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Ranking alternatives with fuzzy weights using maximizing set and minimizing set

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Abstract

Ranking alternatives (both qualitative as well as quantitative) in a multicriterion environment, employing experts opinion (preference structure) using fuzzy numbers and linguistic variables, are presented in this paper. Fuzzy weights ($\tilde{w}_i$) of alternatives (A_i) are computed using standard fuzzy arithmetic. Concept of maximizing set and minimizing set is introduced to decide total utility or ordering value of each of the alternatives. A numerical example is provided at the end to illustrate the method.

Keywords: Fuzzy numbers; Fuzzy order; Fuzzy weights; Maximizing set; Minimizing set; Multicriteria decision making; Ranking alternatives; Utility values

1. Introduction

Consider the problem of ranking m alternatives (A_i; $i = 1, 2, \ldots, m$) by a decision maker (DM). DM wishes to select from amongst m alternatives, with the help of information supplied by n experts (E_j; $j = 1, 2, \ldots, n$) about the alternatives for each of K Criteria (C_k; $k = 1, 2, \ldots, K$) and also the relative importance of each criteria with respect to some overall objective; which one best satisfy the criteria. This is essentially the problem considered in this paper and the methodology proposed is explained in detail.

Many authors have studied different methods of ranking alternatives under fuzzy environment during the last two decades. Jain [14, 15] proposed a method of using the concept of maximizing set to order alternatives. Baas and Kwakernaak [3] proposed the concept of membership level. But Baldwin and Guild [4] indicated that the above two methods suffer from some difficulties for comparing the alternatives and have disadvantages. Adamo [1] introduced α-preference rule using the concept of α-level set. Chang [10] indicated that the method proposed by Adamo may lead to an inappropriate choice and went on to introduce preference function concept of an alternative. In some special cases Chang's preference function

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seem to contradict intuition. Other contributions in this direction include: index of strict preference defined by Watson et al. [19]; three indices proposed by Yager [20–22]; method of Kerre [16]; four grades of dominance studied by Dubois and Prade [12, 13] and the average-value ranking method given by Campos and Gonzalez [8]. A more recent and complete review of fuzzy numbers ranking methods was presented by Bortolan et al. [5] and Zhu and Lee [23]. Requena et al. [18] presented a method of automatic ranking of fuzzy numbers using artificial neural networks (ANN). Other contributions using ANN to rank alternatives are Cano et al. [9] and Requena [17].

Most of these methods suffer at least from one of the following drawbacks: (i) the procedure is computationally complex and therefore the difficulty in implementing the method; (ii) the method is unintuitive, which hinders the implementation process; (iii) the methods may assume only one criteria or one expert (may be the DM himself); (iv) the method may presuppose the existence of some fuzzy relationship or other functional relationships across the alternatives, which is unrealistic or (v) the method may produce crisp ranking from fuzzy data.

To overcome some of these difficulties and to make the problem simple and straightforward, authors, in this paper, have proposed a method which is intuitive in nature, computationally simple and easy to implement. In this method the fuzzy weights of the alternatives are arrived at with the help of the fuzzy information supplied by several experts on alternatives and various important criteria considered in the study. The process of obtaining the fuzzy weights is detailed in the works of Buckley [6, 7] and the same is adopted in this paper. Then the final ranking of the alternatives, according to the utility (or order) values, are determined using the concept of maximizing and minimizing sets proposed by Chen [11].

In the following Section 2, the proposed methodology is explained in detail. A numerical example, some important potential applications and conclusions are presented in Sections 3, 4 and 5, respectively.

2. Methodology

There are a number of issues to be addressed before the final ranking of the alternatives. They are: (i) defining and specifying the types of fuzzy numbers and their membership functions to be used by the experts; (ii) designing the scale of preference structure to be used by the experts; (iii) pooling (or aggregating) and averaging fuzzy numbers across the experts; (iv) computation of fuzzy weights ($\tilde{w}_i$); (v) determination of the total utility values and (vi) final ranking (or ordering) of alternatives. This method can handle any number of hierarchies. But for simplicity, in this paper, only one hierarchy is considered.

2.1. *Fuzzy numbers*

Let $\tilde{a}_i$ be a fuzzy number which is a fuzzy subset of $\mathbb{R}$ (real numbers) and is considered in the form of

$$\tilde{a}_i = \{\alpha_i/\beta_i, \gamma_i/\delta_i\}, \quad i = 1, 2, \ldots, m, \tag{1}$$

where $\alpha < \beta < \gamma < \delta \in \mathscr{L}$, $\mathscr{L}$ is the scale of preference information to be used by the experts.

