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# DRAG ON AN AXISYMMETRIC BODY PERFORMING RECTILINEAR OSCILLATIONS IN A MICROPOLAR FLUID

D. SRINIVASA CHARYA and T. K. V. IYENGAR*

Department of Mathematics and Humanities Regional Engineering College, Warangal-506 004, India

**Abstract**—In this paper, a general formula for the drag experienced by an axisymmetric body oscillating rectilinearly along its axis of symmetry in an incompressible micropolar fluid which is otherwise at rest is derived. The oscillatory flow of an incompressible micropolar fluid arising from the harmonic oscillations of an approximate sphere along its axis of symmetry is also considered. Assuming that the oscillation amplitude is small, the velocity components are obtained in terms of Bessel functions and Gegenbauer's functions. The drag experienced by the body is evaluated by using the formula derived and its variation is studied with respect to micropolarity parameter, frequency parameter and geometric parameter. The oscillatory flow generated by the sphere and spheroid are obtained as special cases. © 1997 Elsevier Science Ltd.

## 1. INTRODUCTION

Payne and Pell discussed the Stokes flow of a viscous liquid past a class of axially symmetric bodies with uniform streaming at infinity parallel to the axis of symmetry, and obtained a general formula for the drag experienced by the body in terms of the stream function [1]. Using the formula in Ref. [1], they obtained expressions for the drag experienced by sphere, prolate spheroid, oblate spheroid, lense shaped body, hemisphere, spherical cap and a pair of separated spheres just by determining the stream function of the flow region without calculating the stress components on the surface of the body. Ramkissoon and Majumdar generalized the above formula of Payne and Pell to the case of incompressible micropolar fluid in their classic paper [2].

In 1987, Lawrence and Weinbaum have derived a general formula for the drag on an axisymmetric body (analogous to that of Payne and Pell) performing rectilinear oscillation along its axis of the symmetry in an incompressible viscous liquid [3]. In the present paper, the authors obtain one such general formula in the case of an incompressible micropolar fluid. The theory of micropolar fluids was introduced by Eringen [4]. The formula is verified in the case of sphere, prolate and oblate spheroids and elliptic cylinder. To illustrate further the special case of an approximate sphere is considered in detail. The stream function of the flow generated by the rectilinear oscillations of an approximate sphere is determined. Using this expression for the stream function and formula developed, the drag on the approximate sphere is evaluated. Two drag parameters $K$ and $K'$ are introduced and their variation is studied through the graphs with respect to micropolarity parameters, frequency parameter and geometric parameter.

This paper is divided in to two sections: Section 2 is devoted to derivation of the general formula and Section 3 to the special case of approximate sphere.

## 2. DRAG ON AN AXIALLY SYMMETRIC BODY

Consider a simply connected axisymmetric smooth body $B_1$ oscillating rectilinearly with the speed of oscillation $Ue^{i\sigma t}$ along the axis of symmetry in an incompressible micropolar fluid.

* Author to whom all correspondence should be addressed.

Assuming the oscillaion amplitude $U$ to be sufficiently small, under the Stokesian assumption, the linearized version of the fluid flow equations is

$$\text{div}\,\mathbf{q} = 0 \tag{2.1}$$

$$\rho \frac{\partial \mathbf{q}}{\partial t} = \nabla \cdot \Pi \tag{2.2}$$

$$\rho j \frac{\partial \boldsymbol{\nu}}{\partial t} = -2k\boldsymbol{\nu} + k\nabla \times \mathbf{q} + \nabla \cdot \mathbf{m} \tag{2.3}$$

where $\Pi$ is the stress vector and $\mathbf{m}$ is the couple stress vector at any point of the fluid.Let $(n, s, \phi)$ be intrisinic coordinate system with scale factors $h_1 = 1$, $h_2 = 1$ and $h_3 = 1/\omega$ and $\mathbf{n}$, $\mathbf{s}$ and $\mathbf{i}_\phi$ be corresponding unit base vectors. The flow generated is axially symmetric and all the flow functions are independent of $\phi$. We can choose the velocity and microrotation vectors as

$$\mathbf{q} = [\mathbf{n}u_n + \mathbf{s}\,u_s]e^{i\sigma t} \tag{2.4}$$

$$\boldsymbol{\nu} = \boldsymbol{\nu}_\phi \mathbf{i}_\phi e^{i\sigma t}. \tag{2.5}$$

We write the velocity components in terms of the stream function as

$$u_n = \frac{-1}{\bar{\omega}}\frac{\partial \Psi}{\partial s}\,; u_s = \frac{1}{\bar{\omega}}\frac{\partial \Psi}{\partial n} \tag{2.6}$$

where

$$\Psi = \psi e^{i\sigma t}. \tag{2.7}$$

Let

$$P = p e^{i\sigma t}. \tag{2.8}$$

Substituting these values in equations (2.2) and (2.3), we get

$$-\frac{i\sigma\rho}{\bar{\omega}}\frac{\partial \psi}{\partial s} = -\frac{\partial p}{\partial n} + \frac{k}{\bar{\omega}}\frac{\partial}{\partial s}[\bar{\omega}\nu_\phi] - \frac{\mu + k}{\bar{\omega}}\frac{\partial}{\partial s}[E^2\psi] \tag{2.9}$$

$$-\frac{i\sigma\rho}{\bar{\omega}}\frac{\partial \psi}{\partial n} = -\frac{\partial p}{\partial s} - \frac{k}{\bar{\omega}}\frac{\partial}{\partial s}[\bar{\omega}\nu_\phi] - \frac{\mu + k}{\bar{\omega}}\frac{\partial}{\partial n}[E^2\psi] \tag{2.10}$$

$$i\rho j\sigma\nu_\phi = -2k\nu_\phi + kE^2\psi + \gamma E^2(\bar{\omega}\nu_\phi). \tag{2.11}$$

From equations (2.9) and (2.10), by eliminating pressure term, we get

$$[(\mu + k)E^4 - i\rho\sigma E^2]\psi - kE^2(\bar{\omega}\nu_\phi) = 0 \tag{2.12}$$

where $E^2$ is the Stokes stream function operator.

