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Reliability Analysis of Machine Tool Structures

A method of estimating the reliability of machine tool structures is developed. The reliability analysis of horizontal milling machines in various failure modes, like static deflection, fundamental natural frequency and chatter stability, is considered for illustration. The table height, distance of the cutter center from the arbor support, damping factor, Young's modulus of the material and the load acting on the cutter and the table are considered as random variables. The finite-element displacement method is used to idealize the structure. The reliability analysis is based on the linearization of a function of several random variables about the mean values of the random variables. The overall reliability of the machine tool structure is found by treating it as a weakest-link system having several failure modes. A sensitivity analysis is also conducted to find the variaion of reliability with a change in the coefficients of variation of different random parameters.

Introduction

When the parameters affecting the strength of a structure and the loads acting on it are statistical in nature, the conventional analysis and design approaches based on the concept of "factor of safety" cannot be used to maintain a proper degree of safety. A more rational criterion, in the presence of random design parameters, will be to base the structural analysis and design on the concept of reliability or probability of failure. Reliability analyses in structural engineering recognize that both loads and strengths have statistical frequency distributions that must be considered in evaluating safety. Since the design parameters like cutting conditions, dimensions of the workpiece and the location of the tool are random in nature in machine tools, an analysis, based on the principles of reliability, becomes important.

Freudenthal [1][1] explained that the most rational way of describing the overall safety of structures is in terms of reliability or probability of failure. In reference [2], Moses and Kinser have demonstrated that an overall level of structural safety can be prescribed in terms of a rational criterion like probability of failure, and minimum-weight structures can be designed to meet the prescribed safety level. In 1970, Moses and Stevenson [3] considered the subject of sensitivity of statistical parameters and presented methods of incorporating reliability analysis into optimum design of trusses and frames. Since then a number of such applications have appeared in literature [4, 5, 6] which show that a probabilistic design is a practical possibility.

Recently, the probabilistic design concepts have also been applied in the design of mechanical systems. Mischke [7] presented a formal relationship between the reliability and the factor of safety of a mechanical element. By treating the factor of safety as a random variable, he used Bienayme-Chabyshev and Camp-Meidell theorems to derive expressions for the mean and the variance of the factor of safety for any specified value of reliability. In reference [8], Rao has developed a probability based design method for the design of mechanical power transmission systems like gear trains. By idealizing the transmission system as a weakest-link kinematic chain (similar to a weakest-link structure), the design has been made to achieve a specified reliability with respect to bending and surface-wear modes of failure.

In this paper, a method of analysing the reliability of machine tool structures is developed. More specifically, the reliability analysis of horizontal milling machines in various failure modes is considered for illustrating the method. The finite-element method, using triangular plate elements and frame elements, is used to idealize the machine tool structure. The reliability of the structure against the various response quantities or failure modes is found by taking the table height, h, the distance of the cutter center from arbor support, c, the damping factor, ζ, the Young's modulus of the material of the structure, E, and the load on the cutter and the table, P, as random variables. The response quantities considered are: 1 the maximum static deflection of the cutter center in any direction, d_c, 2 the first natural frequency of vibration, ω_1, and 3 the minimum negative inphase cross receptance of the cutter center relative to the table, G_{MIN}.

The location of the table and the cutter centre depend on the size of the workpiece. Since the size of the workpiece will vary for different jobs, h and c are taken as random variables. From the present knowledge, the modal damping factors of structures, particularly in machine tool structures where joints are involved, cannot be estimated precisely. Therefore modal damping factors, ζ_i, are taken as random variables. In actual practice, the material properties vary, hence the Young's modulus, E, is considered as a random variable in this work. Finally the magnitude of the load, P, is also taken as a probabilistic quantity since the cutting forces depend on various cutting conditions

[1] Numbers in brackets designate References at end of paper.

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such as feed, depth of cut, speed and the materials of the tool and the workpiece. All the random design parameters are assumed to follow normal distribution. This assumption of normal distribution is justified from the central limit theorem, and also it simplifies the computations involved. Finally the overall reliability of the machine tool structure is calculated from the reliabilities in various failure modes by considering the structure as a weakest-link system.

Probability of Failure of a Weakest-Link Chain

A weakest link is a series model in which the failure of any one link constitutes the failure of the whole chain. Since the failure of a machine tool structure in any one of the failure modes is considered as a failure of the whole system, the machine tool structure has to be idealized as a weakest-link chain. Fig. 1 shows the fundamental case which consists of a single member of strength R subject to a load L, along with the probability density functions of R and L. Here the strength R represents the allowable value of any response quantity like material strength, deflection, natural frequency or cross receptance, and the load L represents the induced value of the corresponding response quantity.

The probability of failure is given by

$$P_f = P(R < L) = \int_{-\infty}^{\infty} F_R(l) \cdot f_L(l) \cdot dl$$
$$= 1 - \int_{-\infty}^{\infty} F_L(r) \cdot f_R(r) \cdot dr \quad (1)$$

where $f_X(x)$ and $F_X(x)$ represent the probability density and distribution functions, respectively.

