C}{\partial\alpha} + (\mu + k)\frac{\partial}{\partial\alpha}(\nabla_1^2\psi) \tag{2.12}$$

$$\rho j\frac{\partial C}{\partial t} = -2kC + k\nabla_1^2\psi + \gamma\nabla_1^2 C \tag{2.13}$$

in which $h_\alpha = h_\beta = h$ and

$$\nabla_1^2 = \frac{1}{h^2}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right) \tag{2.14}$$

In view of the harmonic oscillation of the cylinder we may take

$$(\psi(\alpha, \beta, t), C(\alpha, \beta, t), p(\alpha, \beta, t)) = (f(\alpha, \beta), g(\alpha, \beta), P(\alpha, \beta)) \exp(i\sigma t) \qquad (2.15)$$

From equations (2.11–2.15) it follows easily that

$$\frac{\partial P}{\partial \alpha} = i\rho\sigma\frac{\partial f}{\partial \beta} + k\frac{\partial g}{\partial \beta} - (\mu + k)\frac{\partial}{\partial \beta}(\nabla_1^2 f) \qquad (2.16)$$

$$\frac{\partial P}{\partial \beta} = -i\rho\sigma\frac{\partial f}{\partial \alpha} - k\frac{\partial g}{\partial \alpha} + (\mu + k)\frac{\partial}{\partial \alpha}(\nabla_1^2 f) \qquad (2.17)$$

$$(i\rho j\sigma + 2k)g = \gamma\nabla_1^2 g + k\nabla_1^2 f \qquad (2.18)$$

Eliminating P from (2.16) and (2.17), we have

$$i\rho\sigma\nabla_1^2 f + k\nabla_1^2 g - (\mu + k)\nabla_1^2\nabla_1^2 f = 0 \qquad (2.19)$$

and eliminating g using (2.18) and (2.19) we obtain the following differential equation for $f(\alpha, \beta)$:

$$\gamma(\mu + k)\nabla_1^6 f - \{k(2\mu + k) + i\rho\sigma(\gamma + j\mu + jk)\}\nabla_1^4 f + i\rho\sigma(i\rho j\sigma + 2k)\nabla_1^2 f = 0 \qquad (2.20)$$

The function $g(\alpha, \beta)$ is expressible in terms of $f(\alpha, \beta)$ in the form

$$k(2k + i\rho j\sigma)g = \gamma(\mu + k)\nabla_1^4 f + (k^2 - i\rho\sigma\gamma)\nabla_1^2 f \qquad (2.21)$$

Let the complex numbers a^2 and b^2 be defined by the equations

$$\gamma(\mu + k)(a^2 + b^2) = k(2\mu + k) + i\rho\sigma(\gamma + j\mu + jk) \qquad (2.22)$$

$$\gamma(\mu + k)a^2b^2 = i\rho\sigma(i\rho\sigma j + 2k) \qquad (2.23)$$

The roots with their real parts positive are denoted by a and b. Equation (2.20) can be written in the form

$$\nabla_1^2(\nabla_1^2 - a^2)(\nabla_1^2 - b^2)f = 0 \qquad (2.24)$$

and it is possible to write f in the form

$$f = f_0 + f_1 + f_2 \qquad (2.25)$$

where

$$\nabla_1^2 f_0 = 0, \qquad \nabla_1^2 f_1, \qquad \nabla_1^2 f_2 = b^2 f_2 \qquad (2.26)$$

The superposition of the three solutions in (2.26) to make up the solution f is on the tacit understanding that a^2 and b^2 are distinct. The possibility of resonance ($a^2 = b^2$) cannot be ruled out in the case of micropolar fluids and this arises when

$$(\gamma/j) = (2\mu + k)(\mu + k)/(2\mu + 3k)$$
$$\rho\omega = (2\mu + k)(2\mu + 3k)/[2(\mu + k)j] \qquad (2.27)$$

The case of resonance which arises also in circular and spherical geometry is significant and merits separate discussion.

Equation (2.21) directly expresses g in terms of the derivatives of f and specifically it can be expressed in terms of f_1 and f_2 in the form

$$k(2k + i\rho j\sigma)g(\alpha, \beta) = \gamma(\mu + k)(a^4 f_1 + b^4 f_2) + (k^2 - i\rho\sigma\gamma)(a^2 f_1 + b^2 f_2) \qquad (2.28)$$

The problem of the oscillation of the elliptic cylinder therefore rests on the solution of the differential equations in (2.26) subject to the prescribed regularity requirements at infinity and hyperstick or superadherence condition on the boundary. It may be observed that while alternative types of boundary conditions involving the microrotation vector are possible, the hypersticks condition is the most common among them all.

3. OSCILLATION PARALLEL TO THE MAJOR AXIS

The elliptic coordinates (α, β) are defined by the relation

$$x + iy = c\cosh(\alpha + i\beta) \qquad (3.1)$$

and the cylinder is given by $\alpha = \alpha_0$. The scale factors of the frame are

$$h_\alpha = h_\beta = h = c(\cosh^2\alpha - \cos^2\beta)^{1/2} \qquad (3.2)$$

The adherence condition on the cylinder means that the velocity of the fluid element on the cylinder is $U\exp(i\sigma t)$ parallel to the major axis and the microrotation vector on the cylinder equals $\frac{1}{2}\,\mathrm{curl}(\bar{\mathbf{q}}_{\mathrm{boundary}})$ and this is equal to zero. Therefore, on the boundary $\alpha = \alpha_0$, we have the conditions

$$u(\alpha, \beta, t) = \frac{Uc}{h}\sinh\alpha\cos\beta\exp(i\sigma t)$$

$$v(\alpha, \beta, t) = -\frac{Uc}{h}\cosh\alpha\sin\beta\exp(i\sigma t)$$

$$C(\alpha, \beta, t) = 0\,(\text{i.e.})\,g(\alpha, \beta) = 0 \qquad (3.3)$$

These are equivalent to the statements

$$f(\alpha, \beta) = -Uc\,\sinh\alpha\,\sin\beta$$

$$\frac{\partial f}{\partial \alpha} = -Uc\,\cosh\alpha\,\sin\beta \qquad (3.4)$$

and

$$\gamma(\mu + k)\nabla_1^4 f + (k^2 - i\rho\sigma\gamma)\nabla_1^2 f = 0 \qquad (3.5)$$

on $\alpha = \alpha_0$. Far away from the body, as $\alpha \to \infty$, the velocity and microrotation vanish.