Equation (2.11) can also be written in the form

$$(2k + i\rho j\sigma)\bar{\omega}\nu_\phi = \gamma E^2(\bar{\omega}\nu_\phi) + kE^2\psi. \tag{2.13}$$

From equations (2.12) and (2.13), by eliminating the function $\nu_\phi$, we can obtain the following differential equation for the stream function $\psi$

$$[\gamma(\mu + k)E^6 - \{k(2\mu + k) + i\rho\sigma(\gamma + j\mu + jk)\}E^4 + i\rho\sigma(2k + i\rho j\sigma)E^2]\psi = 0. \tag{2.14}$$

The function $\nu_\phi$ is expressible in terms of $\psi$ in the form

$$k(2k + i\rho j\sigma)(\omega\nu_\phi) = [\gamma(\mu + k)E^4 + (k^2 - i\rho\sigma\gamma)E^2]\psi. \tag{2.15}$$

The equation (2.14) can also be written in the form

$$E^2(E^2-\alpha^2)(E^2-\beta^2)\psi=0 \tag{2.16}$$

where $\alpha^2$, $\beta^2$ are the such that

$$\alpha^2+\beta^2=\frac{k(2\mu+k)+i\rho\sigma(\gamma+j\mu+jk)}{\gamma(\mu+k)} \tag{2.17}$$

$$\alpha^2\beta^2=\frac{i\rho\sigma(2k+i\rho j\sigma)}{\gamma(\mu+k)}. \tag{2.18}$$

The problem thus reduces to the determination of the two scalar functions $\psi(r,\theta)$ and $\nu_\phi(r,\theta)$ which are governed by the equations (2.15) and (2.16) subject to the following conditions:

(1) Far away from the oscillating body there is practically no flow and the functions $\psi$, $\nu_\phi$ tend to zero.

(2) At the boundary of the oscillating body we have the hyperstick or superadherence condition and the velocity of a fluid element on the body equals that of the oscillating body, while the microrotation of the fluid element is zero.

Using equation (2.6), the equations of motion can be written as

$$\nabla\cdot\Pi=i\rho\sigma\nabla\times\left[i_\phi\frac{1}{\bar\omega}\psi\right]. \tag{2.19}$$

We shall integrate the above equation over the domain $D$ bounded by $B_1$ and a large concentric sphere $B_2$ of radius $R$. We convert the volume integrals to surface integrals using the Gauss divergence theorem and we retain only the $z$-component of the vector equation:

$$\int_{B_1}\!\!\int i_z\cdot\Pi\cdot\mathbf{n}\,\mathrm{d}S+\int_{B_2}\!\!\int i_z\cdot\Pi\cdot\mathbf{n}\,\mathrm{d}S+i\rho\sigma\int_{B_1}\!\!\int i_z\cdot\mathbf{n}\times\left[i_\phi\frac{\psi}{\bar\omega}\right]\mathrm{d}S$$

$$+i\rho\sigma\int_{B_2}\!\!\int i_z\cdot\mathbf{n}\times\left[i_\phi\frac{\psi}{\bar\omega}\right]\mathrm{d}S=0 \tag{2.20}$$

where $i_z$ is the unit vector along $z$-axis and $\mathbf{n}$ is the local outward normal to the domain $D$. The first integral of equation (2.20) is simply force exerted by the body on the fluid, $-F$. The second integral of equation (2.20) can be evaluated individually as below.

Following Happel and Brenner [5], the second integral can be put in the form:

$$\iint i_z\cdot\Pi\cdot\mathbf{n}\,\mathrm{d}S=\pi(2\mu-k)\int\bar\omega\frac{\partial}{\partial n}(E^2\psi)\,\mathrm{d}S$$

$$+\frac{3k\pi}{2}\int\bar\omega^3\frac{\partial}{\partial n}\left[\frac{1}{\bar\omega^2}E^2\psi\right]\mathrm{d}S-\pi k\int\bar\omega^3\frac{\partial}{\partial n}\left[\frac{1}{\bar\omega}\nu_\phi\right]\mathrm{d}S-i\pi\rho\sigma\int\bar\omega\frac{\partial\psi}{\partial n}\,\mathrm{d}S. \tag{2.21}$$

The solution of equation (2.16), subject to $\mathbf{q}$, $\Pi\to0$ as $r\to\infty$ in terms of spherical harmonics is

$$\psi=\psi_1^P+\psi_2^D+\psi_3^D \tag{2.22}$$

with

$$\psi_1^P=\sum A_n r^{-n+1}\vartheta_n(\zeta) \tag{2.23}$$

$$\psi_2^D=\sum B_n R_n^{(1)}(r)\vartheta_n(\zeta) \tag{2.24}$$

$$\psi_3^D = \sum C_n R_n^{(2)}(r)\vartheta_n(\zeta) \tag{2.25}$$

where

$$R_n^{(1)}(r) = r^n\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{n-1}\frac{1}{r}\mathrm{e}^{-\alpha r} \tag{2.26}$$

$$R_n^{(2)}(r) = r^n\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{n-1}\frac{1}{r}\mathrm{e}^{-\beta r} \tag{2.27}$$

If there are no sources, then $A_0$ is zero. $R_n^{(1)}(r)$ and $R_n^{(2)}(r)$ are exponentially small at large $r$, and hence we neglect them in equation (2.22) if $r$ is large enough. Then equation (2.21) reduces to:

$$\iint i_z\cdot\Pi\cdot\mathbf{n}\mathrm{d}S = -\,i\pi\rho\sigma\int\bar{\omega}\frac{\partial\psi^P}{\partial r}\,\mathrm{d}S. \tag{2.28}$$

Using the property of $\vartheta_n(\zeta)$:

$$\int_{-1}^{1}\vartheta_n(\zeta)\,\mathrm{d}\zeta = 2\delta_{n0} + (2/3)\delta_{n2} \tag{2.29}$$

where $\delta_{mn}$ is Kronecker delta, the above equation can be written as

$$\iint i_z.\Pi.\mathbf{n}\mathrm{d}S = \frac{2}{3}\pi i\rho\sigma A_2 \tag{2.30}$$

which is independent of $R$. The third integral in equation (2.20) using the condition $\mathbf{q}\to U_0 i_z \mathrm{e}^{\mathrm{i}\sigma t}$ takes the value $i\rho\sigma UV$, where $V$ is the volume of the body. The fourth integral in equation (2.20) is evaluated using the procedure that is similar to that just outlined for the second integral, and its value is determined to be $(4/3)\pi i\rho\sigma\, A_2$. $A_2$ depends on $i\rho\sigma$ and the geometry of the body and that term seems to have no simple interpretation. Using equations (2.22)–(2.25), we may express $A_2$ in the form

$$A_2 = 2\lim_{r\to\infty}\frac{r^3\psi}{\bar{\omega}^2}. \tag{2.31}$$

Substituting the above results in equation (2.20), we obtain the force on the body as

$$F = i\rho\sigma UV + 4\pi i\rho\sigma\lim_{r\to\infty}\frac{r^3\psi}{\bar{\omega}^2}. \tag{2.32}$$

This is analogous to the result of Lawrence and Weinbaum [3].