If several loads act simultaneously on the structural system as shown in Fig. 2(a), the failure probability is given by

$$P_f = 1 - \int_{-\infty}^{\infty} \left[\prod_{j=1}^{p} F_{L_j}(r) \right] f_R(r) \cdot dr \quad (2)$$

If a single member is subjected to several load conditions as shown in Fig. 2(b), the probability of failure can be determined from the relation

$$P_f = 1 - \int_{-\infty}^{\infty} \left\{ \prod_{i=1}^{q} [1 - F_{R_i}(l_i)] \right\} f_L(l) \cdot dl \quad (3)$$

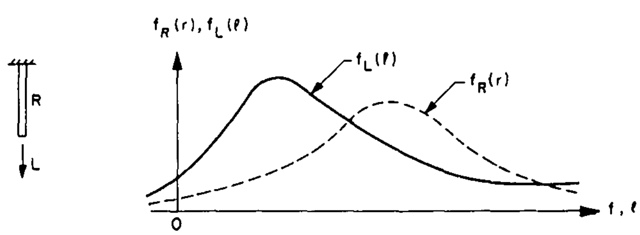
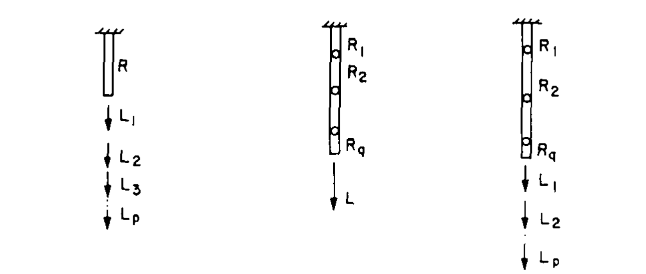


Fig. 1 A structural system consisting of one member and one load



(a) A System Consisting of One Member And p Loads.

(b) A System Consisting of q Members And One Load

(c) A System Consisting of q Members And p Loads.

Fig. 2 Weakest-link system

where l_i is the force induced in ith member due to the load l. Since this is a weakest-link chain, the term in braces in equation (3) represents the probability of survival of the chain and is based on all links surviving under the load $L = l$. This term is evaluated from the products of probabilities of individual links surviving under the load $L = l$. Equation (3) is often approximated as

$$P_f \approx 1 - \prod_{i=1}^{q} (1 - P_{f_i}) \quad (4)$$

where P_{f_i} denotes the probability of failure of ith link. Finally the probability of failure of a multicomponent, multiload system shown in Fig. 2(c) is given by

$$P_f = 1 - \int_{L_1} \int_{L_2} \cdots \int_{L_p} \left[\prod_{i=1}^{q} \{1 - F_{R_i}(l_{im})\} \right]$$
$$\times f_{L_1}(l_1) \cdot f_{L_2}(l_2) \cdots f_{L_p}(l_p) \cdot dl_1 \cdot dl_2 \cdots dl_p \quad (5)$$

where l_{im} is the total force induced in ith member due to the loads l_1, l_2, ···, l_p. Since the computation of the exact probability of failure is a complex probabilistic problem, equation (4) is often approximated as

$$P_f \approx 1 - \prod_{i=1}^{q} \prod_{j=1}^{p} (1 - P_{f_{ij}}) \quad (6)$$

where $P_{f_{ij}}$ denotes the probability of failure of ith member under jth load.

Computation of Reliability in a Particular Failure Mode

The reliability R_o of a system is taken as one minus the probability of failure P_f. If R is the resistance and L is the load acting in the specified failure mode, the reliability of the system can be analysed as a single-member/single-load problem. For simplicity the resistance and the load are assumed to be normally distributed so that

$$f_L(l) = \frac{1}{\sqrt{2\pi} \cdot \sigma_L} \exp \left[-\frac{1}{2} \left(\frac{l - \bar{L}}{\sigma_L} \right)^2 \right] \quad (7)$$

and

$$f_R(r) = \frac{1}{\sqrt{2\pi} \cdot \sigma_R} \exp \left[-\frac{1}{2} \left(\frac{r - \bar{R}}{\sigma_R} \right)^2 \right] \quad (8)$$

where $\bar{L}$ and $\bar{R}$ represent the mean values and σ_L and σ_R the standard deviations of L and R, respectively. Although equation (1) is applicable, the following simpler procedure is used to find the reliability of the system in this case. By defining a new random variable, ξ, as

$$\xi = R - L, \quad (9)$$

the reliability of the system can be expressed as

$$R_o = P(\xi \geq 0) = \int_0^{\infty} f_\xi(\eta) \cdot d\eta, \quad (10)$$

where $f_\xi(\eta)$ is the density function of ξ given by

$$f_\xi(\eta) = \frac{1}{\sqrt{2\pi} \cdot \sigma_\xi} \exp \left[-\frac{1}{2} \left(\frac{\eta - \bar{\xi}}{\sigma_\xi} \right)^2 \right], \quad (11)$$