The function $f_0(\alpha, \beta)$ is harmonic (cf. equation 2.26), and the boundary conditions (3.4), (3.5) suggest the solution of the form

$$f_0(\alpha, \beta) = \sum_{n=1}^{\infty} C_n \exp(-n\alpha)\sin n\beta \qquad (3.6)$$

The function $f_1(\alpha, \beta)$ satisfies the differential equation

$$(\nabla_1^2 - a^2)f_1 = 0 \qquad (3.7)$$

and this can be written as

$$\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} - \frac{a^2c^2}{2}(\cosh 2\alpha - \cos 2\beta)\right]f_1 = 0 \tag{3.8}$$

Putting

$$f_1(\alpha, \beta) = R(\alpha)S(\beta) \tag{3.9}$$

we find that

$$R''(\alpha) - (\lambda + (a^2c^2/2)\cos 2\alpha)R = 0 \tag{3.10}$$

$$S''(\beta) + (\lambda + (a^2c^2/2)\cos 2\beta)S = 0 \tag{3.11}$$

in which λ is the separation constant. The Mathieu equation (3.11) has the periodic solutions [10]

$$ce_m(\beta, -a^2c^2/4), \qquad se_m(\beta, -a^2c^2/4) \tag{3.12}$$

corresponding to a discrete set of values of λ which are functions of $(a^2c^2/4)$. In this problem we need only the solutions se_m correspond to the characteristic numbers $\lambda = a_{2m+1}$, $\lambda = b_{2m+2}$ and have the Fourier sine series expansions

$$se_{2m+1}(\beta, -a^2c^2/4) = \sum_{r=0}^{\infty} (-1)^{m+r} A_{2r+1}^{(2m+1)} \sin(2r+1)\beta \tag{3.13}$$

$$se_{2m+2}(\beta, -a^2c^2/4) = \sum_{r=0}^{\infty} (-1)^{m+r} B_{2r+2}^{(2m+2)} \sin(2r+2)\beta \tag{3.14}$$

In these Fourier expansions, the coefficients A, B are functions of the parameters $(-a^2c^2/4)$.

The solutions of the modified Mathieu equation (3.10) that correspond to the solutions in (3.13), (3.14) and vanish as $\alpha \to \infty$ are given by the functions $Gek_m(\alpha, -a^2c^2/4)$ which are representable in the form [10, p. 248]:

$$Gek_{2m+1}(\alpha, -a^2c^2/4) = [p'_{2m+1}/(\pi A_1^{(2m+1)})] \sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)} \{I_r(ace^{-\alpha}/2)K_{r+1}(ace^{\alpha}/2) + I_{r+1}(ace^{-\alpha}/2)K_r(ace^{\alpha}/2)\} \tag{3.15}$$

$$Gek_{2m+2}(\alpha, -a^2c^2/4) = [s'_{2m+2}/\pi B_2^{(2m+2)}] \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} \{I_r(ace^{-\alpha}/2)K_{r+2}(ace^{\alpha}/2) + I_{r+2}(ace^{-\alpha}/2)K_r(ace^{\alpha}/2)\} \tag{3.16}$$

in which I and K denote the modified Bessel functions. The solution $f_1(\alpha, \beta)$ is thus representable in the form

$$f_1(\alpha, \beta) = \sum_{n=1}^{\infty} D_n Gek_n(\alpha, -a^2c^2/4) se_n(\beta, -a^2c^2/4) \tag{3.17}$$

The function $f_2(\alpha, \beta)$ can similarly be represented in the form

$$f_2(\alpha, \beta) = \sum_{n=1}^{\infty} E_n Gek_n(\alpha, -b^2c^2/4) se_n(\beta, b^2c^2/4) \tag{3.18}$$

The solution of (2.24) is obtained by the addition of f_0, f_1 and f_2 given in (3.6), (3.17) and (3.18) respectively.

The constants $\{C_n\}$, $\{D_n\}$, $\{E_n\}$ are determined by the boundary conditions (3.4), (3.5) and to enforce these, it is essential to recast the expansions of f_1 and f_2 in (3.17) and (3.18) into Fourier series involving sine terms of β. This is done using the expansions for se_{2m+1} and se_{2m+2} given in (3.13) and (3.14). We may write

$$Gek_m(\alpha, -a^2c^2/4) se_m(\beta, -a^2c^2/4) = \sum_{n=1}^{\infty} F_{mn}(\alpha) \sin n\beta \tag{3.19}$$

and then it is easily seen from (3.13), (3.14) and (3.19) that

$$F_{2m+1,n}(\alpha) = \begin{cases} 0, (n = 2, 4, 6, \ldots) \\ (-1)^{m+r} A_{2r+1}^{(2m+1)} Gek_{2m+1}(\alpha, -a^2c^2/4), (n = 2r+1, r = 0, 1, 2, \ldots) \end{cases} \tag{3.20}$$

$$F_{2m+2,n}(\alpha) = \begin{cases} 0, (n = 1, 3, 5, \ldots) \\ (-1)^{m+r} B_{2r+2}^{(2m+2)} Gek_{2m+2}(\alpha, -a^2c^2/4), (n = 2r+2, r = 0, 1, 2, \ldots) \end{cases} \tag{3.21}$$

In a similar way, we can have

$$Gek_m(\alpha, -b^2c^2/4) se_m(\beta, -b^2c^2/4) = \sum_{n=1}^{\infty} G_{mn}(\alpha) \sin n\beta \tag{3.22}$$

The functions $G_{mn}(\alpha)$ are defined exactly as $F_{mn}(\alpha)$ given in (3.20), (3.2) with the parameter $b^2c^2/4$ instead of $a^2c^2/4$. The function $f(\alpha, \beta)$ can then be represented in the form

$$f(\alpha, \beta) = \sum_{n=1}^{\infty} \left(C_n \exp(-n\alpha) + \sum_{m=1}^{\infty} D_m F_{mn}(\alpha) + \sum_{m=1}^{\infty} E_m G_{mn}(\alpha)\right) \sin n\beta \tag{3.23}$$