### *2.1 Special cases*

#### *2.1.1 Sphere*

The stream function in the case of a sphere oscillating rectilinearly in an incompressible micropolar fluid is [6]

$$\psi = \left\{C_1\left(\frac{1}{r}+\alpha_1\right)\mathrm{e}^{-\alpha r} + C_2\left(\frac{1}{r}+\beta_1\right)\mathrm{e}^{-\beta r} + \frac{iB_1}{\rho\sigma}\frac{1}{r}\right\}\mathrm{e}^{\mathrm{i}\sigma t}\sin^2\theta \tag{2.33}$$

and using equation (2.32), the drag experienced by the sphere is

$$(4/3)\pi a^2\left(i\rho\sigma a U - \frac{3B_1}{a^2}\right) \tag{2.34}$$

which is the same as in Ref. [6].

*2.1.2 Prolate spheroid*

The stream function in the case of a prolate spheroid oscillating rectilinearly in an incompressible micropolar fluid is [7]

$$\psi = c\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty}[A_n Q_n^{(1)}(s)P_n^{(1)}(t) + B_n R_{1n}^{(3)}(i\alpha c,s)S_{1n}^{(1)}(i\alpha c,t)$$

$$+ C_n R_{1n}^{(3)}(i\beta c,s)S_{1n}^{(1)}(i\beta c,t)] \quad (2.35)$$

and using equation (2.32), the drag experienced by the prolate spheroid is

$$(4/3)i\rho\sigma s_0\sqrt{s_0^2-1}[(1/2)Uc + Q_1^{(1)}(s_o)A_1] \quad (2.36)$$

which is the same as in Ref. [7].

*2.1.3 Oblate spheroid*

The stream function in the case of an oblate spheroid oscillating rectilinearly in an incompressible micropolar fluid is [7]

$$\psi = c\sqrt{(\tau^2+1)(1-t^2)}\sum_{n=1}^{\infty}[A_n Q_n^{(1)}(i\tau)P_n^{(1)}(t) + B_n R_{1n}^{(3)}(\alpha c,i\tau)S_{1n}^{(1)}(\alpha c,t)$$

$$+ C_n R_{1n}^{(3)}(\beta c,i\tau)S_{1n}^{(1)}(\beta c,t)] \quad (2.37)$$

and using equation (2.32), the drag experienced by the oblate spheroid is

$$(4/3)i\rho\sigma\tau_0\sqrt{\tau_0^2+1}[(1/2)Uc + Q_1^{(1)}(i\tau_o)A_1] \quad (2.38)$$

which is same as in Ref. [7].

*2.1.4 Elliptic cylinder*

The stream function in the case of an elliptic cylinder oscillating rectilinearly in an incompressible micropolar fluid [8]

(a) Parallel to major axis is

$$\psi = \sum_{n=1}^{\infty}[C_n e^{-n\alpha} + \sum_{m=1}^{\infty} D_m F_{mn}(\alpha) + \sum_{m=1}^{\infty} E_m G_{mn}(\alpha)]\sin n\beta \quad (2.39)$$

and using equation (2.32), the drag is

$$i\pi\rho\sigma c L^2[Uc\sinh\alpha_0\cosh\alpha_0 + C_1] \quad (2.40)$$

which is the same as in Ref. [8].

(b) Parallel to minor axis is

$$\psi = \sum_{n=1}^{\infty}[C_n^* e^{-n\alpha} + \sum_{m=1}^{\infty} D_m^* F_{mn}^*(\alpha) + \sum_{m=1}^{\infty} E_m^* G_{mn}^*(\alpha)]\sin n\beta \quad (2.41)$$

and using the formula (2.32), the drag is

$$i\pi\rho\sigma c L^2[Uc\sinh\alpha_0\cosh\alpha_0 + C_1^*] \quad (2.42)$$

which is the same as in Ref. [8].

## 3. THE RECTILINEAR OSCILLATIONS OF AN APPROXIMATE SPHERE IN AN INCOMPRESSIBLE MICROPOLAR FLUID

In this section we study the oscillatory flow of an incompressible micropolar fluid arising from the rectilinear oscillations of an approximate sphere along its axis of symmetry $\theta = 0$. We find the drag on the approximate sphere making use of the formula developed in Section 2.

### *3.1 Statement of the problem*

Let $(r, \theta, \phi)$ denote a spherical polar coordinate system with $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ as the corresponding unit base vectors and $h_1 = 1$, $h_2 = r$ and $h_3 = r \sin\theta$ as the scale factors.

Consider an approximate sphere in an infinite expanse of an incompressible micropolar fluid oscillating rectilinearly along its axis of symmetry and with the speed of oscillation $Ue^{i\omega t}$. We assume that the oscillation amplitude $U$ is small and omit the second-order terms in the equations of motion. Ignoring the body force and body couple, the basic equations of the flow can be written in the form

$$\operatorname{div}\mathbf{q} = 0 \tag{3.1}$$

$$\rho\frac{\partial \mathbf{q}}{\partial t} = -\nabla p - k\,\operatorname{curl}\boldsymbol{\nu} - (\mu + k)\operatorname{curl}\operatorname{curl}\mathbf{q} \tag{3.2}$$

$$\rho j\frac{\partial \boldsymbol{\nu}}{\partial t} = 2k\boldsymbol{\nu} + k\,\operatorname{curl}\mathbf{q} - \gamma\operatorname{curl}\operatorname{curl}\boldsymbol{\nu} + (\alpha + \beta + \gamma)\operatorname{grad}(\operatorname{div}\boldsymbol{\nu}). \tag{3.3}$$

Since the flow generated by the oscillation is axially symmetric all the flow field functions are independent of the coordinate variable $\phi$. We may choose the velocity and microrotation of the flow in the form

$$\mathbf{q} = [u(r,\theta)\mathbf{e}_r + v(r,\theta)\mathbf{e}_\theta]e^{i\omega t} \tag{3.4}$$

and

$$\boldsymbol{\nu} = [C(r,\theta)\mathbf{e}_\phi]e^{i\omega t}. \tag{3.5}$$

We introduce the Stokes stream function $\psi(r, \theta)$ by means of the equations

$$u = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta};\; v = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{3.6}$$

and write the pressure in the form

$$p(r,\theta) = P(r,\theta)e^{i\omega t}. \tag{3.7}$$

As seen in Section 2, the problem reduces to the determination of the two scalar functions $\psi(r, \theta)$ and $C(r, \theta)$ which are governed by equations (2.16) and (2.15) subject to the following boundary conditions:

$$u = U\cos\theta,\; v = -U\sin\theta \text{ and } C = 0 \tag{3.8}$$

on the boundary.