and $\bar{\xi}$ is the mean value and σ_ξ is the standard deviation of ξ. If R and L are independent, the expressions for $\bar{\xi}$ and σ_ξ are given by

$$\bar{\xi} = \bar{R} - \bar{L} \quad (12)$$

and

$$\sigma_\xi = (\sigma_R{}^2 + \sigma_L{}^2)^{1/2}. \quad (13)$$

Equations (10) and (11) give

$$R_o = \frac{1}{\sqrt{2\pi} \cdot \sigma_\xi} \int_0^{\infty} \exp \left[-\frac{1}{2} \left(\frac{\eta - \bar{\xi}}{\sigma_\xi} \right)^2 \right] \cdot d\eta \quad (14)$$

By defining a standard normal variate, Z, as

$$Z = \frac{\eta - \bar{\xi}}{\sigma_\xi}, \quad (15)$$

Equation (14) can be rewritten as

$$R_o = \frac{1}{\sqrt{2\pi}} \int_{Z=z}^{\infty} \exp\left(-\frac{1}{2}Z^2\right) \cdot dZ, \tag{16}$$

where the lower limit of integration, z, is given by

$$z = -\frac{\bar{\xi}}{\sigma_\xi} = -\left[\frac{\overline{R} - \overline{L}}{(\sigma_R{}^2 + \sigma_L{}^2)^{1/2}}\right]. \tag{17}$$

Once the value of z is calculated, the corresponding reliability R_o can be determined from equation (16). This value can be obtained more readily from the standard normal tables [9].

In order to find the reliability of a structural system using equations (17) and (16), the mean values and standard deviations of the generalized load L and the generalized resistance R must be known. Since L and R generally depend on several other random design parameters, one has to determine $\overline{L}$, $\overline{R}$, σ_L, and σ_R in terms of the means and standard deviations of the random design parameters. In general, if Y is a nonlinear function of several random variables $x_1, x_2, \cdots, x_s$, the approximate values of the mean and the variance of Y can be found by linearizing Y about mean values of $x_1, x_2, \cdots, x_s$ using a Taylor's series expansion. The expressions of $\overline{Y}$ *and* σ_Y are given by

$$\overline{Y} \approx Y(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_s) \tag{18}$$

and

$$\sigma_Y \approx \left(\sum_{i=1}^{s} \left[\frac{\partial Y}{\partial x_i}\bigg|_{(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_s)}\right]^2 \sigma_{x_i}{}^2\right)^{1/2} \tag{19}$$

where the random variables $x_1, x_2, \cdots, x_s$ are assumed to have zero correlation.

Since the reliabilities against the response quantities d_c, ω_1, and G_{MIN} are to be found, the partial derivatives of d_c, ω_1, and G_{MIN} with respect to the random design parameters h, c, ζ, E, and P (evaluated at the mean values of the design variables) are required. These partial derivatives are found by using a finite-difference scheme in this work.

Computation of Response Quantities

In order to find the reliability of the machine tool structure in various failure modes, the expected values and standard deviations of the response quantities d_c, ω_1, and G_{MIN} have to be computed. Since the reliability analysis of milling machine structures is considered in this work, the horizontal milling machine structure is idealized by using triangular plate elements and frame elements. Since the structure is a three-dimensional one, both in-plane and bending effects are included in the analysis. The local displacement variations in the plate element are taken as

$$u(x, y) = a_1 + a_2x + a_3y \tag{20}$$

$$v(x, y) = a_4 + a_5x + a_6y \tag{21}$$

$$w(x, y) = a_7 + a_8x + a_9y + a_{10}x^2 + a_{11}xy + a_{12}y^2 + a_{13}x^3 + a_{14}(x^2y + xy^2) + a_{15}y^3 \tag{22}$$

where u, v, and w indicate the components of displacement along the local x, y, and z directions, and the xy-plane represents the plane of the plate. The displacements u, v, w, θ_x, and θ_y are taken as degrees of freedom at each of the nodes of the triangle. For the frame element, u, v, w, θ_x, θ_y, and θ_z are taken as degrees of freedom at each of the two nodes, where θ_i represents rotation about the axis $i (i = x, y, z)$. The triangular plate elements are used to idealize the column, overarm and table, and the frame elements are used to model the ribs on the overarm, the overarm joint with the column, the arbor and the arbor support.