2.2. *Scale of preference structure* ($\mathscr{L}$)

We consider the scale $\mathscr{L} = \{\ell_1, \ell_2, \ell_3, \ldots, L\}$ of preference information to be used by the experts. This scale is assumed to be finite, linearly ordered and $\ell_1 < \ell_2 < \cdots < L$. This $\mathscr{L}$ can be an ordinal, an exact, a ratio, an interval scale or a combination of these scales. It may be easier for the experts to express their preferences in ordinal values (linguistic variables), especially when there are more number of alternative and qualitative criteria and when some of the criteria vaguely understood or imprecisely defined. In this case the evaluation process may be very much subjective, but it seems more appropriate to use ordinal scale than any other scales.

For example, consider the problem of ranking river-basin planning and development alternatives with multiple criteria. This is a complex large-scale problem. This may contain a large number of both quantitative and qualitative criteria. For comparison amongst alternatives with respect to some of the criteria, experts may prefer ordinal values rather than numbers. Suppose the DM asks the experts to rank the alternatives with respect to some criteria like environmental quality improvements or recreational facilities, etc., using a scale of integers from $\ell_1 = 0$ (worst) to $L = 10$ (best). At this stage the DM may specify the standard linguistic variables to be used by the experts. These linguistic variables could be W = worst; VP = very poor; P = poor; BA = below average; M = medium (average); AA = above average; H = high; VH = very high and B = best (as given for the numerical example in Section 3) or the DM may ask the experts to specify their own preference structure. If the experts are in confusion to assign fuzzy numbers to these linguistic variables, DM, under proper interpretation, could help them to assign fuzzy numbers (see numerical example at the end). Otherwise the DM could specify standard fuzzy numbers for these linguistic variable. If all the criteria are expressed in ordinal values the procedure suggested by Buckley [6] can be adopted. For more details of this river-basin planning problem, where both linguistic variables and numbers are used by the experts, one can refer the work of the authors presented in [2].

2.3. Membership functions

The normalized membership function of an alternative a_i is considered in the form of

$$\mu_{\tilde{a}_i}(x) = \begin{cases} 0, & x < \alpha_i, \\ (x-\alpha_i)/(\beta_i-\alpha_i), & \alpha_i < x < \beta_i, \\ 1, & \beta_i < x < \gamma_i, \\ (\delta_i - x)/(\delta_i - \gamma_i), & \gamma_i < x < \delta_i, \\ 0, & x > \delta_i. \end{cases} \tag{2}$$

The membership function $\mu_{\tilde{a}_i}(x)$ is graphically represented in Fig. 1. From this it can be understood that $\mu_{\tilde{a}_i}(x)$ is a straight line segment over the interval (α_i, β_i); (β_i, γ_i) and (γ_i, δ_i).

Let the experts (E_j; $j = 1, 2, \ldots, n$) assign fuzzy numbers to the alternatives (A_i; $i = 1, 2, \ldots, m$) for each of K criteria (C_k; $k = 1, 2, \ldots, K$) and also to each criteria. Let

$$\tilde{a}_{ij}^k = (\alpha_{ij}^k/\beta_{ij}^k, \gamma_{ij}^k/\delta_{ij}^k) \tag{3}$$

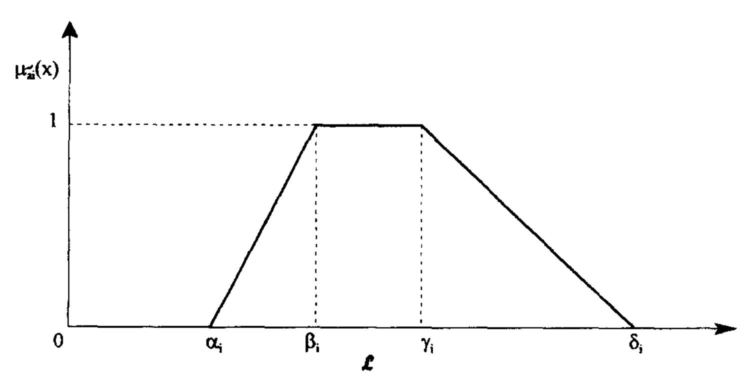

Fig. 1. Graphical representation of the membership function $\mu_{\tilde{a}_i}(x)$ of a fuzzy number $\tilde{a}_i$.

be the fuzzy number assigned to alternative A_i by expert E_j for criteria C_k. This means that $\tilde{a}_{ij}^k$ measures how well A_i satisfies C_k for expert E_j. For each criterion k, the corresponding membership function can be represented as $\mu_{A_i}^j(x)$ (similar to Eq. (2)) and this data can be expressed in the matrix form as

$$R_k = \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \overset{\begin{matrix} E_1 & E_2 & \cdots & E_n \end{matrix}}{\begin{bmatrix} \\ \mu_{A_i}^j(x) = \tilde{a}_{ij}^k \in \mathscr{L} \\ \\ \end{bmatrix}}. \tag{4}$$