Equivalently, these can be written in the form

$$\psi_r = Ur^2\sin^2\theta,\; \psi_\theta = -Ur^2\sin\theta\cos\theta \text{ and } C = 0 \tag{3.9}$$

on the boundary.

### *3.2 Solution of the problem*

From equation (2.16), we can obtain the solution $\psi$ by superposing the solutions of

$$E^2\psi = 0 \tag{3.10}$$

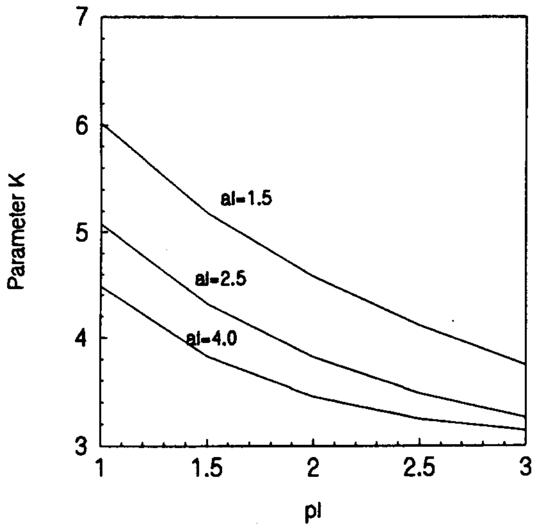


Fig. 1. Parameter $K$ vs $pl$ for different values of $al$ ($pt=0.4$, $pj=0.6$, $\epsilon=0.0$).

$$(E^2-\alpha^2)\psi=0 \tag{3.11}$$

$$(E^2-\beta^2)\psi=0 \tag{3.12}$$

in the form $\psi=\psi_1+\psi_2+\psi_3$.

Following the technique of separation of variables, it can be seen that the solutions of equations (3.10)–(3.12) are, respectively,

$$\psi_1=\sum_{n=2}^{\infty}[A_nr^n+B_nr^{-n+1}][A_n^*\vartheta_n(\zeta)+B_n^*H_n(\zeta)] \tag{3.13}$$

$$\psi_2=\sum_{n=2}^{\infty}\sqrt{r}[C_nK_{n-1/2}(\alpha r)+D_nI_{n-1/2}(\alpha r)][C_n^*\vartheta_n(\zeta)+D_n^*H_n(\zeta)] \tag{3.14}$$

$$\psi_3=\sum_{n=2}^{\infty}\sqrt{r}[E_nK_{n-1/2}(\beta r)+F_nI_{n-1/2}(\beta r)][E_n^*\vartheta_n(\zeta)+F_n^*H_n(\zeta)] \tag{3.15}$$

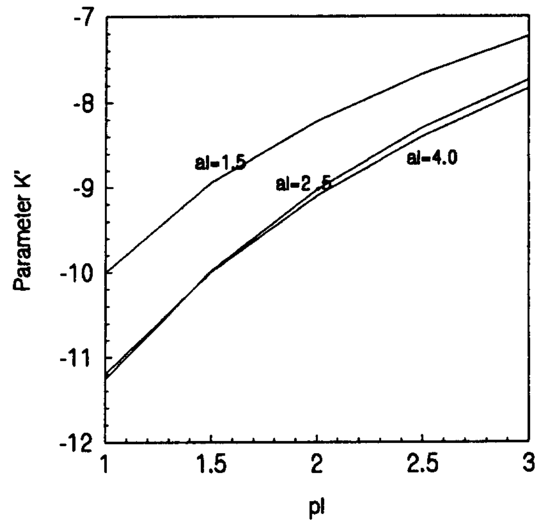


Fig. 2. Parameter $K'$ vs $pl$ for different values of $al$ ($pt=0.4$, $pj=0.6$, $\epsilon=0.0$).

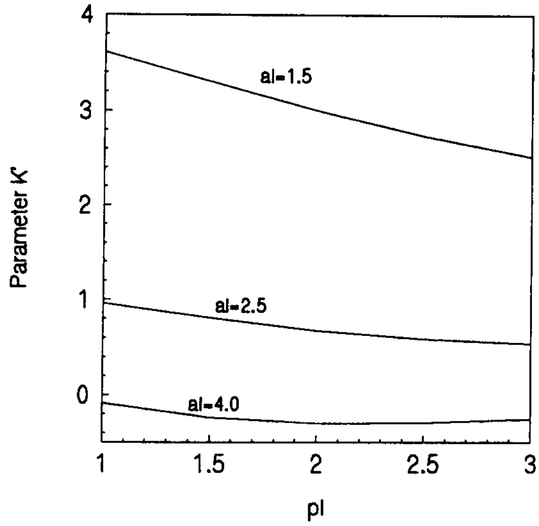


Fig. 3. Parameter $K$ vs $pl$ for different values of $al$ ($pt=0.4$, $pj=0.6$, $\epsilon=0.05$).

where $\zeta=\cos\theta$, $\vartheta_n(\zeta)$ and $H_n(\zeta)$ are Gagenbauer functions of the first and second kinds and $K_{n-1/2}(\alpha r)$, $I_{n-1/2}(\alpha r)$ are modified Bessel functions of the third and fourth kinds. Using the regularity conditions at infinity, we obtain

$$\psi=[B_2r^{-1}+C_2\sqrt{r}K_{3/2}(\alpha r)+E_{3/2}\sqrt{r}K_{3/2}(\beta r)]\vartheta_2(\zeta) + \sum_{n=3}^{\infty}[B_nr^{-n+1}+C_n\sqrt{r}K_{n-1/2}(\alpha r)+E_n\sqrt{r}K_{n-1/2}(\beta\gamma)]\vartheta_n(\zeta) \tag{3.16}$$

$$C=\frac{1}{r\sin\theta}[\{C_2\sqrt{r}A_\alpha^*K_{3/2}(\alpha r)+E_2\sqrt{r}A_\beta^*K_{3/2}(\beta r)\}\vartheta_2(\zeta) + \sum_{n=3}^{\infty}\{C_nA_\alpha^*\sqrt{r}K_{n-1/2}(\alpha r)+E_nA_\beta^*\sqrt{r}K_{n-1/2}(\beta r)\}\vartheta_n(\zeta)] \tag{3.17}$$

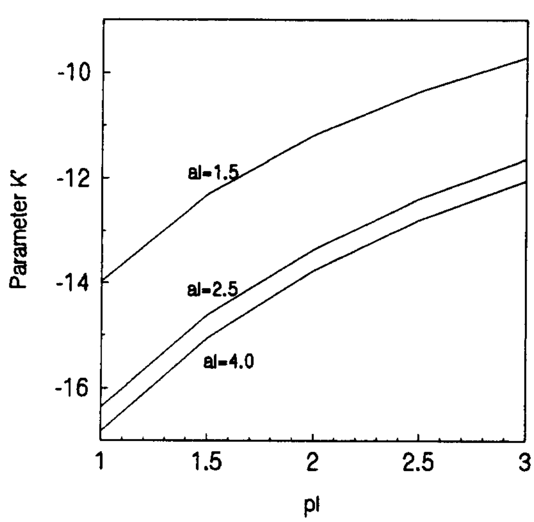


Fig. 4. Parameter $K'$ vs $pl$ for different values of $al$ ($pt=0.4$, $pj=0.6$, $\epsilon=0.05$).