The element stiffness and mass matrices of the two types of finite elements are given in reference [11]. The transformation of element matrices to global coordinate system and their assembly to derive the master matrices follow standard procedures of structural analysis [11]. Since the orientation of cutting forces in up milling are generally more unfavourable from the points of view of static rigidity of cutter center and chatter stability, only up milling is considered in the displacement analysis of the milling machine structure. The static forces acting on the milling machine are assumed to correspond to the following cutting parameters: diameter of milling cutter = 0.1 m, width of engagement = 0.09 m, number of teeth on milling cutter = 12, feed per tooth = 0.0001 m, angle of engagement = 30 deg, helix angle of milling cutter = 25 deg, material of the workpiece = mild steel. These parameters give the horizontal, vertical and axial forces on the milling cutter as 8237.88, 1647.58, and 1647.58 N, respectively [12].

The response quantity d_c is found by solving the equilibrium equations

$$[K]\mathbf{Y} = \mathbf{P} \tag{23}$$

where $[K]$ is the master stiffness matrix, $\mathbf{Y}$ is the displacement vector and $\mathbf{P}$ is the load vector. Equations (23) are solved by using the Cholesky decomposition of symmetric band matrices, storing only the upper triangular matrix, followed by forward and backward elimination technique. To find the response quantity, ω_1, the linear eigenvalue problem

$$[K]\mathbf{Y} = \omega^2[M]\mathbf{Y} \tag{24}$$

is solved where ω is the natural frequency of vibration and $[M]$ is the master mass matrix of the structure. In this work, equation (24) is solved to find the first few eigenvalues and eigenvectors by using the Rayleigh-Ritz subspace iteration algorithm developed by Bathe and Wilson for large structural systems [13]. The receptances of the cutter centre relative to the table of the horizontal milling machine have been obtained by using modal coordinates taking the damping matrix as a linear combination of the stiffness and mass matrices. The damping factors are assumed to have a value of 0.06 for the first few modes in the present work. From the dynamic analysis, the negative inphase cross receptance of the cutter centre relative to the table is taken as the response quantity G_{MIN}.

The expected values $\overline{d}_c$, $\overline{\omega}_1$, and $\overline{G}_{\mathrm{MIN}}$ are obtained by analyzing the structure at the expected values of the random parameters, while the standard deviations σ_{d_c}, σ_{ω_1}, and $\sigma_{G_{\mathrm{MIN}}}$ are found by computing the rates of change of the response quantities with respect to random parameters and using equation (19).

Numerical Results

To illustrate the procedure developed, the reliability analysis of the horizontal milling machine structure shown in Fig. 3 is considered in each of the failure modes. The finite-element idealization of the structure is shown in Fig. 4. The reliabilities of the machine tool structure are found at two design points[2]; one at $X_1 = 0.50$ m, $X_2 = 0.028$ m, $X_3 = 0.32$ m, $X_4 = 0.028$ m, $X_5 = 0.42$ m, and $X_6 = 0.03$ m, and the other at $X_1 = 0.50119$ m, $X_2 = 0.00987$ m, $X_3 = 0.28826$ m, $X_4 = 0.00906$ m, $X_5 = 0.41149$ m, and $X_6 = 0.00896$ m. The permissible mean values of the response quantities are taken as:

(*a*) deflection of the cutter center, $(\overline{d}_c)_{\max} = 0.00009$ m

(*b*) fundamental natural frequency, $(\omega_1)_{\min} = 850$ rad/sec and

(*c*) minimum negative cross receptance of cutter center relative to the table, $|\overline{G}_{\mathrm{MIN}}|_{\max} = 3.059 \times 10^{-9}$ m/N.

The two design points are selected such that $d_c = 0.0000661$ m, $\omega_1 = 1003$ rad/sec, and $|G_{\mathrm{MIN}}| = 0.6628 \times 10^{-9}$ m/N at the first point and $d_c = 0.0000893$ m, $\omega_1 = 913$ rad/sec, and $|G_{\mathrm{MIN}}| = 2.039 \times 10^{-9}$ m/N at the second point. Thus the response quantities in the case of the second design can be seen to be closer to their permissible values than in the case of the first design.

The values of the partial derivatives of the response quantities with respect to the random design parameters are given in Table 1. The reliability results obtained by using a coefficient of variation, V_{x_i}, of 0.05 for all the random variables are shown in Table 2. From these

[2] X_1 = depth of column at bottom, X_2 = thickness of the overarm, X_3 = width of the machine, X_4 = thickness of the column and table, X_5 = depth of column at the top, X_6 = square cross-sectional dimensions of the ribs on the overarm and its joint with the column.