From this we have

$$\nabla_1^4 f + \frac{k^2 - i\rho\sigma\gamma}{\gamma(\mu + k)} \nabla_1^2 f = \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} D_m \left(a^4 + \frac{k^2 - i\rho\sigma\gamma}{\gamma(\mu + k)} a^2\right) F_{mn}(\alpha) + \sum_{m=1}^{\infty} E_m \left(b^4 + \frac{k^2 - i\rho\sigma\gamma}{\gamma(\mu + k)} b^2\right) G_{mn}(\alpha) \right\} \sin n\beta \tag{3.24}$$

The boundary conditions (3.4), (3.5) can now be enforced and we have the following muster of linear algebraic equations in the constants $\{C_n\}$, $\{D_n\}$, $\{E_n\}$:

$$C_n \exp(-n\alpha_0) + \sum_{m=1}^{\infty} D_m F_{mn}(\alpha_0) + \sum_{m=1}^{\infty} E_m G_{mn}(\alpha_0) = -Uc \sinh \alpha_0 \delta_{n1} \tag{3.25}$$

$$-nC_n \exp(-n\alpha_0) + \sum_{m=1}^{\infty} D_m F'_{mn}(\alpha_0) + \sum_{m=1}^{\infty} E_m G'_{mn}(\alpha_0) = -Uc \cosh \alpha_0 \delta_{n1} \tag{3.26}$$

$$\sum_{m=1}^{\infty} D_m\left(a^4 + \frac{k^2 - i\rho\sigma\gamma}{\gamma(\mu + k)}a^2\right)F_{mn}(\alpha_0) + \sum_{m=1}^{\infty} E_m\left(b^4 + \frac{k^2 - i\rho\sigma\gamma}{\gamma(\mu + k)}b^2\right)G_{mn}(\alpha_0) = 0 \qquad (n \geqslant 1) \tag{3.27}$$

The prime on F_{mn} and G_{mn} in (3.26) indicates differentiation with respect to α. Elimination of C_n from equations (3.25), (3.26) leads to the linear set of equations in the constants $\{D_m\}$, $\{E_m\}$:

$$\sum_{m=1}^{\infty} D_m[nF_{mn}(\alpha_0) + F'_{mn}(\alpha_0)] + \sum_{m=1}^{\infty} E_m[nG_{mn}(\alpha_0) + G'_{mn}(\alpha_0)] = -Uc\exp(\alpha_0)\delta_{n1} \tag{3.28}$$

The systems of equations (3.27) and (3.28) can be used to determine the constants D_m, E_m and then the constants C_n are obtainable from (3.25). This leads to the determination of the solution $f(\alpha, \beta)$. The function $g(\alpha, \beta)$ is thereafter determined from (2.28). The velocity and microrotation components are then determined from the stream function $\psi(\alpha, \beta, t) = f(\alpha, \beta)$ $\exp(i\sigma t)$ and the function $g(\alpha, \beta)\exp(i\sigma t)$.

Pressure

The pressure distribution $p(\alpha, \beta, t) = P(\alpha, \beta)\exp(i\sigma t)$ is determined from (2.16) and (2.17). From (2.16), we have

$$\frac{\partial P}{\partial \alpha} = i\rho\sigma\frac{\partial}{\partial \beta}(f_0 + f_1 + f_2) + (2k + i\rho j\sigma)^{-1} \cdot\left\{\gamma(\mu + k)\frac{\partial}{\partial \beta}(a^4 f_1 + b^4 f_2) + (k^2 - i\rho\sigma\gamma)\frac{\partial}{\partial \beta}(a^2 f_1 + b^2 f_2)\right\} - (\mu + k)\frac{\partial}{\partial \beta}(a^2 f_1 + b^2 f_2) \tag{3.29}$$

It can be seen that $\partial f_1/\partial\beta$ and $\partial f_2/\partial\beta$ in the above add up to zero. Hence

$$\frac{\partial P}{\partial \alpha} = i\rho\sigma\sum_{n=1}^{\infty} nC_n\exp(-n\alpha)\cos n\beta \tag{3.30}$$

and on integration

$$P(\alpha, \beta) = -i\rho\sigma\sum_{n=1}^{\infty} C_n\exp(-n\alpha)\cos n\beta \tag{3.31}$$

The integration constant in (3.31) is chosen equal to 0.

Drag on the cylinder

The stress tensor t_{ij} can be evaluated using (2.4) and the nonvanishing components of it are $t_{\alpha\alpha}$, $t_{\alpha\beta}$, $t_{\beta\alpha}$, $t_{\beta\beta}$, t_{zz}. The stress vector on the cylinder is $t_{\alpha\alpha}\bar{\mathbf{e}}_\alpha + t_{\alpha\beta}\bar{\mathbf{e}}_\beta + t_{\alpha z}\bar{\mathbf{e}}_z$ and the drag per length L of the cylinder is

$$D = D_p + D_f = cL\int_0^{2\pi}(t_{\alpha\alpha}\sinh\alpha\cos\beta - t_{\alpha\beta}\cosh\alpha\sin\beta)_{\alpha=\alpha_0}\,d\beta \tag{3.32}$$

where the pressure drag D_p and the friction drag D_f are given by

$$D_p = cL_0^{2\pi}(t_{\alpha\alpha}\sinh\alpha)_{\alpha=\alpha_0}\cos\beta\,d\beta$$

$$D_f = -cL\int_0^{2\pi}(t_{\alpha\beta}\cosh\alpha)_{\alpha=\alpha_0}\sin\beta\,d\beta \tag{3.33}$$

