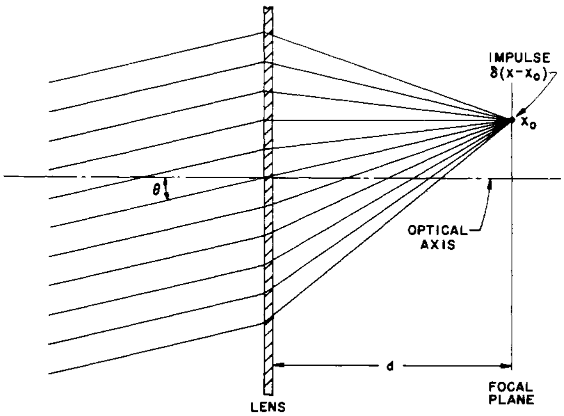

Fig. 1. Definitions of lens and impulse.

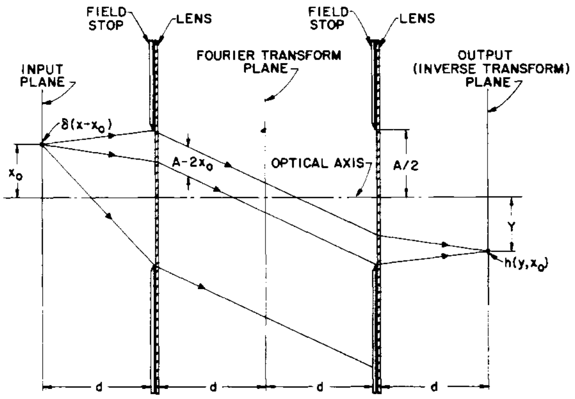

Fig. 2. Vignetting in the coherent processor.

Consider the special case when the input function is band-limited or if a band-limiting filter is placed in the Fourier-transform plane. Then the impulse response is shown to be input-position invariant for a limited range of input positions. In particular, assume that a filter is used to force the Fourier transform to be zero for spatial frequencies $|\omega| \geq B$. Then for certain values of x_0, the position of the input impulse, the effective width of the collimated beam is limited by this Fourier-plane filter and hence does not change with x_0. From geometric arguments similar to those used above, it can be shown that the impulse response is input-position invariant if

$$x_0 \leq \frac{A}{2} - B.$$

This illustrates an interesting tradeoff. As the bandwidth B approaches zero, the impulse response becomes input-position invariant over the maximum usable input range, $|x| \leq A/2$. Conversely, as the bandwidth approaches its maximum, $B \rightarrow A/2$, the impulse response becomes input-position sensitive everywhere on the input plane, and is given by (4).

In conclusion, the effects of vignetting by finite lens apertures on one-dimensional optical Fourier processors were considered. It was shown that for the general coherent optical processor, the usual convolution integral (2) cannot be applied because the impulse response is input-position sensitive. If an input-position invariant impulse response is desired, then the bandwidth of the Fourier-transform plane must be limited. In that case, there is a tradeoff between input size and bandwidth. It was also shown that those parts of the input which extend beyond the size of the lens apertures make no contribution to the output.

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Transient Response Evaluation from the State Transition Matrix

Abstract—A novel method for obtaining the transient response of a linear time-invariant system is presented. The main advantages of this method are that it eliminates the evaluation of the eigenvalues and also involves a minimal number of numerical steps. The method is illustrated by a specific example.

Introduction

For a linear time-invariant system described by the vector matrix equation

$$\dot{X} = AX + u(t), \tag{1}$$

the solution can be written as

$$X(t) = e^{AT}X_0(0) + \int_0^t e^{A(t-\tau)}u(\tau)\, d\tau. \tag{2}$$

$X(t)$ is the n-vector $\{x_1(t), x_2(t), \cdots x_n(t)\}$ specifying the state of the system, A is the constant coefficient matrix, and $u(t)$ is the input vector.