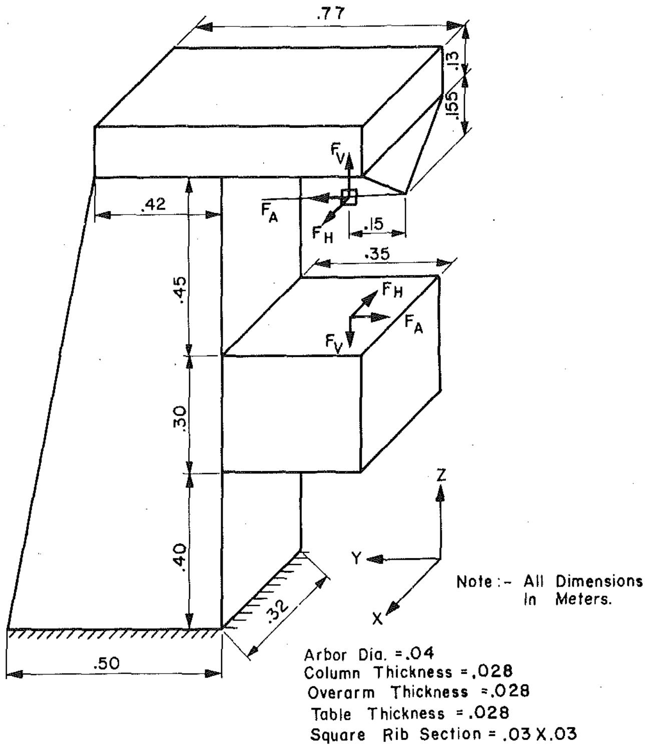


Fig. 3 Horizontal knee-type milling machine

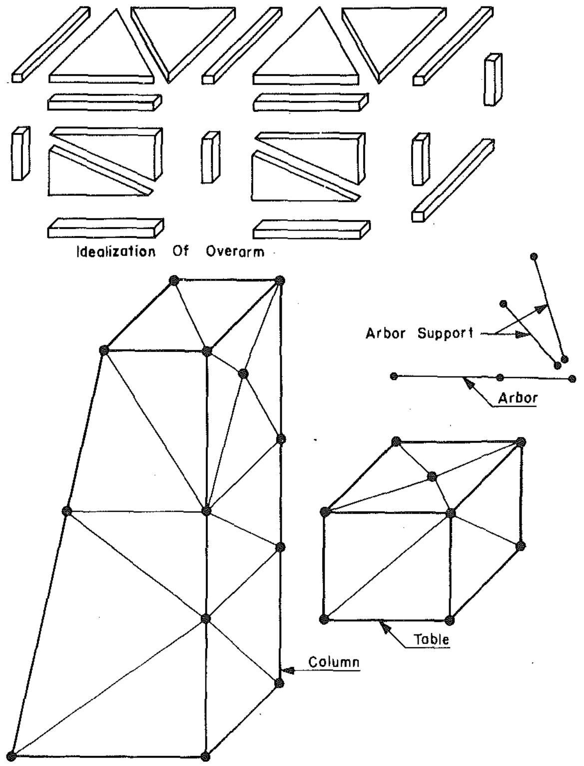


Fig. 4 Finite-element idealization of milling machine (shown in Fig. 3)

Table 1 Partial derivatives of d_c, ω_1 and G_{MIN} with respect to the random design parameters

Partial derivative	Random variable x_i					
	h		c		ξ	
	At design point 1[a]	At design point 2[b]	At point 1	At point 2	At point 1	At point 2
1	2	3	4	5	6	7
$\frac{\partial d_c}{\partial x_i}$	$+0.32 \times 10^{-3}$	$+0.76 \times 10^{-4}$	$+0.56 \times 10^{-4}$	-0.43×10^{-4}	0.0	0.0
$\frac{\partial \omega_1}{\partial x_i}$	−182.0	−254.0	0.0	0.0	0.0	0.0
$\frac{\partial G_{MIN}}{\partial x_i}$	$+2.957 \times 10^{-9}$	$+9.687 \times 10^{-9}$	$+2.754 \times 10^{-9}$	$+9.484 \times 10^{-9}$	-12.240×10^{-9}	-4.793×10^{-9}

Random variable x_i			
E		P	
At point 1	At point 2	At point 1	At point 2
8	9	10	11
-0.4487×10^{-15}	-0.5711×10^{-15}	$+0.7546 \times 10^{-8}$	$+0.9687 \times 10^{-8}$
$+0.3671 \times 10^{-8}$	$+0.3263 \times 10^{-10}$	0.0	0.0
-0.5928×10^{-20}	-2.392×10^{-20}	0.0	0.0

[a] Point 1: $X_1 = 0.50$ m, $X_2 = 0.028$ m, $X_3 = 0.32$ m, $X_4 = 0.028$ m, $X_5 = 0.42$ m, $X_6 = 0.03$ m
[b] Point 2: $X_1 = 0.50119$ m, $X_2 = 0.00987$ m, $X_3 = 0.28826$ m, $X_4 = 0.00006$ m, $X_5 = 0.41149$ m, $X_6 = 0.00896$ m

results, it can be seen that the reliabilities are lower at the second design point compared to those at the first point. This is to be expected in this case, since the response quantities are very near to their critical values (and hence the design is less safer) in the case of second design.