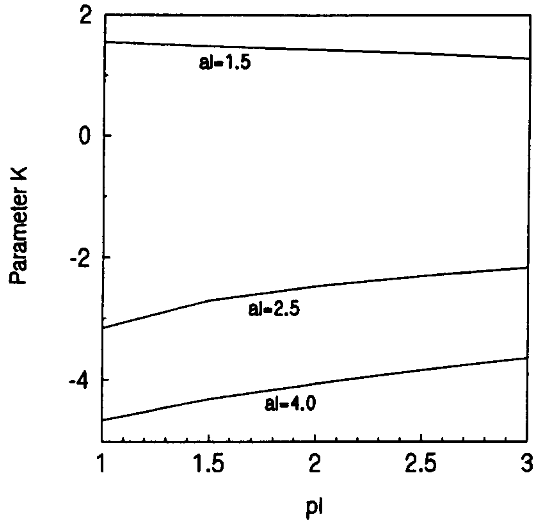


Fig. 5. Parameter $K$ vs $pl$ for different values of $al$ ($pt=0.4$, $pj=0.6$, $\epsilon=0.1$).

where

$$A_\alpha^* = \frac{(\mu+k)\alpha^2 - i\rho\omega}{k};\ A_\beta^* = \frac{(\mu+k)\beta^2 - i\rho\omega}{k}. \tag{3.18}$$

Let us introduce the following nondimensionalization scheme before proceeding to the implementation of the boundary conditions to determine the arbitrary constants in the expressions of $\psi$ and $C$:

$$\psi = Ua^2\tilde{\psi};\ B_n = Ua^{n+1}\tilde{B}_n;\ C_n = Ua^{3/2}\tilde{C}_n,\ E_n = Ua^{3/2}\tilde{E}_n;\ C = U/a\tilde{C}. \tag{3.19}$$

Introducing these in equations (3.16) and (3.17) and then dropping the tildes, the expressions for $\psi$ and $C$ in nondimensional form, respectively, become

$$\psi = [B_2 r^{-1} + C_2\sqrt{r}K_{3/2}(a\alpha r) + E_{3/2}\sqrt{r}K_{3/2}(a\beta r)]\,\vartheta_2(\zeta) - \sum_{n=3}^{\infty} [B_n r^{-n+1} + C_n\sqrt{r}K_{n-1/2}(a\alpha r) + E_n\sqrt{r}K_{n-1/2}(a\beta r)]\,\vartheta_n(\zeta) \tag{3.20}$$

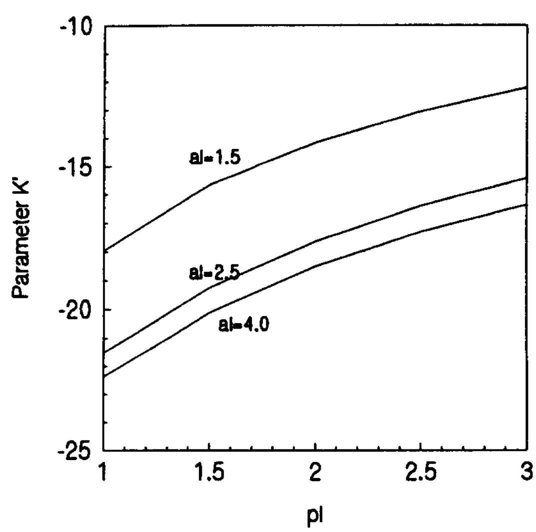


Fig. 6. Parameter $K'$ vs $pl$ for different values of $al$ ($pt=0.4$, $pj=0.6$, $\epsilon=0.1$).

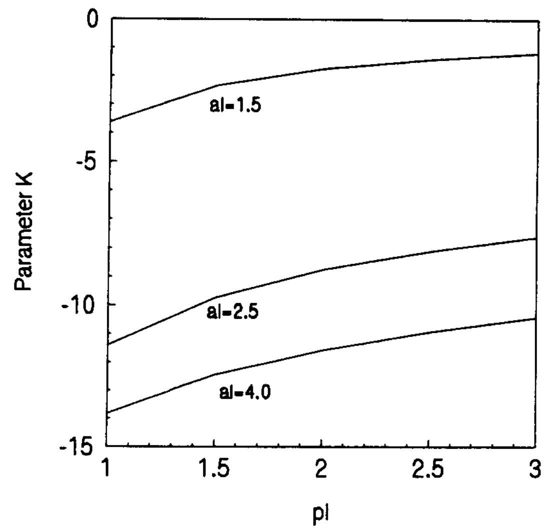


Fig. 7. Parameter $K$ vs $pl$ for different values of $al$ ($pt=0.4$, $pj=0.6$, $\epsilon=0.2$).

$$C=\frac{1}{r\sin\theta}\,[\{C_2\sqrt{r}A_\alpha^* a^2K_{3/2}(a\alpha r)+E_2\sqrt{r}A_\beta^* a^2K_{3/2}(a\beta r)\}\vartheta_2(\zeta)$$

$$+\sum\{C_nA_\alpha^*\sqrt{r}a^2K_{n-1/2}(a\alpha r)+E_nA_\beta^*\sqrt{r}a^2K_{n-1/2}(a\beta r)\}\vartheta_n(\zeta)]. \tag{3.21}$$

### *3.3 Determination of arbitrary constants*

We first propose to develop the solutions corresponding to the boundary $r=a[1+\beta_m\vartheta_m(\zeta)]$ and assume that the coefficient $\beta_m$ is sufficiently small, so that squares and higher powers of $\beta_m$ may be neglected, and we replace $(r/a)^k$ by $1+k\beta_m\vartheta_m(\zeta)$ where $k$ is positive or negative.