Similarly, let

$$\tilde{c}_{kj} = (\varepsilon_{kj}/\zeta_{kj}, \eta_{kj}/\theta_{kj}) \tag{5}$$

be fuzzy number given to criteria C_k by expert E_j. Thus, $\tilde{c}_{kj}$ indicates the importance of C_k for expert E_j with respect to an overall objective. The membership function of these fuzzy numbers can be represented as $\mu_{C_k}(x)$ and in matrix form this data can be shown as

$$R = \begin{matrix} C_1 \\ C_2 \\ \vdots \\ C_K \end{matrix} \overset{\begin{matrix} E_1 & E_2 & \cdots & E_n \end{matrix}}{\begin{bmatrix} \\ \mu_{C_k}(x) = \tilde{c}_{kj} \in \mathscr{L} \\ \\ \end{bmatrix}}. \tag{6}$$

2.4. Fuzzy weights

Given the data R_k and R, the DM computes the fuzzy weights ($\tilde{w}_i; i = 1, 2, \ldots, m$) of the alternatives. The fuzzy weights for each of the alternatives can be arrived at by pooling, averaging or aggregating across experts. This task can be achieved in two ways. They are "pool first" and "pool last" procedures.

In pool-first procedure, the first step is to find the averages of fuzzy numbers across all the experts first as shown in Eq. (7). For this purpose let us consider $\oplus$ and $\odot$ as fuzzy addition and multiplication, respectively, as defined in [11]. Then

$$\tilde{p}_{ik} = (1/n) \odot (\tilde{a}_{i1}^k \oplus \tilde{a}_{i2}^k \oplus \cdots \oplus \tilde{a}_{in}^k) \quad \text{and} \quad \tilde{q}_k = (1/n) \odot (\tilde{c}_{k1} \oplus \tilde{c}_{k2} \oplus \cdots \oplus \tilde{c}_{kn}), \quad \tilde{p}_{ik}, \tilde{q}_k \in \mathscr{L}. \tag{7}$$

These fuzzy numbers given in Eq. (7) are simply the row averages of matrices given in Eqs. (4) and (6), respectively. $\tilde{p}_{ik}$ is the fuzzy ranking of A_i for criteria C_k and $\tilde{q}_k$ is the fuzzy ranking of C_k. The next step is then to determine the fuzzy weights of the alternatives ($\tilde{w}_i$). To compute these weights, multiply $\tilde{p}_{ik}$ and $\tilde{q}_k$ and find the average over all criteria as shown in Eq. (8). That is

$$\tilde{w}_i = (1/\mathrm{KL}) \odot \{(\tilde{p}_{i1} \odot \tilde{q}_1) \oplus (\tilde{p}_{i2} \odot \tilde{q}_2) \oplus \cdots \oplus (\tilde{p}_{iK} \odot \tilde{q}_K)\}. \tag{8}$$

In pool-last method, fuzzy weights ($\tilde{w}_{ij}$) for alternative A_i for each of the expert E_j are computed first. This means that $\tilde{w}_{ij}$ is the fuzzy average over all the criteria and is given in Eq. (9).

$$\tilde{w}_{ij} = (1/\mathrm{KL}) \odot \{(\tilde{a}_{ij}^1 \odot \tilde{c}_{1j}) \oplus (\tilde{a}_{ij}^2 \odot \tilde{c}_{2j}) \oplus \cdots \oplus (\tilde{a}_{ij}^K \odot \tilde{c}_{Kj})\}. \tag{9}$$

The fuzzy weights $\tilde{w}_{ij}$ are then pooled across all the experts to obtain final weights ($\tilde{w}_i'$) of the alternatives as shown in Eq. (10). That is

$$\tilde{w}_i' = (1/n) \odot (\tilde{w}_{i1} \oplus \tilde{w}_{i2} \oplus \cdots \oplus \tilde{w}_{in}). \tag{10}$$

Here one can note that $\tilde{w}_i$ and $\tilde{w}_i'$ can differ in their support and hence may produce different rankings to the alternatives. In the further discussions, we limit ourselves to the pool-first procedure. This procedure can easily be extended to pool-last method.