In many practical problems, one would be interested not only in the reliability of a structure for a particular set of data, but also in knowing how the reliability changes with a variation in the standard deviations of the various design parameters. Hence the variation of reliabilities of the milling machine structure against d_c, ω_1, and G_{MIN} for various coefficients of variation of the design parameters, V_{x_i}, are shown in Figs. 5–7, respectively. From these figures, it can be seen that the reliability in the deflection failure mode is most sensitive with respect to V_P and least with respect to V_ξ at both of the design points. The reliability against the fundamental frequency is most sensitive to a change in V_E and least to a change in V_c, V_ξ, and V_P. Similarly, the reliability against the response quantity, G_{MIN}, can be observed

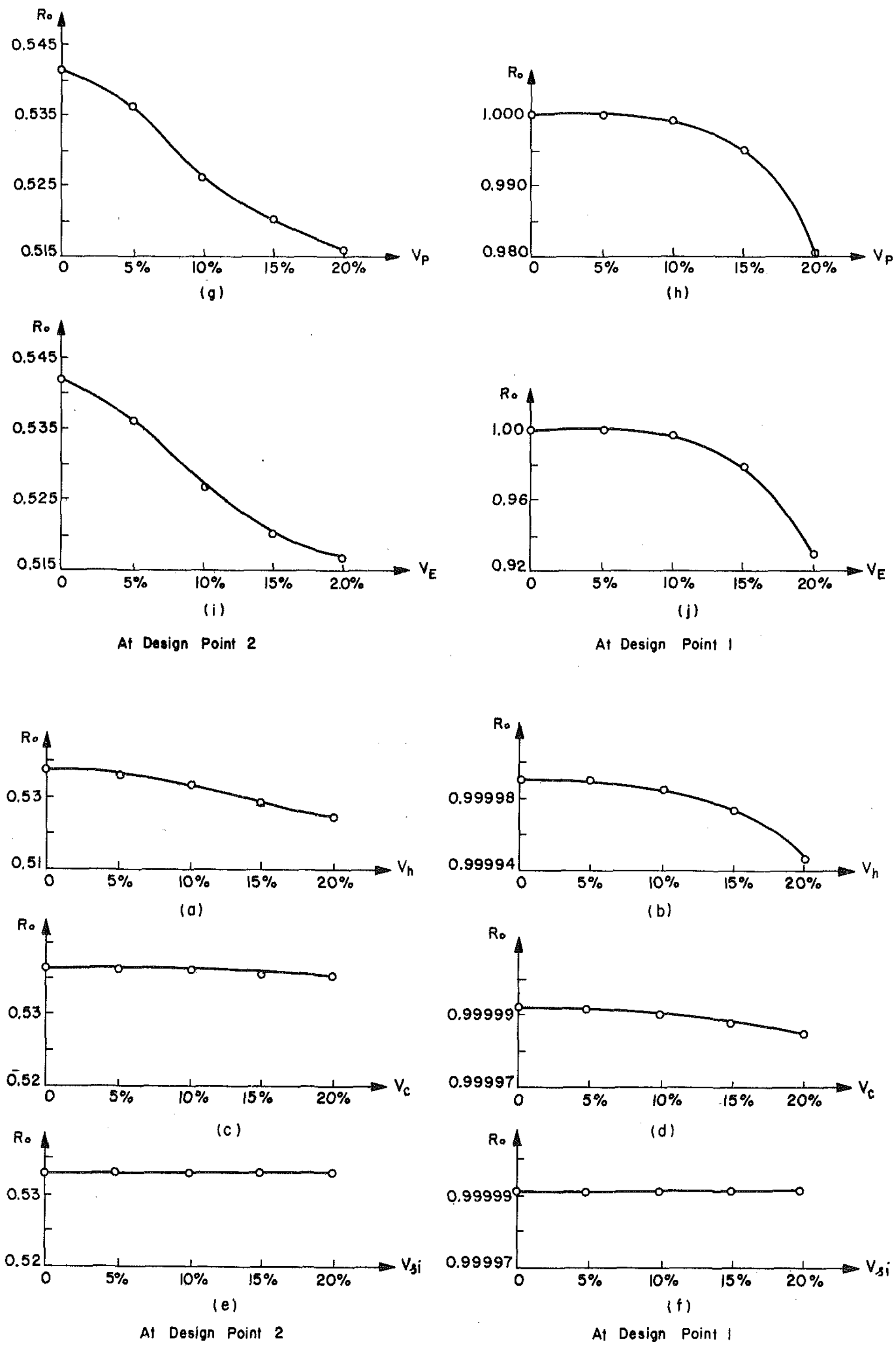


Fig. 5 Reliability versus deflections

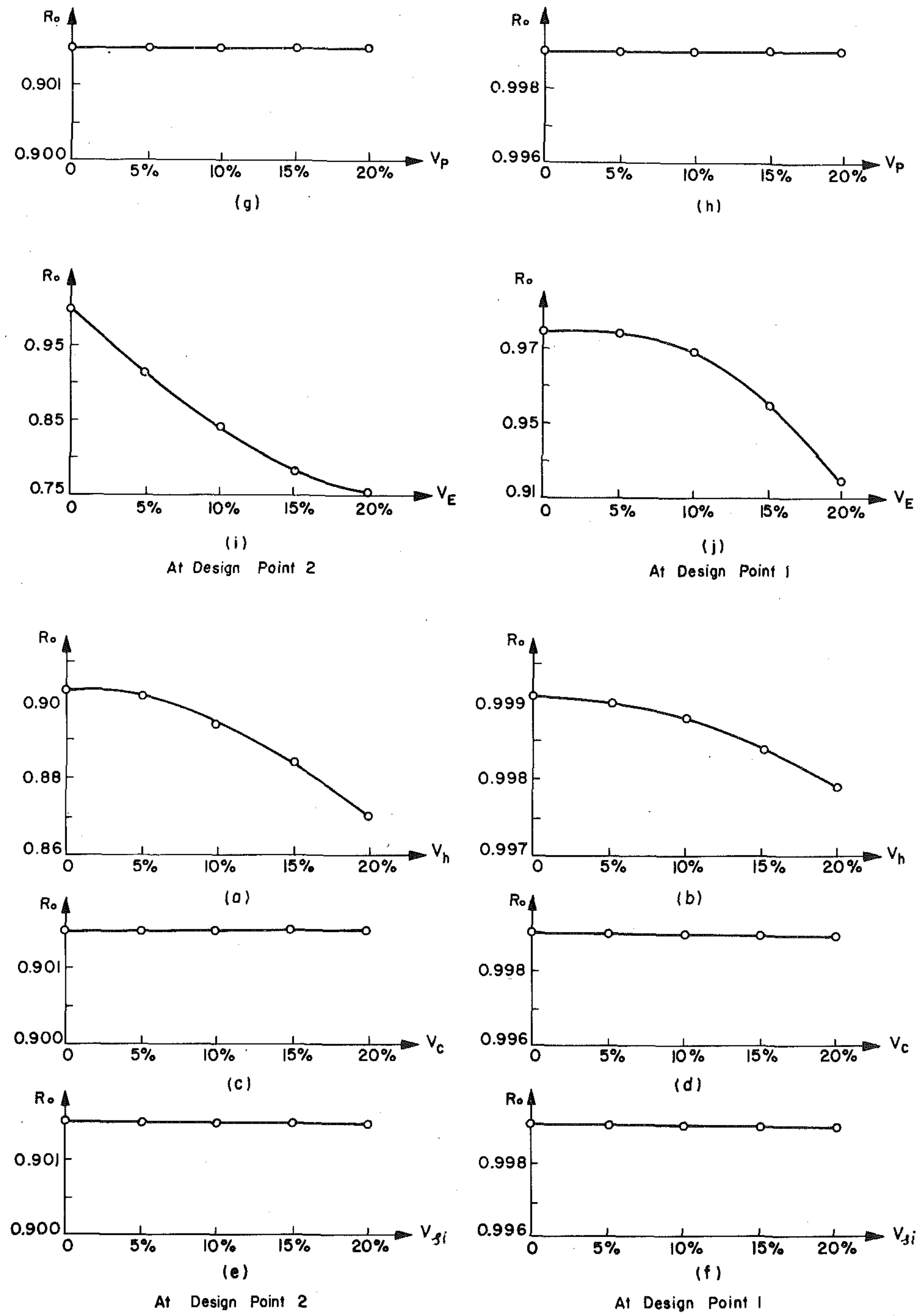


Fig. 6 Reliability versus first natural frequency

to be most sensitive to V_h and least to V_P.

If the machine tool structure is considered as a weakest-link chain, each link representing one particular failure mode, the overall reliability of the system can be calculated from equation (4). This gives the overall system reliability as $(R_o)_{\text{overall}} = 0.9989564688$ for the first design and $(R_o)_{\text{overall}} = 0.4816022724$ for the second design.

Conclusions

A method of analyzing the reliability of complex machine tool structures, using finite-element idealization, is developed. The reliability analysis procedure is more realistic, since, in practice, most of the parameters influencing the machine tool performance, like table height, cutting forces and structural properties, are really random in nature. Although normal distribution has been assumed in this work, the basic approach will not change even if the variables follow some other type of distribution. The reliability analysis procedure developed in this work can be coupled with formal optimization methods and the problem of design of machine tool structures can be stated as follows [10]:

Find **X** which minimizes $F(\mathbf{X})$ subject to the constraint

$$R_o(\mathbf{X}) \geq k$$

where **X** = vector of design variables

F = objective function to be minimized (like weight, static or dynamic rigidity)