From (2.4) we have

$$t_{\alpha\alpha} = -p + (2\mu + k)e_{\alpha\alpha} \tag{3.34}$$

$$t_{\alpha\beta} = (2\mu + k)e_{\alpha\beta} + k(\omega_z - v_z) \tag{3.35}$$

and elementary but long calculation shows that on the boundary of the cylinder

$$e_{\alpha\alpha} = 0 \tag{3.36}$$

$$e_{\alpha\beta} = -\left(\frac{i\rho\sigma}{2(\mu + k)}\right)\left\{Uc\sin\alpha_0\sin\beta + \sum_{n=1}^{\infty} C_n(\exp(-n\alpha_0)\sin n\beta\right\}\exp(i\sigma t) \tag{3.37}$$

and

$$\omega_z - v_z = e_{\alpha\beta} \tag{3.38}$$

It follows that

$$D_p = i\pi\rho\sigma cLC_1\sin\alpha_0\exp(-\alpha_0)\exp(i\sigma t) \tag{3.39}$$

while

$$D_f = i\pi\rho\sigma cL\cosh\alpha_0[Uc\sinh\alpha_0 + C_1\exp(-\alpha_0)]\exp(i\sigma t) \tag{3.40}$$

The total drag on the cylinder is thus

$$D = i\pi\rho\sigma cL(Uc\sinh\sigma_0\cosh\alpha_0 + C_1)\exp(\tag{3.41}$$

The couple stress m_{ij} defined in (2.5) has the nonvanishing components $m_{z\alpha}$, $m_{\alpha z}$, $m_{\beta z}$ and $m_{z\beta}$. The couple stress vector on the boundary is $m_{\alpha z}\bar{e}_z$ and the couple on the cylinder about the axis is zero.

Limiting case

By allowing α_0 to $+0$, we have the case of a flat plate harmonically oscillating along its edge. The stream function $\psi = f(\alpha,\beta)\exp(i\sigma t)$ is found from (3.23) and the constants $\{C_n\}$, $\{D_m\}$, $\{E_m\}$ are determined from the linear equations after chosing $\alpha_0 = 0$. The pressure drag in this case is zero and the drag

$$D = D_f = i\pi\rho cLC_1\exp(i\sigma t). \tag{3.42}$$

4. OSCILLATIONS PARALLEL TO MINOR AXIS

This runs similarly in all details to the problem of oscillation parallel to the major axis and results are only briefly stated.

Let the elliptic coordinates be now defined by the relation

$$x + iy = c\sinh(\alpha + i\beta) \tag{4.1}$$

The cylinder is given by $\alpha = \alpha_0$ and

$$h = c(\sinh^2\alpha + \cos^2\beta)^{1/2} \tag{4.2}$$

The boundary conditions are

$$\begin{aligned} f(\alpha, \beta) &= -Uc\cosh\alpha\sin\beta \\ \partial f/\partial\alpha &= -Uc\sinh\alpha\sin\beta \end{aligned} \tag{4.3}$$

and

$$\gamma(\mu + k)\nabla_1^4 f + (k^2 - i\rho\sigma\gamma)\nabla_1^2 f = 0 \tag{4.4}$$

on $\alpha = \alpha_0$. The differential equation for $f(\alpha, \beta)$ is (2.24) as before and the solution is the sum of the three functions $f_0(\alpha, \beta), f_1(\alpha, \beta), f_2(\alpha, \beta)$ respectively satisfying the three differential equations in (2.26). We have

$$f_0(\alpha, \beta) = \sum_{n=1}^{\infty} C_n^* \exp(-n\alpha)\sin n\beta \tag{4.5}$$

The solutions $f_1(\alpha, \beta)$, $f_2(\alpha, \beta)$ are of a different type compared to those in the previous section. The differential equation for $f_1(\alpha, \beta)$ is

$$\left\{\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} - \frac{a^2c^2}{2}(\cosh 2\alpha + \cos 2\beta)\right\} f_1 = 0 \tag{4.6}$$

which differs from (3.8). Taking $f_1(\alpha, \beta) = R(\alpha)S(\beta)$, we see that

$$R''(\alpha) - \left(\lambda + \frac{a^2c^2}{2}\cosh 2\alpha\right)R(\alpha) = 0 \tag{4.7}$$

$$S''(\beta) + \left(\lambda - \frac{a^2c^2}{2}\cos 2\beta\right)S(\beta) = 0 \tag{4.8}$$

The solutions of (4.8) are $ce_m(\beta, a^2c^2/4)$ and $se_m(\beta, a^2c^2/4)$ corresponding to a discrete set of values of the separation constant λ and in the problem we need the functions se_m only. They correspond to the characteristic numbers $\lambda = b_{2m+1}, b_{2m+2}$ and have the Fourier sine series expansions

$$se_{2m+1}(\beta, a^2c^2/4) = \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)} \sin(2r + 1)\beta \tag{4.9}$$

$$se_{2m+2}(\beta, a^2c^2/4) = \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} \sin(2r + 2)\beta \tag{4.10}$$

The solutions of (4.7) that correspond to the solutions in (4.9), (4.10) and which vanish as $\alpha \to \infty$ are the modified Mathieu functions Gek_{2m+1}^*, Gek_{2m+2}^* given by

$$\begin{aligned} &Gek_{2k+1}^*(\alpha, -a^2c^2/4) \\ &\quad = (s'_{2m+1}/(\pi B_1^{(2m+1)}) \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)} \{I_r(ace^{-\alpha}/2)K_{r+1}(ace^{\alpha}/2) \\ &\qquad + I_{r+1}(ace^{-\alpha}/2)K_r(ace^{\alpha}/2)\} \end{aligned} \tag{4.11}$$

$$Gek^*_{2m+2}(\alpha, -a^2c^2/4)$$

$$= [s'_{2m+2}/(\pi B_2^{(2m+2)})] \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} \{I_r(ace^{-\alpha}/2)K_{r+2}(ace^{\alpha}/2) - I_{r+2}(ace^{-\alpha}/2)K_r(ace^{\alpha}/2)\}. \quad (4.12)$$

The functions $f_1(\alpha, \beta)$ and $f_2(\alpha, \beta)$ can be expressed in the form

$$f_1(\alpha, \beta) = \sum_{n=1}^{\infty} D_n^* Gek_n^*(\alpha_1 - a^2c^2/4) se_n(\beta, a^2c^2/4) \quad (4.13)$$

and

$$f_2(\alpha, \beta) = \sum_{n=1}^{\infty} E_n^* Gek_n^*(\alpha, -b^2c^2/4) se_n(\beta, b^2c^2/4) \quad (4.14)$$