If the oscillating body were the sphere $r=a$, the expression for $\psi$ is given by [6]

$$\psi=[B_2r^{-1}+C_2\sqrt{r}K_{3/2}(\alpha ar)+E_2\sqrt{r}K_{3/2}(\beta ar)]\,\vartheta_2(\zeta) \tag{3.22}$$

only.

Comparing equation (3.20) with the above expression, the terms involving $B_n$, $C_n$, $E_n$ for $n>2$ in equation (3.20) are the extra terms here which are not present in $\psi$ for the sphere. The body in the present problem is an approximate sphere and the motion is expected not to be far

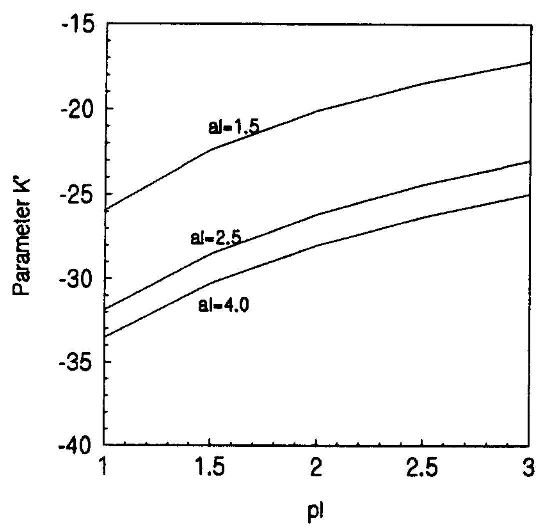


Fig. 8. Parameter $K'$ vs $pl$ for different values of $al$ ($pt=0.4$, $pj=0.6$, $\epsilon=0.2$).

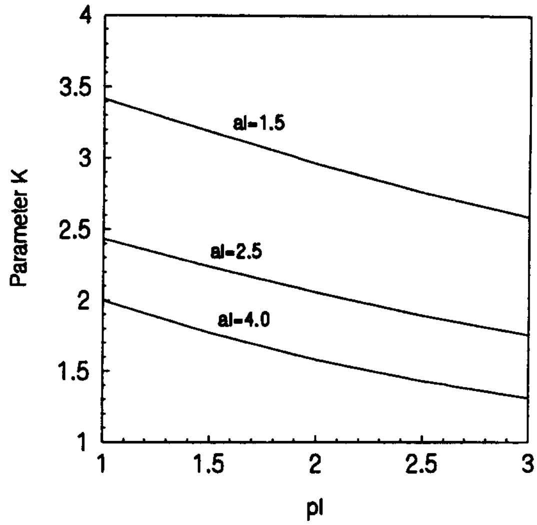


Fig. 9. Parameter $K$ vs $pl$ for different values of $al$ ($pt=0.8$, $pj=0.6$, $\epsilon=0.05$).

different from that which occurs when the body is a sphere. All the coefficients $B_n$, $C_n$, $E_n$ for $n>2$ will be of the order $\beta_m$. Therefore in these terms involving $B_n$, $C_n$, $E_n$ we disregard the departure from a spherical form and set $r=1$ ($r$ is nondimensional) while implementing the boundary conditions.

The condition equation (3.9) implies that

$$-[B_2+C_2K_{3/2}(a\alpha)+E_2K_{3/2}(a\beta)+1]P_1(\zeta)+[B_2-2]\beta_m\vartheta_m(\zeta)P_1(\zeta) + \sum[B_n+C_nK_{n-1/2}(a\alpha)+E_nK_{n-1/2}(a\beta)]P_{n-1}(\zeta)=0 \quad (3.23)$$

$$[-B_2-C_2\{K_{3/2}(a\alpha)+a\alpha K_{1/2}(a\alpha)\}-E_2\{K_{3/2}(\alpha\beta)+a\beta K_{1/2}(a\beta)\}+2]\vartheta_2(\zeta) + [2B_2+C_2K_{3/2}(a\alpha)+E_2K_{3/2}(a\beta)+2]\beta_m\vartheta_m(\zeta)\vartheta_2(\zeta) + \sum[(1-n)B_n-C_n\{(n-1)K_{n-1/2}(a\alpha)+a\alpha K_{n-3/2}(a\alpha)\}-E_n\{(n-1)K_{n-1/2}(a\beta) + a\beta K_{n-3/2}(a\beta)\}]\vartheta_n(\zeta)=0 \quad (3.24)$$

and

$$\{C_2A^*_\alpha K_{3/2}(a\alpha)+E_2A^*_\beta K_{3/2}(a\beta)\}\vartheta_2(\zeta)-\{C_2A^*_\alpha K_{3/2}(a\alpha)+E_2A^*_\beta K_{3/2}(a\beta)\}\beta_m\vartheta_m(\zeta)\vartheta_2(\zeta) + \sum[C_nA^*_\alpha K_{n-1/2}(a\alpha)+E_nA^*_\beta K_{n-1/2}(a\beta)]\vartheta_n(\zeta)=0. \quad (3.25)$$

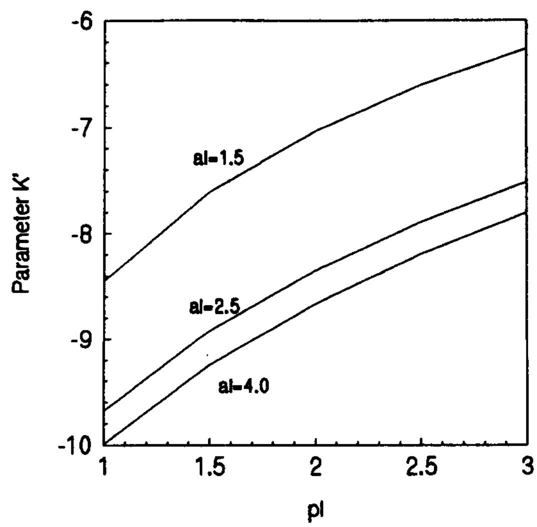


Fig. 10. Parameter $K'$ vs $pl$ for different values of $al$ ($pt=0.8$, $pj=0.6$, $\epsilon=0.05$).