R_o = reliability

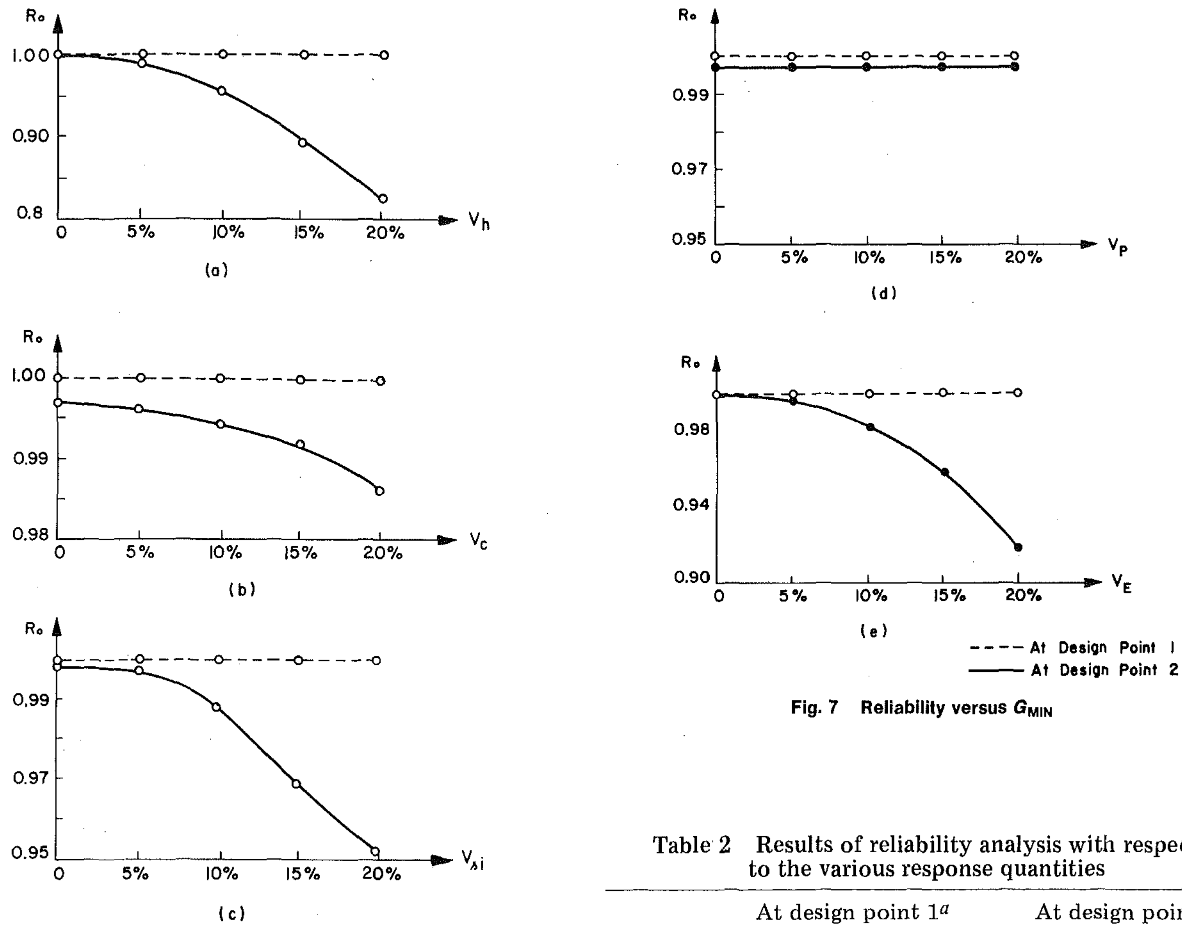


Fig. 7 Reliability versus G_{MIN}

k = constant denoting the minimum required reliability. The sensitivity analysis of the reliability of the machine tool structure is expected to give the machine tool structural designer an insight into the reliability behavior of the structure so that he can alter the design to satisfy any specified reliability condition.

Table 2 Results of reliability analysis with respect to the various response quantities

Failure mode (response quantity)	At design point 1[a] Lower limit of integration (z)	At design point 1[a] Reliability (R_o)	At design point 2[b] Lower limit of integration (z)	At design point 2[b] Reliability (R_o)
d_c	4.30	0.999991467	0.0902	0.535936
ω_1	3.08	0.998964996	1.29	0.901474
G_{MIN}	13.12	0.999999999	2.73	0.996833

[a] Design point 1: X_1 = 0.50 m, X_2 = 0.028 m, X_3 = 0.32 m, X_4 = 0.028 m, X_5 = 0.42 m, X_6 = 0.03 m

[b] Design point 2: X_1 = 0.50119 m, X_2 = 0.00987 m, X_3 = 0.28826 m, X_4 = 0.00906 m, X_5 = 0.41149 m, X_6 = 0.00896 m

Permissible values: $(\bar{d}_c)_{max}$ = 0.00009 m, $(\bar{\omega}_1)_{min}$ = 850 rad/sec, $|\bar{G}_{MIN}|_{max}$ = 3.059 × 10^{-9} m/N

NOTE: Coefficient of variation of all the design variables = 0.05

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