These functions can be put into alternative forms involving a series of sines in β as in the earlier section and the solution $f(\alpha, \beta)$ may be cast into the form

$$f(\alpha, \beta) = \sum_{n=1}^{\infty} \left\{ C_n^* \exp(-n\alpha) + \sum_{m=1}^{\infty} D_m^* F_{mn}^*(\alpha) + \sum_{m=1}^{\infty} E_m^* G_{mn}^*(\alpha) \right\} \sin n\beta \quad (4.15)$$

The functions $F_{mn}^*(\alpha)$, $G_{mn}^*(\alpha)$ are defined in a way analogous to the corresponding relations of the previous section:

$$F_{2m+1,n}^*(\alpha) = \begin{cases} 0, (n = 2, 4, 6, \ldots) \\ B_{2r+1}^{(2m+1)} Gek_{2m+1}^*(\alpha, -a^2c^2/4), (n = 2r+1, r = 0, 1, 2, \ldots) \end{cases} \quad (4.16)$$

$$F_{2m+2,n}^*(\alpha) = \begin{cases} 0, (n = 1, 3, 5, \ldots) \\ B_{2r+2}^{(2m+2)} Gek_{2m+2}^*(\alpha, -a^2c^2/4), (n = 2r+2, r = 0, 1, 2, \ldots) \end{cases} \quad (4.17)$$

$$G_{2m+1,n}^*(\alpha) = \begin{cases} 0, (n = 2, 4, 6, \ldots) \\ B_{2r+1}^{(2m+1)} Gek_{2m+1}^*(\alpha, -b^2c^2/4), (n = 2r+1, r = 0, 1, 2, \ldots) \end{cases} \quad (4.18)$$

$$G_{2m+2,n}^*(\alpha) = \begin{cases} 0, (n = 1, 3, 5, \ldots) \\ B_{2r+2}^{(2m+2)} Gek_{2m+2}^*(\alpha, -b^2c^2/4), (n = 2r+2, r = 0, 1, 2, \ldots) \end{cases} \quad (4.19)$$

The constants $\{C_n^*\}$, $\{D_m^*\}$, $\{E_m^*\}$ in (4.15) are determined from the boundary conditions (4.3) and (4.4). The linear equations involving the unknowns $\{C_n^*\}$, $\{D_m^*\}$, $\{E_m^*\}$ are given by

$$C_n^* \exp(-n\alpha_0) + \sum_{m=1}^{\infty} D_m^* F_{mn}^*(\alpha_0) + \sum_{m=1}^{\infty} E_m^* G_{mn}^*(\alpha_0) = -Uc \cosh \alpha_0 \delta_{n1} \quad (4.20)$$

$$-nC_n^* \exp(-n\alpha_0) + \sum_{m=1}^{\infty} D_m^* F_{mn}^{*\prime}(\alpha_0) + \sum_{m=1}^{\infty} E_m^* G_{mn}^{*\prime}(\alpha_0) = -Uc \sinh \alpha_0 \delta_{n1} \quad (4.21)$$

$$\sum_{m=1}^{\infty} D_m^* \left(a^4 + \frac{k^2 - i\rho\sigma}{\gamma(\mu + k)} a^2 \right) F_{mn}^*(\alpha_0) + \sum_{m=1}^{\infty} E_m^* \left(b^4 + \frac{k^2 - i\rho\sigma\gamma}{\gamma(\mu + k)} b^2 \right) G_{mn}^*(\alpha_0) = 0 \quad (4.22)$$

Elimination of the constants C_n^* from the two equations (4.20), (4.21) leads to the set of linear equations

$$\sum_{m=1}^{\infty} D_m^*\{nF_{mn}^*(\alpha_0) + F_{mn}^{*\prime}(\alpha_0)\} + \sum_{m=1}^{\infty} E_m^*\{nG_{mn}^*(\alpha_0) + G_{mn}^{*\prime}(\alpha_0)\} = -Uc\exp(\alpha_0)\delta_{n1} \quad (4.23)$$

From equations (4.22) and (4.23) it should, in principle, be possible to determine the constants $\{D_m^*\}$ and $\{E_m^*\}$ and thereafter the constants $\{C_n^*\}$ from (4.20). This leads to the determination of the solution $f(\alpha, \beta)$ and the function $g(\alpha, \beta)$ will then be available from (2.28).

Pressure

The pressure $p(\alpha, \beta, t) = P(\alpha, \beta)\exp(i\sigma t)$ is determined from equations (2.16) and (2.17). After elementary though lengthy calculation, we find that

$$P(\sigma, \beta, t) = -i\rho\sigma \sum_{n=1}^{\infty} C_n^* \exp(-n\alpha)\cos n\beta \exp(i\sigma t) \quad (4.24)$$

Drag on the cylinder

It is seen that on the cylinder, the strain velocity component $e_{\alpha\alpha} = 0$ and

$$e_{\alpha\beta} = -\frac{i\rho\sigma}{2(\mu + k)}(Uc\cosh\alpha_0 \sin\beta + \sum_{n=1}^{\infty} C_n^* \exp(-n\alpha_0)\sin n\beta)\exp(i\sigma t) \quad (4.25)$$

Further, $\omega_z - v_z = e_{\alpha\beta}$ on $\alpha = \alpha_0$. The stress components on the cylinder are therefore

$$t_{\alpha\alpha} = i\rho\sigma \sum_{n=1}^{\infty} (C_n^* \exp(-n\alpha_0)\cos n\beta)\exp(i\sigma t) \quad (4.26)$$

$$t_{\alpha\beta} = -(i\rho\sigma)\left(Uc\cosh\alpha_0 \sin\beta + \sum_{n=1}^{\infty} C_n^* \exp(-n\alpha_0)\sin n\beta\right)exp(i\sigma t) \quad (4.27)$$

The pressure drag per unit length L of the cylinder is

$$D_p = cL\cosh\alpha_0 \int_0^{2\pi} (t_{\alpha\alpha})_{\alpha=\alpha_0} \cos\beta \, d\beta = i\pi\rho\sigma c L C_1^* \cosh\alpha_0 \exp(-\alpha_0)\exp(i\sigma t) \quad (4.28)$$

The friction drag is

$$\begin{aligned} D_1 &= -cL\sinh\alpha_0 \int_0^{2\pi} (t_{\alpha\beta})_{\alpha=\alpha_0} \sin\beta \, d\beta \\ &= i\pi\rho\sigma c L \sinh\alpha_0 (Uc\cosh\alpha_0 + C_1^* \exp(-\alpha_0))\exp(i\sigma t) \end{aligned} \quad (4.29)$$

The drag D on the cylinder is

$$\begin{aligned} D &= D_p + D_f \\ &= i\pi\rho\sigma c L(Uc\sinh\alpha_0 \cosh\alpha_0 + C_1^*)\exp(i\sigma t) \end{aligned} \quad (4.30)$$

The nonvanishing couple stress components are $m_{z\alpha}$, $m_{\alpha z}$, $m_{z\beta}$, $m_{\beta z}$. The couple stress vector on the boundary of the cylinder is $m_{\alpha z}\bar{\mathbf{e}}_z$ and the couple on the body is seen to be zero.