In the pool-first procedure, the fuzzy weight w_i can easily be computed using standard fuzzy arithmetic as shown below. Let $\alpha_{ik}, \beta_{ik}, \gamma_{ik}$ and δ_{ik} be averages across experts of $\alpha_{ij}^{\kappa}, \beta_{ij}^{\kappa}, \gamma_{ij}^{\kappa}$ and δ_{ij}^{κ}, respectively. Similarly let ε_k, ζ_k, η_k and θ_k be defined as the averages across experts of ε_{kj}, ζ_{kj}, η_{kj} and θ_{kj}, respectively. That is

$$\alpha_{ik} = (\Sigma \alpha_{ij}^{\kappa})/n, \quad j = 1, 2, \ldots, n, \quad \text{and} \quad \varepsilon_k = (\Sigma \varepsilon_{kj})/n, \quad j = 1, 2, \ldots, n. \tag{11}$$

Similar expressions can be written for $\beta_{ik}, \gamma_{ik}, \delta_{ik}, \zeta_k, \eta_k$ and θ_k. Let the fuzzy weight $\tilde{w}_i$ be described as

$$\tilde{w}_i = (\alpha_i[L_{i1}, L_{i2}]/\beta_i, \gamma_i/\delta_i[U_{i1}, U_{i2}]). \tag{12}$$

The graph of the membership function of w_i is: zero to the left of α_i; $L_{i1}y^2 + L_{i2}y + \alpha_i = x$ on $[\alpha, \beta_i]$; horizontal line ($y = 1$) between $[\beta_i, \gamma_i]$; $U_{i1}y^2 + U_{i2}y + \alpha_i = x$ on $[\gamma_i, \delta_i]$ and zero to the right of δ_i (here we assume that the x-axis is horizontal and the y-axis vertical). Where the terms in the Eq. (12) are given by Eqs. (13) and (14). Theorems related to these equations, the proofs and properties are well described in the works of Dubois and Prade [13] and Buckley [7].

$$\alpha_i = (\Sigma \alpha_{ik} \varepsilon_k)/\mathrm{KL}, \qquad \beta_i = (\Sigma \beta_{ik} \zeta_k)/\mathrm{KL}, \qquad \gamma_i = (\Sigma \gamma_{ik} \eta_k)/\mathrm{KL}, \qquad \delta_i = (\Sigma \delta_{ik} \theta_k)/\mathrm{KL}. \tag{13}$$

$$\begin{aligned} L_{i1} &= \{\Sigma(\beta_{ik} - \alpha_{ik})(\zeta_k - \varepsilon_k)\}/\mathrm{KL}, \qquad L_{i2} = [\Sigma\{\alpha_{ik}(\zeta_k - \varepsilon_k) + \varepsilon_k(\beta_{ik} - \alpha_{ik})\}]/\mathrm{KL}, \\ U_{i1} &= \{\Sigma(\delta_{ik} - \gamma_{ik})(\theta_k - \eta_k)\}/\mathrm{KL}, \qquad U_{i2} = -[\Sigma\{\delta_{ik}(\theta_k - \eta_k) + \theta_k(\delta_{ik} - \gamma_{ik})\}]/\mathrm{KL}. \end{aligned} \tag{14}$$

2.5. Ranking of alternatives

We now need to calculate the final ranking of the alternatives. The method proposed here is to use the concept of maximizing set and minimizing set, so as to find the order of the fuzzy weights. This method distinguishes the alternatives clearly. Fuzzy weights can have triangular or trapezoidal or two-sided parabolic drum-like shaped or any other appropriate-shaped membership functions. In this paper we present ranking of fuzzy weights with two-sided parabolic drum-like shaped membership function which is defined as

$$\mu_{\tilde{w}_i}(x) = \begin{cases} 0, & x < \alpha_i, \\ -L_{i2}/2L_{i1} + \{(L_{i2}/2L_{i1})^2 + (x - \alpha_i)/L_{i1}\}^{1/2}, & \alpha_i < x < L_{i1}y^2 + L_{i2}y + \alpha_i, \\ w_i, & L_{i1}y^2 + L_{i2}y + \alpha_i < x < U_{i1}y^2 + U_{i2}y + \delta_i, \\ -U_{i2}/2U_{i1} + \{(U_{i2}/2U_{i1})^2 + (x - \delta_i)/U_{i1}\}^{1/2}, & U_{i1}y^2 + U_{i2}y + \delta_i < x < \delta_i, \\ 0, & x > \delta_i. \end{cases} \tag{15}$$

With the definition of fuzzy weights in Eq. (12), the membership function $\mu_{\tilde{w}_i}(x)$ was restricted to the normal form, that is there exists at least one support point (x_0) with value $\mu_{\tilde{w}_i}(x_0) = 1$. But in many cases we cannot restrict the membership function to the normal form. So we must find a more general form of fuzzy numbers and it is given in Eq. (15).