In the above equation e^{At}, which is a function of time, is the state transition matrix. This matrix completely establishes the state of a system in the time domain and for a system with no external input,

$$X(t) = e^{At}X(0). \tag{3}$$

The state transition matrix e^{At} can be expanded as an infinite series in terms of powers of t and that of matrix A as follows:

$$e^{At} = \sum_{n=0}^{n=\infty} \frac{1}{n!} A^n t^n \tag{4}$$

For a chosen interval of time T, which need not be very small, the series of equations (4) can be approximated by the first few terms retaining any desired accuracy. If e^{AT} is the matrix thus evaluated, the response vector $X(kT)$ for $k = 1, 2, 3, \cdots$ is given as

$$X(kT) = e^{kAT}X(0) \tag{5}$$

for a force-free system.

This was the method adopted by Liou,[1] and possesses distinct advantages over the analytical methods for the actual evaluation of the transient response. While the difficult step of the calculation of eigenvalues of the matrix A requiring the roots of an nth-order polynomial is eliminated in Liou's method, the requirement of convergence at e^{At} in (4) requires that for an interval T, a large number of terms are to be used unless T is chosen sufficiently small. In the example given by Liou, 10 terms in the summation are required, necessitating the evaluation of matrices A, A^2, $A^3, \cdots, A^{10}$. The method becomes a step-by-step procedure requiring the knowledge of the vector $X(\overline{n-1}\ T)$ to obtain $X(nT)$ from the equation

$$X(nT) = e^{AT}X(\overline{n-1}\ T). \tag{6}$$

The method described below is a modification of Liou's procedure making use of the Cayley-Hamilton theorem which states that every matrix satisfies its own characteristic equation. The entire procedure is illustrated in a series of steps as follows.

Step 1: For a given square matrix A of an nth-order system, compute $A^2, A^3, \cdots, A^{n-1}$.

Step 2: Set up the characteristic equation

$$(-1)^n|A - \lambda I| = \lambda^n + p_1\lambda^{n-1} + \cdots + p_n = 0 \tag{7}$$

where $p1, p2, \cdots, pn$ can be computed by the expansion of the determinant.

Step 3: Every square matrix satisfies its own characteristic equation. Hence,

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[1] M. L. Liou, "A novel method of evaluating transient response," *Proc. IEEE*, vol. 54, pp. 20–23, January 1966.

$$A^n = -p_1A^{n-1} - p_2A^{n-2} - \cdots - p_nI. \tag{8}$$

Successive multiplication of (8) by A gives rise to the following equations expressing A^{n+q} in terms of $A, A^2, \cdots, A^{n-1}$.

$$A^{n+q} = \alpha_{0q}I + \alpha_{1q}A + \alpha_{2q}A^2 + \cdots + \alpha_{n-1q}A^{n-1} \qquad q = 0, 1, 2, \cdots. \tag{9}$$

where $\alpha_{00} = -p_n, \alpha_{10} = -p_{n-1}, \cdots, \alpha_{n-1,0} = -p_1$, and $\alpha_{0q}, \alpha_{1q}, \cdots, \alpha_{n-1,q}$ are obtained from the recurrence relationships explained in Table I.

Step 4: For a given choice of T, e^{AT} can be written as

$$\begin{aligned} e^{AT} &= I + TA + \frac{T^2}{2!}A^2 + \frac{T^3}{3!}A^3 + \cdots + \frac{T^n}{n!}A^n + \cdots \\ &= I + TA + \frac{T^2}{2!} + \cdots + \frac{T^{n-1}}{(n-1)!}A^{n-1} + \cdots \\ &\quad + \frac{T^n}{n!}\left[\alpha_{00}I + \alpha_{10}A + \alpha_{20}A^2 + \cdots + \alpha_{n-1,0}A^{n-1}\right] \\ &\quad + \frac{T^{n+1}}{(n+1)!}\left[\alpha_{01}I + \alpha_{11}A + \alpha_{21}A^2 + \cdots + \alpha_{n-1,1}A^{n-1}\right] + \cdots \qquad (10) \\ &= I\left[1 + \frac{\alpha_{00}T^n}{n!} + \frac{\alpha_{01}T^{n+1}}{(n+1)!} + \cdots\right] \\ &\quad + A\left[T + \frac{\alpha_{10}T^n}{n!} + \frac{\alpha_{11}T^{n+1}}{(n+1)!} + \cdots\right] + \cdots \\ &\quad + A^{n-1}\left[\frac{T^{n-1}}{(n+1)!} + \frac{\alpha_{n-1,0}T^n}{n!} + \cdots\right]. \end{aligned}$$

From the already computed matrices $A^2, \cdots, A^{n-1}$ and the α's from Step 3, e^{AT} is quite easily calculated to any desired accuracy without actual series summation of powers of A higher than $(n-1)$.