Limiting case

By allowing α_0 to 0, we have the case of a flat plate performing harmonic oscillation transverse to its plane. The constants $\{C_n^*\}$, $\{D_m^*\}$, $\{E_m^*\}$ are to be obtained from the muster of linear equations (4.20) to (4.22) after the substitution $\alpha_0 = 0$. It is seen that the friction drag in this case is zero and the total drag is given by

$$D = D_f = i\pi\rho\sigma c L C_1^* \exp(i\sigma t) \tag{4.31}$$

Numerical work

For the numerical information on the drag one has to solve systems of linear equations involving the coefficients $\{C_n\}$, $\{D_n\}$, $\{E_m\}$ and $\{C_n^*\}$, $\{D_m^*\}$, $\{E_m^*\}$. The systems are of infinite order and one has to deal with three sets of unknowns, in both the cases of oscillations discussed in the paper. The matrices of the concerned linear systems have necessarily to be truncated and the order of truncation is decided by the extent to which the elements of the matrices can be numerically evaluated. These elements are transcendental functions involving modified Matheu functions and each coefficient of the matrix involves an infinite series expansion. The generic terms of these infinite series involve the coefficients A_r^m and B_r^m connected with the Mathieu functions and the modified Bessel functions I and K. Further the separation constant λ which is the eigenvalue parameter, involves an infinite series expansion in powers of $(ac/2)/(bc/2)$ and the order of truncation of the two linear systems is controlled by the availability in explicit form of the individual terms in the expansion for the parameter λ in the standard source material [10, 11]. For the evaluation of the constants λ, all the known terms in its expansion as presented in the above two references have been utilized.

The functions Gek_n are evaluated for $n = 1, 3, 5, 7, 9$ and in the process of this evaluation, the needed modified Bessel functions I_r and K_r are evaluated for the orders $r = 0, 1$ using the standard expansions for them [12] and for $r = 2, 3, 4, 5$ the well known recurrence relations have been utilized. The functions $F_{mn}(\alpha)$ and $F'_{mn}(\alpha)$ are needed only for odd values of m and n. These are obtained for $m, n = 1, 3, 5, 7, 9$. Thus the evaluation of the constants $\{D_n\}$ and $\{E_n\}$ rests on a truncated system of 10×10 linear algebraic equations and the constants C_n are determined after the determination of $\{D_m\}$ and $\{E_m\}$. Aim to have a larger sized truncation in the evaluation of $\{D_m\}$, $\{E_m\}$ would involve the need for the evaluation of an enlarged set of constants A_n^m, B_n^m and the functions Gek_n, F_{mn} and G_{mn}. The numerical evaluation of the constants $\{C_n^*\}$, $\{D_m^*\}$, $\{E_m^*\}$ in the oscillation of the cylinder parallel to the minor axis is treated on similar lines using a 10×10 linear system involving $\{D_m^*\}$, $\{E_m^*\}$.

In both the instances, viz. oscillation of the cylinder parallel to the major axis and the minor axis, the drag can be written in the form

$$D = -MU\sigma(iK + K')\exp(i\sigma t) \tag{4.33}$$

where

$$M = \pi\rho L c^2 \cosh\alpha_0 \sinh\alpha_0 \tag{4.34}$$

measures the mass of the fluid displaced by a height L of the cylinder. The drag parameters K, K' are defined by

$$-K' - iK = i\left(1 + \frac{C_1}{Uc\cosh\alpha_0\sinh\alpha_0}\right) \tag{4.35}$$

when the oscillation is parallel to the major axis. When the oscillation is parallel to the minor axis, these parameters are defined by

$$-K' - iK = i\left(1 + \frac{C_1^*}{Uc\cosh\alpha_0\sinh\alpha_0}\right) \tag{4.36}$$

Figures 1–20 show the variation of the drag parameters K and K' for various values of the frequency and material parameters of the fluid. The case of nonpolar fluids is also included in the profile of figures. The symbols employed in the figures are as follows:

$$PL = \frac{k(2\mu + k)c^2}{\gamma(\mu + k)}, \qquad PJ = \frac{j(\mu + k)}{\gamma},$$

$$AL = \frac{2(\mu + k)}{k}, \qquad PT = \frac{\rho\sigma c^2}{\mu + k}$$

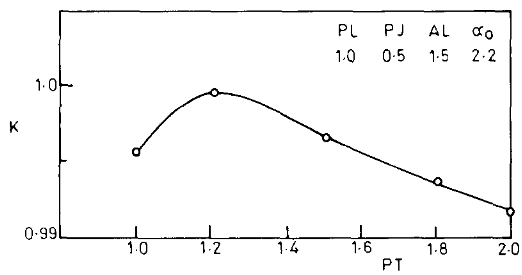

Fig. 1. Variation of K (oscillation along major axis).

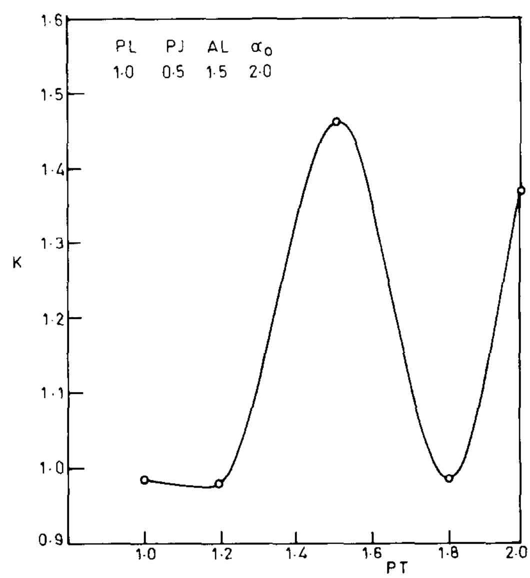

Fig. 2. Variation of K (oscillation along major axis).