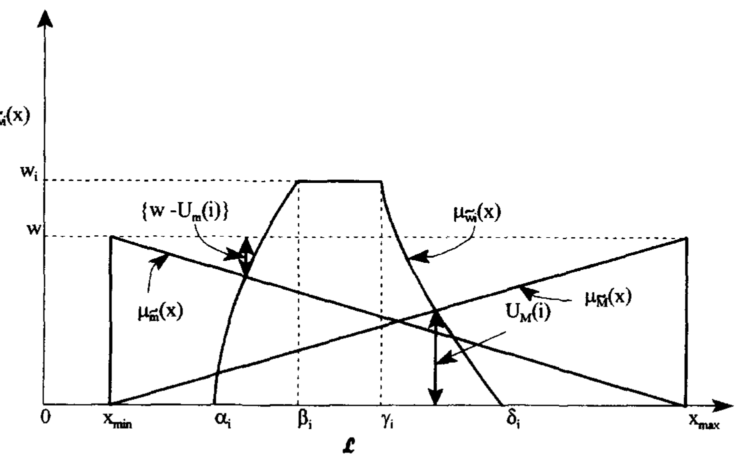

Fig. 2. Graphical representation of $\mu_{\tilde{w}_i}(x)$, $\mu_{\tilde{M}}(x)$ and $\mu_{\tilde{m}}(x)$.

We now define the triangular membership function of maximizing set $\{\mu_{\tilde{M}}(x)\}$ and minimizing set $\{\mu_{\tilde{M}}(x)\}$. These membership functions are, respectively, given by

$$\mu_{\tilde{M}}(x) = \begin{cases} w\{(x - x_{\min})/(x_{\max} - x_{\min})\}^r, & x_{\min} < x < x_{\max}, \\ 0, & \text{otherwise}, \end{cases}$$
$$\mu_{\tilde{m}}(x) = \begin{cases} w\{(x - x_{\max})/(x_{\min} - x_{\max})\}^r, & x_{\min} < x < x_{\max}, \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

where $w = \min_{1<i<m}(w_i)$, $x_{\max} = \sup_{1<i<m}(\delta_i)$, $x_{\min} = \inf_{1<i<m}(\alpha_i)$, (see Fig. 2).

In case: if $r = 1$, we consider maximizing and minimizing sets with linear membership functions; if $r = 2$, we consider maximizing and minimizing sets with convex-curved (risk prone) membership functions, which denotes that DM tends to have an adventurous character, i.e., as the value gets larger, the degree of preference of DM increases rapidly and if $r = \frac{1}{2}$, we consider maximizing and minimizing sets with concave-curved (risk averse) membership functions, which denotes that the DM possesses a conservative preference. In this case, as concavity becomes larger, the degree of preference of DM increases more slowly than the previous case. In general, these three cases cover the three types of preferences: fair, adventurous, conservative – of human beings. Here we present the case when $r = 1$. The graphical representations of $\mu_{\tilde{w}_i}(x)$, $\mu_{\tilde{m}}(x)$ and $\mu_{\tilde{M}}(x)$ are shown in Fig. 2.

Then the right utility value $\{U_M(i)\}$ and the left utility value $\{U_m(i)\}$ of a fuzzy weight $(\tilde{w}_i)$ are, respectively, defined as

$$U_M(i) = \sup_x\{\mu_{\tilde{w}_i}(x) \cap \mu_{\tilde{M}}(x)\} \quad \text{and} \quad U_m(i) = \sup_x\{\mu_{\tilde{w}_i}(x) \cap \mu_{\tilde{m}}(x)\}. \tag{17}$$

It is seen from Fig. 2 that the right utility value is the membership value at the intersection point of $\mu_{\tilde{M}}(x)$ with the right-hand side of $\mu_{\tilde{w}_i}(x)$ and the left utility value is the membership value at the intersection point of $\mu_{\tilde{m}}(x)$ with the left-hand side of $\mu_{\tilde{w}_i}(x)$, respectively.