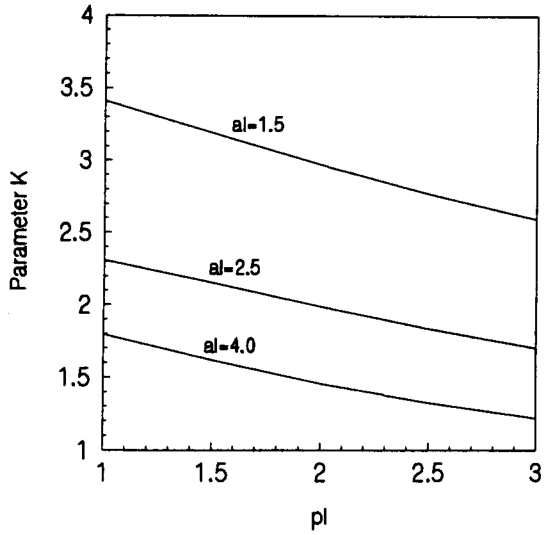


Fig. 11. Parameter $K$ vs $pl$ for different values of $al$ ($pt=0.8$, $pj=0.8$, $\epsilon=0.05$).

Equating leading coefficients to zero in equations (3.23)–(3.25), we obtain

$$B_2 + C_2K_{3/2}(a\alpha) + E_2K_{3/2}(a\beta) + 1 = 0 \tag{3.26}$$

$$-B_2 - C_2[K_{3/2}(a\alpha) + a\alpha K_{1/2}(a\alpha)] - E_2[K_{3/2}(a\alpha) + a\beta K_{1/2}(a\beta)] + 2 = 0 \tag{3.27}$$

$$C_2A^*_\alpha K_{3/2}(a\alpha) + E_2A^*_\beta K_{3/2}(a\beta) = 0 \tag{3.28}$$

and these give the expressions

$$B_2 = [A^*_\alpha K_{3/2}(a\alpha)\{a\beta K_{1/2}(a\beta) - 3K_{1/2}(a\beta)\} + A^*_\beta K_{3/2}(a\beta)\{3K_{3/2}(a\alpha) - a\alpha K_{1/2}(a\alpha)\}]/D(\alpha,\beta) \tag{3.29}$$

$$C_2 = 3K_{3/2}(a\beta)A^*_\beta/D(\alpha,\beta) \tag{3.30}$$

and

$$E_2 = -3K_{3/2}(a\alpha)A^*_\alpha/D(\alpha,\beta) \tag{3.31}$$

where

$$D(\alpha,\beta) = a\alpha A^*_\beta K_{1/2}(a\alpha)K_{3/2}(a\beta) - a\beta A^*_\alpha K_{1/2}(a\beta)K_{3/2}(a\alpha). \tag{3.32}$$

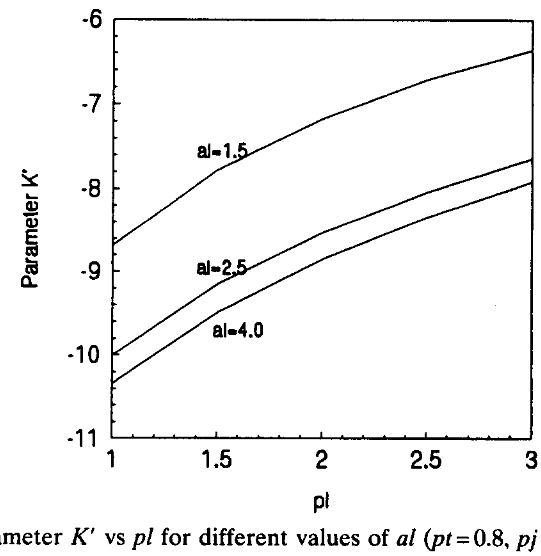


Fig. 12. Parameter $K'$ vs $pl$ for different values of $al$ ($pt=0.8$, $pj=0.8$, $\epsilon=0.05$).

Using these values $B_2$, $C_2$ and $E_2$ in equations (3.23)–(3.25), we obtain

$$\sum[B_n + C_nK_{n-1/2}(a\alpha) + E_nK_{n-1/2}(a\beta)]P_{n-1}(\zeta) = \epsilon_1\beta_m\vartheta_m(\zeta)P_1(\zeta) \quad (3.33)$$

$$\sum[(1-n)B_n - C_n\{(n-1)K_{n-1/2}(a\alpha) + a\alpha K_{n-3/2}(a\alpha)\} - E_n\{(n-1)K_{n-1/2}(a\beta) + a\beta K_{n-3/2}(a\beta)\}]\vartheta_n(\zeta) = \epsilon_2\beta_m\vartheta_m(\zeta)\vartheta_2(\zeta) \quad (3.34)$$

$$\sum[C_nA^*_\alpha K_{n-1/2}(a\alpha) + E_nA^*_\beta K_{n-1/2}(a\beta)]\vartheta_n(\zeta) = 0 \quad (3.35)$$

where

$$\epsilon_1 = [3A^*_\beta K_{3/2}(a\beta)\{K_{3/2}(a\alpha) - a\alpha K_{1/2}(a\alpha)\} - 3A^*_\alpha K_{3/2}(a\alpha)\{K_{3/2}(a\beta) - K_{1/2}(a\beta)\}]/D(\alpha,\beta) \quad (3.36)$$

$$\epsilon_2 = [9K_{3/2}(a\alpha)K_{1/2}(a\beta)(A^*_\alpha - A^*_\beta)]/D(\alpha,\beta). \quad (3.37)$$

To obtain the remaining arbitrary constants in equations (3.33)–(3.35), we use the identities (see Ref. [5], p. 142)

$$\vartheta_m(\zeta)\vartheta_2(\zeta) = \frac{-(m-2)(m-3)}{2(2m-1)(2m-3)}\vartheta_{m-2}(\zeta) + \frac{m(m-1)}{(2m+1)(2m-3)}\vartheta_m(\zeta) - \frac{(m+1)(m+2)}{2(2m-1)(2m-3)}\vartheta_{m+2}(\zeta) \quad (3.38)$$

and

$$\vartheta_m(\zeta)P_1(\zeta) = \frac{(m-2)}{2(2m-1)(2m-3)}P_{m-3}(\zeta) + \frac{1}{(2m+1)(2m-3)}P_{m-1}(\zeta) - \frac{(m+1)}{(2m-1)(2m+1)}P_{m+1}(\zeta) \quad (3.39)$$

then we get

$$B_n = C_n = E_n = 0 \text{ if } n \neq m-2, m, m+2 \quad (3.40)$$

and when $n = m-2$, $m$, $m+2$, we have the following system

$$B_n + C_nK_{n-1/2}(a\alpha) + E_nK_{n-1/2}(\alpha\beta) = \epsilon_1\beta_m a_n \quad (3.41)$$

$$(1-n)B_n - C_n\{(n-1)K_{n-1/2}(a\alpha) + a\alpha K_{n-3/2}(a\alpha)\} - E_n\{(n-1)K_{n-1/2}(a\beta) + a\beta K_{n-3/2}(a\beta)\} = \epsilon_2\beta_m b_n \quad (3.42)$$

$$C_nA^*_\alpha K_{n-1/2}(a\alpha) + E_nA^*_\beta K_{n-1/2}(a\beta) = 0 \quad (3.43)$$

where

$$b_{m-2} = \frac{-(m-2)(m-3)}{2(2m-1)(2m-3)},\ b_m = \frac{m(m-1)}{(2m+1)(2m-3)},\ b_{m+2} = \frac{-(m+1)(m+2)}{2(2m-1)(2m-3)},$$

$$a_{m-2} = \frac{(m-2)}{2(2m-1)(2m-3)},\ a_m = \frac{1}{(2m+1)(2m-3)},\ a_{m+2} = \frac{(m+1)}{(2m-1)(2m+1)}. \quad (3.44)$$