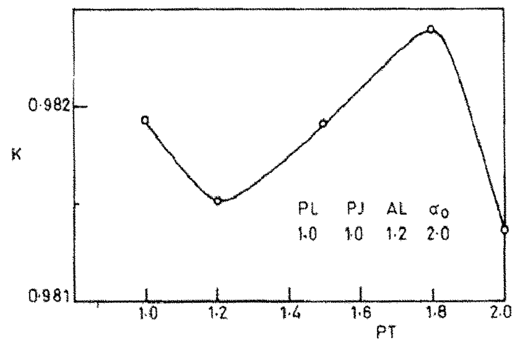

Fig. 3. Variation of K (oscillation along major axis).

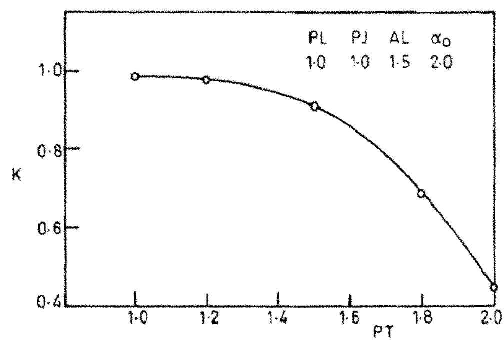

Fig. 4. Variation of K (oscillation along major axis).

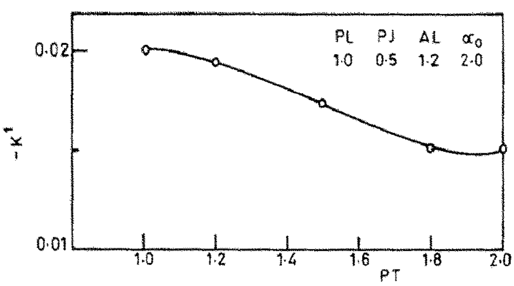

Fig. 5. Variation of K' (oscillation along major axis).

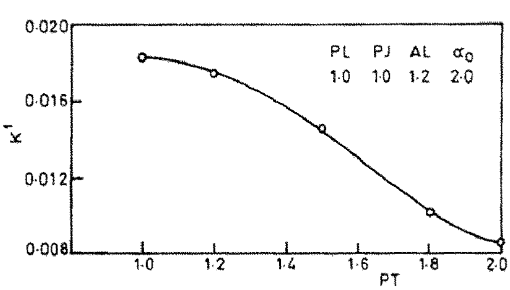

Fig. 6. Variation of K' (oscillation along major axis).

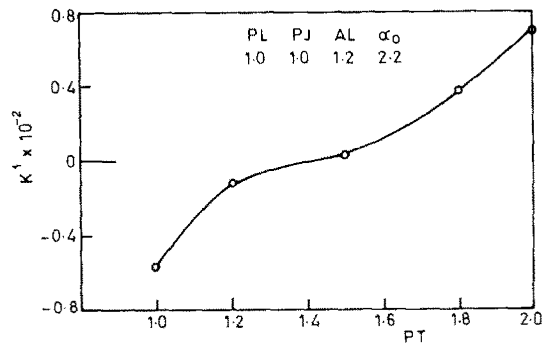

Fig. 7. Variation of K' (oscillation along major axis).

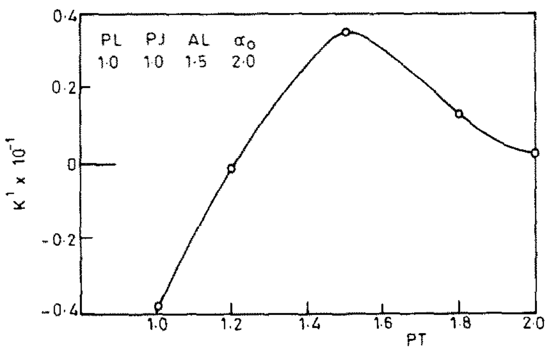

Fig. 8. Variation of K' (oscillation along major axis).

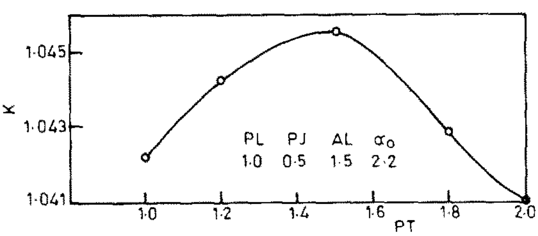

Fig. 9. Variation of K (oscillation along minor axis).

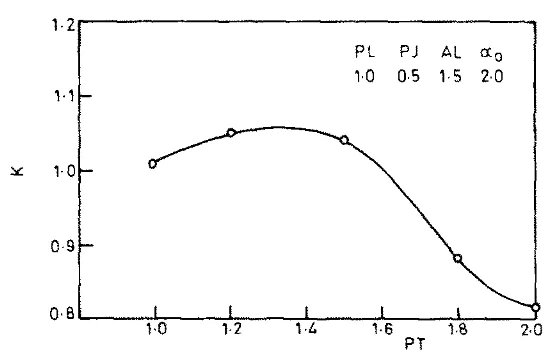

Fig. 10. Variation of K (oscillation along minor axis).

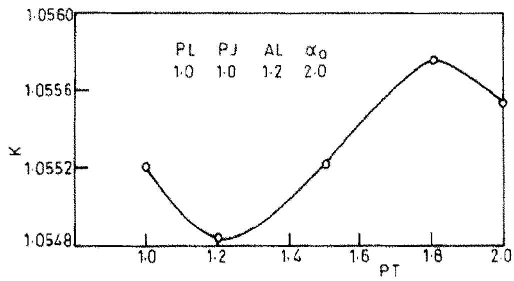

Fig. 11. Variation of K (oscillation along minor axis).