The greater $U_M(i)$, the higher the order of fuzzy weight w_i and greater $U_m(i)$, the smaller the order of fuzzy weight $\tilde{w}_i$. Therefore, we take the average of $U_M(i)$ and $\{w - U_m(i)\}$ in order to find the total utility or order value $U_T(i)$ as shown below

$$U_T(i) = \{U_M(i) + w - U_m(i)]/2. \tag{18}$$

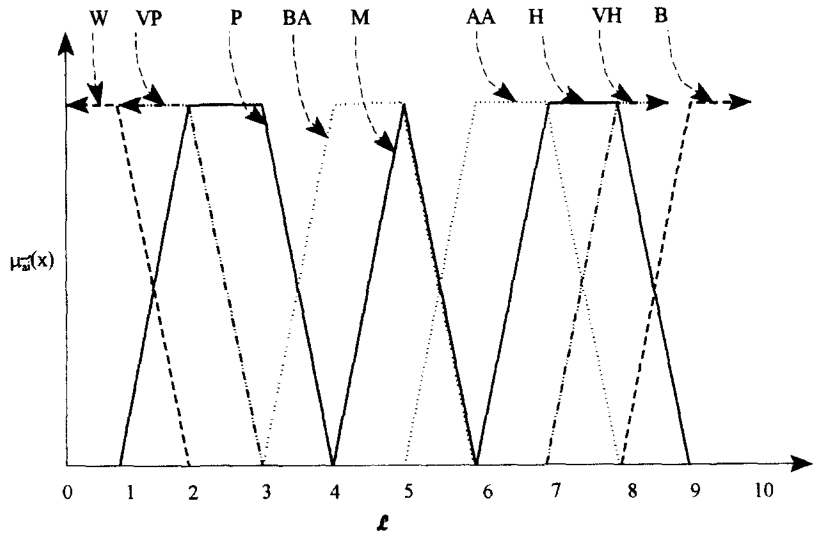

Fig. 3. Graphical representation of DM defined standard fuzzy numbers for the linguistic variables.

In the expanded form $U_T(i)$ can be written as

$$U_T(i) = [-U_{i2}/2U_{i1} - \{(-U_{i2}/2U_{i1})^2 + (X_{iR} - \delta_i)/U_{i1}\}^{1/2} + w + L_{i2}/2L_{i1} - \{(L_{i2}/2L_{i1})^2 + (X_{iL} - \alpha_i)/L_{i1}\}^{1/2}]/2, \quad (19)$$

where

$$X_{iR} = [2x_{\min} - U_{i2}(x_{\max} - x_{\min})/U_{i1}w + ((x_{\max} - x_{\min})/w)^2/U_{i1} - ((x_{\max} - x_{\min})/w) \{(-U_{i2}/U_{i1} + (x_{\max} - x_{\min})/U_{i1}w)^2 + 4(x_{\min} - \delta_i)/U_{i1}\}^{1/2}]/2,$$
$$X_{iL} = [2x_{\max} + L_{i2}(x_{\max} - x_{\min})/L_{i1}w + ((x_{\max} - x_{\min})/w)^2/L_{i1} - ((x_{\max} - x_{\min})/w) \{(L_{i2}/L_{i1} + (x_{\max} - x_{\min})/L_{i1}w)^2 + 4(x_{\max} - \alpha_i)/L_{i1}\}^{1/2}]/2. \quad (20)$$

Using Eqs. (17)–(20), the total utility or order values are calculated and with these values alternatives can be ranked. If two alternatives have the same utility values {i.e., $U_T(1) = U_T(2)$}, we may use the vertices of the graphs of the corresponding membership functions to make the distinction. That is, the vertex further right is the largest, with decreasing size from right to left.

A computer program, RANFUW (**RAN**king **FU**zzy **W**eights), in FORTRAN was developed to implement the method proposed. The steps involved, the flow chart (see Fig. 4) and the output for the numerical example are presented in the following section.

3. Numerical example

A DM wishes to rank four alternatives (A_i; $i = 1,2,3,4$) across two criteria (C_k; $k = 1,2$) using the information supplied by four experts (E_j; $i = 1,2,3,4$). The fuzzy numbers used by the experts have α, β, γ, δ $\in \mathscr{L}(0,1,2,3,\ldots,10)$. For the qualitative (linguistic) evaluation, the experts can use the standard fuzzy numbers suggested by DM or different fuzzy numbers. In this example, let us say that, the experts have used the standard fuzzy numbers specified by DM given in Table 1 (also see Fig. 3). Let us also assume that the DM pools or averages across the experts first before arriving at the final fuzzy weights. The preference