Step 5: Assuming $e^{AT} = M(T)$ as evaluated in Step 4, compute M^2, $M^3, \cdots, M^{n-1}$ and evaluate the characteristic equation of M,

$$(-1)^n|M - \lambda I| = \lambda^n + m_1\lambda^{n-1} + \cdots + m_{n-1} = 0, \tag{11}$$

by expansion of the determinant.

Step 6:

$$X(kT) = M^k(T)X(0). \tag{12}$$

The matrices $M, M^2, \cdots, M^{n-1}$ have already been computed and M^k for $k \geq n$ can be obtained as follows. From (11),

$$M^n = -m_1M^{n-1} - m_2M^{n-2} - \cdots - m_nI \tag{13}$$

and

$$M^{n+p} = \beta_{0p}I + \beta_{1p}M + \cdots + \beta_{n-1,p}M^{n-1} \tag{14}$$

for $p = 0, 1, 2, 3 \cdots$ where $\beta_{00} = -m_n$, $\beta_{10} = -m_{n-1} \cdots \beta_{n-1,0} = -m_1$.

Table I also outlines the scheme for computation of $\beta_{0p}, \beta_{1p} \cdots \beta_{n-1,p}$, with α changed to β, q changed to p, and A changed to M.

Step 7: For $k \geq n$ $X(kT) = M^{n+p}X(0)$ where $p = k - n \geq 0$. Therefore

$$\begin{aligned} X(kT) &= \{\beta_{0p}I + \beta_{1p}M + \beta_{2p}M^2 + \cdots + \beta_{n-1,p}M^{n-1}\}X(0) \\ &= \beta_{0p}X(0) + \beta_{1p}X(T) + \beta_{2p}X(2T) + \cdots + \beta_{n-1,p}X(\overline{n-1}\,T). \end{aligned} \tag{15}$$

Equation (15) expresses the vector $x_1, x_2, \cdots, x_n$ at any instance of time $t = kT$ as a linear combination of n-vectors $X(0)$, $X(T) \cdots X(\overline{n-1}\,T)$ at $t = 0, T, \cdots, \overline{n-1}\,T$, respectively.

It should be emphasized that to compute the vector $X(t)$ at any instant, it is not necessary to compute its values at all instants up to t but only to find the coefficients $\beta_{0p}, \beta_{1p}, \cdots$ of (15) explained in Step 6. Thus, the method is not a step-by-step computation for the vector $X(t)$ but only for the coefficients β_{0p}, β_{1p}, etc. The example given below illustrates the procedure.

TABLE I

$\alpha_{0q} = -p_n\alpha_{0,q-1}$
$\alpha_{1q} = \alpha_{0,q-1} - p_{n-1}\alpha_{1,q-1}$
$\alpha_{2q} = \alpha_{1,q-1} - p_{n-2}\alpha_{2,q-1}$
$\alpha_{n-1q} = \alpha_{n-2,q-1} - p_1\alpha_{n-1,q-1}$

Example 1

The system considered is a third-order system described by $\dot{X} = AX$ where

$$X(0^+) = \begin{bmatrix} 2 \\ -2.5 \\ 3.75 \end{bmatrix}.$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.75 & -2.75 & -3 \end{bmatrix}.$$

This is the same example considered by Liou.