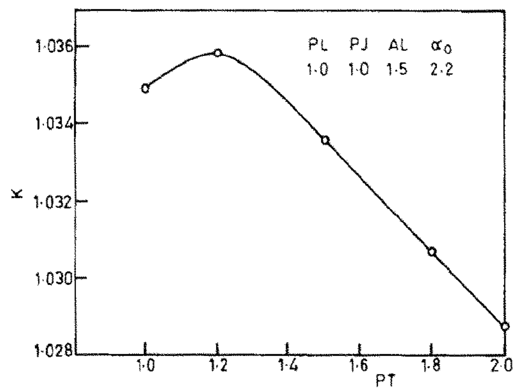

Fig. 12. Variation of K (oscillation along minor axis).

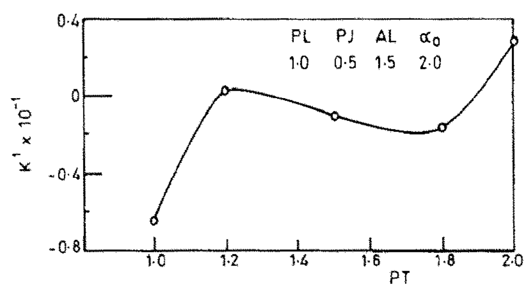

Fig. 13. Variation of K' (oscillation along minor axis).

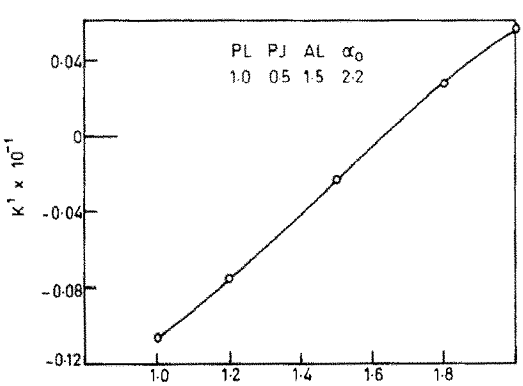

Fig. 14. Variation of K' (oscillation along minor axis).

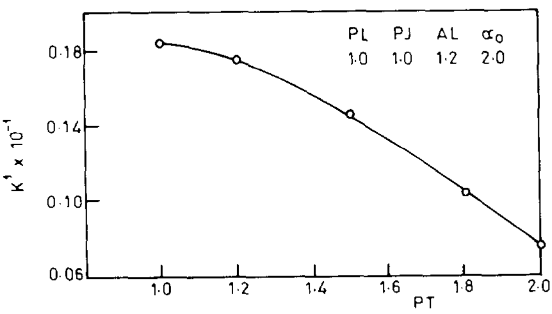

Fig. 15. Variation of K' (oscillation along minor axis).

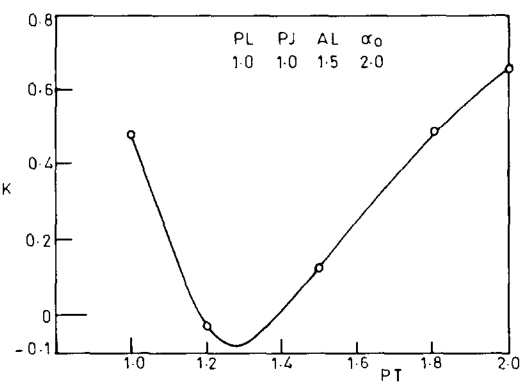

Fig. 16. Variation of K' (oscillation along minor axis).

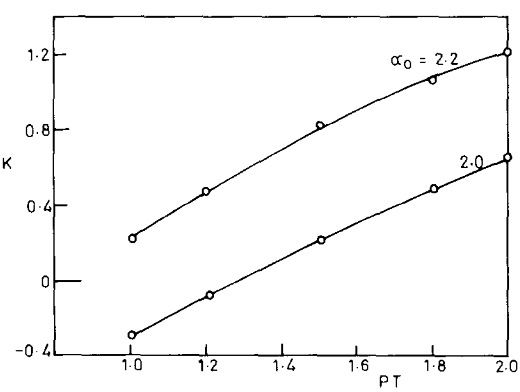

Fig. 17. Variation of K (oscillation along major axis; nonpolar case).

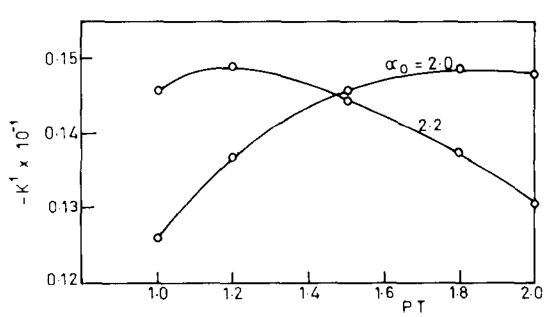

Fig. 18. Variation of K' (oscillation along major axis; nonpolar case).

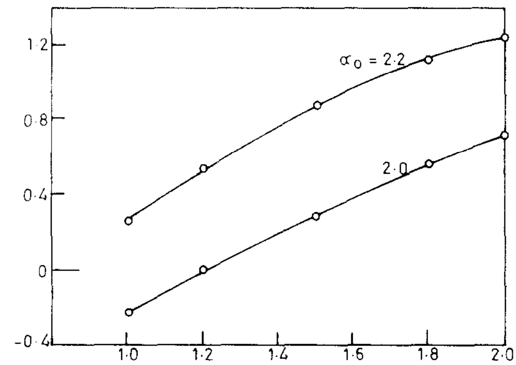

Fig. 19. Variation of K (oscillation along minor axis; nonpolar case).

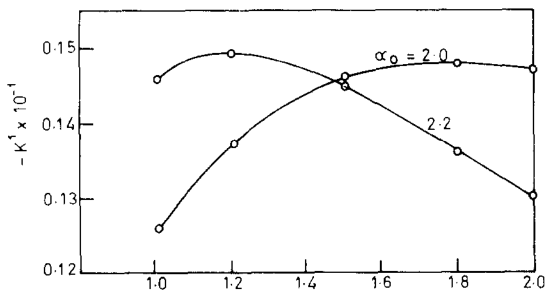

Fig. 20. Variation of K' (oscillation along minor axis; nonpolar case).

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