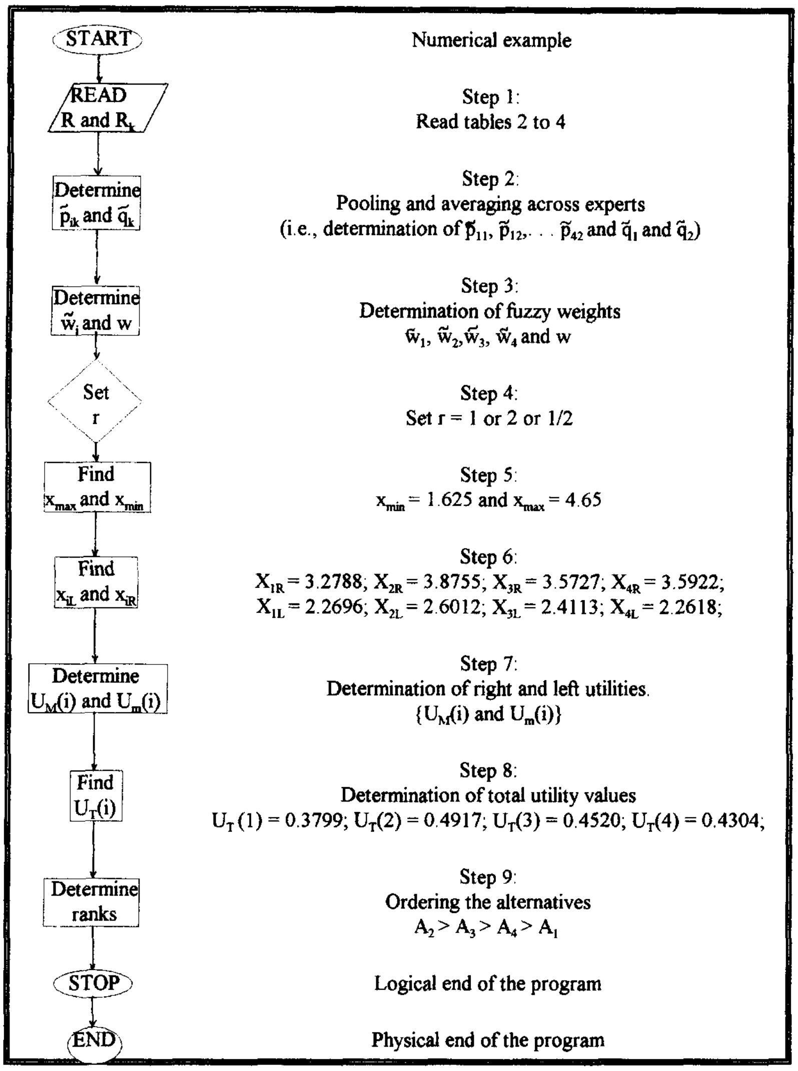

Fig. 4. Flow chart of the program RANFUW (with results of the numerical example).

information given by the experts regarding the criteria is given in Table 2. The preferences of the experts for the alternatives for criteria C_1 and C_2 are summarized in Tables 3 and 4, respectively.

Table 2 shows that experts E_1 and E_3 consider C_1 is more important than C_2. But for E_2, C_2 is more important. Now $\tilde{q}_1$ and $\tilde{q}_2$ (which are criterion's fuzzy ranking across the experts) are computed. They are $\tilde{q}_1 = (\frac{5}{6}, \frac{7}{8})$ and $\tilde{q}_2 = (\frac{3}{4}, 4.5/5)$. From these values it is clear that C_1 turns out to be the most important. Table 3 shows that expert E_2 rank second alternative highest in terms of fuzzy numbers and A_4 is considered medium in terms of linguistic evaluation (here if the expert is not satisfied with the standard fuzzy numbers specified for the linguistic variables by the DM, he can specify his own preference structure for the qualitative aspects). For the second criterion (see Table 4) all the experts believe that alternative A_1 and A_2 have approximately the same ranking. E_4 has given his evaluation for the criteria and alternatives in linguistic terms. A_4 is a qualitative alternative and, therefore, all the experts gave linguistic evaluation

Table 1
DM-defined standard fuzzy numbers for the linguistic variables (ordinal values)

S. no.	Linguistic variable (Ordinal value)	Fuzzy number (DM defined)
1	W = Worst	(0/0, 1/2)
2	VP = Very poor	(0/0, 2/3)
3	P = Poor	(1/2, 3/4)
4	BA = Below average	(3/4, 5/6)
5	M = Average	(4/5, 5/6)
6	AA = Above average	(5/6, 7/8)
7	H = High	(6/7, 8/9)
8	VH = Very high	(7/8, 10/10)
9	B = Best	(8/9, 10/10)

Table 2
Ranking of criteria by experts

	E_1	E_2	E_3	E_4
C_1	(6/7, 8/9)	(4/5, 6/7)	(5/6, 7/8)	AA
C_2	(3/4, 5/6)	(6/7, 7/7)	(0/1, 1/1)	BA

Table 3
Ranking of alternatives for criteria C_1 by experts

C_1	E_1	E_2	E_3	E_4
A_1	(4/5, 5/6)	(2/2, 2/2)	(7/8, 8/9)	M
A_2	(4/5, 5/5)	(7/8, 3/9)	(5/5, 6/7)	AA
A_3	(6/7, 7/8)	(4/4, 5/5)	(0/1, 2/2)	BA
A_4	H	M	H	M