The expressions for $B_n$, $C_n$ and $E_n$ for $n = m-2$, $m$, $m+2$ are given by

$$B_n = [a_n\epsilon_1\{a\alpha A^*_\beta K_{n-1/2}(a\beta)K_{n-3/2}(a\alpha) - a\beta A^*_\alpha K_{n-1/2}(a\alpha)K_{n-3/2}(a\beta) + (n-1)A^*_\beta K_{n-1/2}(a\alpha)K_{n-1/2}(a\beta) - (n-1)A^*_\alpha K_{n-1/2}(a\alpha)K_{n-1/2}(\alpha\beta)\} + b_n\epsilon_2\{A^*_\beta - A^*_\alpha\}K_{n-1/2}(a\alpha)K_{n-1/2}(a\beta)]/D(\alpha,\beta) \quad (3.45)$$

$$C_n = -\{(n-1)\epsilon_1 a_n + \epsilon_2 b_n\} A^*_\beta K_{n-1/2}(a\beta)/D(\alpha,\beta) \tag{3.46}$$

and

$$E_n = \{(n-1)\epsilon_1 a_n + \epsilon_2 b_n\} A^*_\alpha K_{n-1/2}(a\alpha)/D(\alpha,\beta). \tag{3.47}$$

Thus the stream function $\psi$ is determined completely. In case the equation of approximate sphere is $r=a[1+\Sigma\beta_m\vartheta_m(\zeta)]$, we employ the same technique as above and determine the corresponding arbitrary constants in the expansion for $\psi$.

### *3.4 Determination of drag*

The drag experienced by the body is given by

$$D = i\rho\omega UV + 4\pi i\rho\omega \lim_{r\to\infty} \frac{r\psi}{\sin^2\theta} \tag{3.48}$$

where $V$ is the volume of the body. In the present case, using the above formula the drag is seen to be

$$D = 4\pi a i\rho\omega(1+\beta_2)/3 + 4\pi i\rho\omega[B_2 + (1/5)B_2''\beta_2 + (2/35)B_2'\beta_4]e^{i\omega t} \tag{3.49}$$

where

$$B'_{m-2} = a_{m-2}, B_{m-2}; B''_m = a_m B_m; B'''_{m+2} = a_{m+2}B_{m+2}. \tag{3.50}$$

It is interesting to note that though the boundary surface is $r=a[1+\sum_{m=2}^{\infty}\beta_m\vartheta_m(\zeta)]$, the coefficients $\beta_2$ and $\beta_4$ only contribute to the drag. This implies that the drag on the approximate sphere is relatively insensitive to the details of the surface geometry. This is in tune with the observations made in Refs [9, 10].

If $\beta_2=\beta_4=\epsilon$, then the drag is

$$D = (4\pi/3)i\rho\omega[(1+3B_2) + (1+(3/5)B_2'' + (6/35)B_2')\epsilon]z e^{i\omega t}. \tag{3.51}$$

The nondimensional drag on the body can be expressed as

$$(1+3B_2) + (1+(3/5)B_2'' + (6/35)B_2')\epsilon = -K - iK'. \tag{3.52}$$

The drag parameters $K$ and $K'$ are numerically evaluated for various values of $al=(\mu+k)/k$, $\epsilon$, $pt=\rho\omega/(\mu+k)$, $pl=k(2\mu+k)/[\gamma(\mu+k)]$ and $pj=j(\mu+k)/\gamma$ and their variation is presented in the graphs of Figs 1–12.

Figures 1 and 2 pertain to the case of a perfect sphere ($\epsilon=0$). Figures 3–12 concern an approximate sphere ($\epsilon>0$). As the deformation parameter $\epsilon$ increases, both $K$ and $K'$ decrease algebraically. Furthermore, with an increase in $pl$, a decrease in $K$ and increase in $K'$ is noticed. Keeping $al$, $pj$ and $\epsilon$ fixed, and increasing the frequency parameter $pt$, an increase in both $K$ and $K'$ is observed for the selected range of parameter values.

### *3.5 Special cases*

#### *3.5.1 Sphere*

If $\beta_m=0$ for $m\geq 2$, we obtain the case of the sphere. In this case the drag simplifies to

$$D = [(4\pi a i\rho\omega)/3 + 2\pi i\rho\omega B_2]e^{i\omega t} \tag{3.53}$$

which is same as the drag on the sphere as obtained by Lakshmana Rao and Bhujanga Rao [6].

#### *3.5.2 Oblate spheroid*

Consider the oblate spheroid given by

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{a^2(1-\epsilon)^2} = 1 \tag{3.54}$$

whose equatorial radius is "$a$" in which $\epsilon$ is so small that $\epsilon^2$ and higher powers may be neglected. Following Happel and Brenner [5] its polar equation can be put in the form $r=c[1+2\epsilon\vartheta_2(\zeta)]$ where $c=a(1-\epsilon)$ (see Ref. [5], p. 144). This is similar to $r=c[1+\beta_2\vartheta_2(\zeta)]$ where $a=c$ and $\beta_2=2\epsilon$.

Using equations (3.20) and (3.21), the expressions for $\psi(r, \theta)$ and $C(r, \theta)$ can be determined. Utilizing equation (3.48), the drag is seen to be

$$D = (4\pi ai\rho\omega/3)[(1+3B_2)+(4-6B_2+(6/5)B_2'')\epsilon]. \quad (3.55)$$

It is to be noted that the evaluation of the drag on an oblate spheroid is based on the neglect of $\epsilon^2$ and higher terms, while the drag formula obtained by Lakshmana Rao and Iyengar [7] is based on exact analysis. However, the numerical determination of the drag in Ref. [7] is through the solution of a truncated system of simultaneous equations, the details of which can be found in Ref. [7].

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