Step 1:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -0.75 & -2.75 & -3 \\ +2.25 & +7.5 & +6.25 \end{bmatrix}.$$

Step 2: The characteristic equation of A is

$$\lambda^3 + 3\lambda^2 + 2.75\lambda + 0.75 = 0.$$

Step 3:

$$\begin{aligned} A^2 &= -0.75I - 2.75A - 3A^2 \\ A^4 &= 2.25I + 7.5A + 6.25A^2 \\ A^5 &= 4.6875I - 14.9375A + 11.25A^2. \end{aligned}$$

Step 4: For $T = 0.1$

$$\begin{aligned} M = e^{AT} &= 0.999883996I + 0.09957171A + 0.00452513A^2 \\ &= \begin{bmatrix} 0.999884 & 0.0995717 & 0.00452513 \\ -0.00339385 & 0.987441 & 0.0859963 \\ -0.0644972 & -0.239884 & 0.729451 \end{bmatrix}. \end{aligned}$$

Step 5:

$$M^2 = \begin{bmatrix} 0.999138 & 0.196795 & 0.016388 \\ -0.012291 & 0.954072 & 0.147631 \\ -0.1123515 & -0.4182765 & 0.561178 \end{bmatrix}.$$

The characteristic equation of M is obtained by direct expansion of $|M - \lambda I|$ as

$$\lambda^3 - 2.716775\lambda^2 + 2.458239\lambda - 0.7408182 = 0.$$

Step 6: The coefficients are calculated as shown in the scheme of Table I for $n = 3$ with the starting values $\beta_{00} = 0.7408182$, $\beta_{10} = -2.458239$, $\beta_{20} = 2.716775$.

Step 7: The values of $x_1(t)$ computed at intervals of 0.1 are given in Table II and compared with the exact solution and also the solution obtained by Liou.

Conclusions

A simple and straightforward method is developed for obtaining the transient response of a linear time-invariant system with no external input.

TABLE II

$t=nT$	$x(nT)$	Exact solution	Results from Liou's method
0.0	2.00000	2.00000	2.00000
0.1	1.76781	1.76781	1.76781
0.2	1.56774	1.56774	1.56775
0.3	1.39515	1.39515	1.39515
0.4	1.24603	1.24604	1.24604
0.5	1.11700	1.11700	1.11701
0.6	1.00514	1.00515	1.00515
0.7	0.907977	0.907979	1.907982
0.8	0.823377	0.823379	0.823383
0.9	0.749536	0.749538	0.749542
1.0	0.684908	0.684912	0.684914

The method avoids the necessity of calculating the eigenvalues of the system requiring the roots of a polynomial and ultimately reduces to simple steps of multiplication and addition which can be done even on a desk calculator. The method consists of accurate evaluation of the state transition matrix e^{AT} for a chosen time interval T and makes use of certain recurrence relationships which hold for the exponential matrix. The method is quite accurate and the number of multiplications and additions per step is small compared to other methods.

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Observation of the Current and Voltage Waveforms of the Si IMPATT Diode

Abstract—**Current and voltage waveforms of the Si IMPATT diode were observed directly by means of the oscillating circuit using microstrip line. The results indicate that the conventional small-signal theory cannot be applied to the observed type of oscillation. The oscillation starts at the bias voltage just above the breakdown voltage of the diode; then along with its buildup, the bias voltage is lowered owing to the auto-bias effect, to reach a steady value considerably below the breakdown voltage. Large amplitude oscillation of high efficiency is expected over a wide frequency range.**

The circuit configuration and measurement apparatus used to observe the current waveform are shown in Fig. 1. This system is essentially the same as that used in a previous report to observe the current and voltage waveforms of the Gunn diode.[1] The voltage waveform is picked up through a differential probe with capacitive coupling, and fed to a sampling scope, whose recorder output is integrated by a Miller integrator and plotted on an X-Y recorder. The characteristic of this detecting circuit was checked by verifying that the voltage pulse with fast rise time (0.25 ns) applied to the circuit was exactly reproduced on the X-Y recorder.

Typical voltage and current waveforms observed are shown in Fig. 2. The essential feature of this type of oscillation is its involvement with numerous higher harmonic components. The voltage applied to the diode is lower than the breakdown voltage on the average, and exceeds it only instantaneously in each cycle. As seen in the figure, the dc current level is approximately zero. From these facts, it can be expected that this type of oscillation is highly efficient.