Table 4
Ranking of alternatives for criteria C_2 by experts

C_2	E_1	E_2	E_3	E_4
A_1	(3/4, 5/5)	(4/5, 5/6)	(5/5, 6/7)	M
A_2	(4/5, 5/6)	(5/6, 6/7)	(5/5, 6/7)	AA
A_3	(6/7, 7/8)	(6/7, 7/8)	(8/8, 8/8)	H
A_4	P	VP	M	AA

(see Tables 3 and 4). The fuzzy rankings $\tilde{p}_{ik}$ can then be determined as

$$\tilde{p}_{11} = (4.25/5.00, 5.00/5.75), \quad \tilde{p}_{12} = (4.00/4.75, 5.00/6.00),$$

$$\tilde{p}_{21} = (5.25/6.00, 6.50/7.25), \quad \tilde{p}_{22} = (4.75/5.50, 6.00/7.00),$$

$$\tilde{p}_{31} = (3.25/4.00, 4.75/5.25), \quad \tilde{p}_{32} = (6.50/7.25, 7.50/8.25),$$

$$\tilde{p}_{41} = (5.00/6.00, 6.50/7.50), \quad \tilde{p}_{42} = (2.50/3.25, 4.25/5.25).$$

From these values it is clear that for criterion C_1, alternatives A_2 and A_4 are ranked higher. For criteria C_2, A_4 is ranked lower than other alternatives. Using Eqs. (13) and (14) final fuzzy weights ($\tilde{w}_i$) for alternatives (A_i) are calculated and represented in the form of Eq. (12) as shown below:

$$\tilde{w}_1 = (1.6625[0.0750, 0.7125]/2.4500, 2.8750/3.8000[0.0625, -0.9875]),$$

$$\tilde{w}_2 = (2.0250[0.0750, 0.8000]/2.9000, 3.6250/4.6500[0.0625, -1.0875]),$$

$$\tilde{w}_3 = (1.7875[0.0750, 0.7875]/2.6500, 3.3500/4.1625[0.0625, -0.9562]),$$

$$\tilde{w}_4 = (1.6250[0.0875, 0.7375]/2.4500, 3.2312/4.3125[0.0750, -1.1562]).$$

Let $\mu_{\tilde{M}}(x)$ and $\mu_{\tilde{m}}(x)$ be triangular membership functions (i.e., $r = 1$) and $w = w_i = 1.00$ (for $i = 1, 2, 3, 4$); $x_{\min} = 1.6250$ and $x_{\max} = 4.6500$. Then we find

$$X_{1R} = 3.2788; \quad X_{2R} = 3.8755, \quad X_{3R} = 3.5727, \quad X_{4R} = 3.5922,$$

$$X_{1L} = 2.2696; \quad X_{2L} = 2.6012, \quad X_{3L} = 2.4113, \quad X_{4L} = 2.2618$$

and

$$U_T(1) = 0.3799, \quad U_T(2) = 0.4917, \quad U_T(3) = 0.4520 \quad U_T(4) = 0.4304.$$

Therefore, the final ranking of the alternatives are:

Ranking:	1	2	3	4,
Alternative:	A_2	A_3	A_4	A_1.

4. Potential applications

Some of the potential applications of this method include:

(i) Grant proposal: Proposals – Alternatives; the agency awarding grants – DM; and the people who review the grants – Experts.

(ii) Environmental hazards: Chemicals that are harmful to environment – Alternatives; government agency – DM; and scientists whose expertise in this area is sought – Experts.

(iii) Energy development: Different types (nuclear, thermal, solar, hydro wind, etc.) of power – Alternatives; government (or private) agency – DM; and high-ranking officials in energy related industry – Experts.

(iv) River basin planning: Planning and development strategies – Alternatives; scientists and water resources specialists – Experts; and government agency – DM.

5. Conclusions

The methodology presented in this paper was successfully applied to a real-life situation, where ranking of river-basin planning and development alternatives is required and the results are presented by the authors elsewhere [2]. In this paper, a ranking methodology for a multiple-criterion decision-making problem is presented. Both qualitative and quantitative aspects can be handled in this method, employing experts opinion (preference structure) using fuzzy numbers and linguistic variables. Concepts of maximizing set and minimizing set were developed and used to arrive at the utility values to rank the alternatives. This method is very simple, straightforward and it overcomes the limitations of earlier methods mentioned in this paper. It is intuitive, computationally simple and easy to implement and has lot of potential for making policy decisions in a large-scale, real-life and complex problems.

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