As shown in the current waveforms in Fig. 3, shortening the cavity length leads to an increase in the oscillation frequency, and amplitude re-

Manuscript received Nobember 25, 1968.

[1] H. Yanai, T. Ikoma, and H. Torizuka, "Current observation of the electron-transferred oscillation," presented at the 7th Internat'l Conf. on Microwave and Optical Generation and Amplification, September 1968.

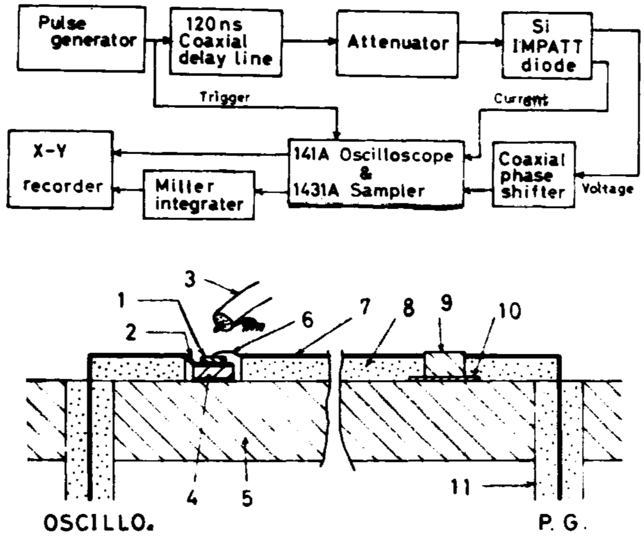

Fig. 1. Circuit configuration and measurement apparatus.
1. Diode
2. Au ribbon
3. Probe
4. Resistor
5. Ground plane
6. Au lead
7. Strip conductor
8. Teflon
9. Al foil
10. Polyethylene
11. Coaxial cable

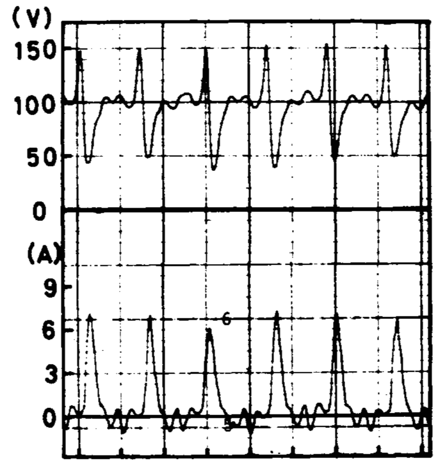

Fig. 2. Voltage and current waveforms. Breakdown voltage at 130 V. Horizontal: 1 ns/div. Upper trace: voltage waveform. Lower trace: current waveform.

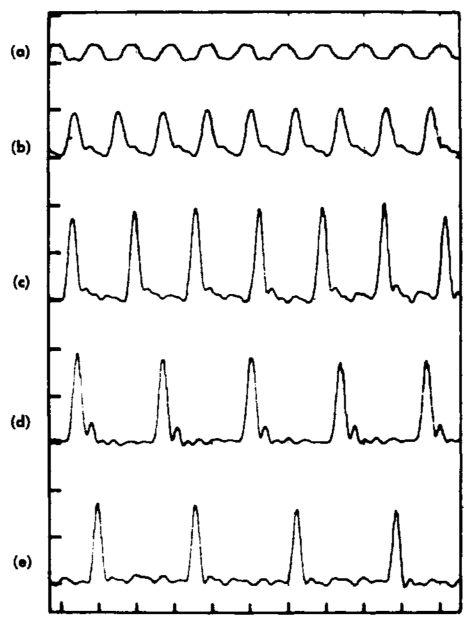

Fig. 3. Current waveforms with changes in cavity length. (a) 6 cm. (b) 8 cm. (c) 12 cm. (d) 16 cm. (e) 20 cm. Vertical: 5A/div. Horizontal: 0.5 ns/div.