$-\partial\psi/\partial x \equiv -\psi_x$, ν is the kinematic viscosity of the liquid and ∇ is the two-dimensional gradient operator.

Cantwell [1] has reported a 10-parameter Lie group of space-time coordinates transformations that leaves (1) invariant. Infinitesimally, this 10-parameter Lie group is described by

$$\begin{cases} \tilde{x} = x + \epsilon\xi + O(\epsilon^2), \\ \tilde{y} = y + \epsilon\rho + O(\epsilon^2), \\ \tilde{t} = t + \epsilon\tau + O(\epsilon^2), \\ \tilde{\psi} = \psi + \epsilon\eta + O(\epsilon^2), \end{cases} \tag{2}$$

where

$$\begin{cases} \xi = ax + by + cty + f_1(t) + d \\ \rho = -bx + ay - ctx + f_2(t) + e \\ \tau = 2at + h \\ \eta = 1/2c(x^2 + y^2) - \dot{f}_2(t)x + \dot{f}_1(t)y + s(t) + p. \end{cases} \tag{3}$$

a, b, c, d, e, h, p are constants and $f_1(t)$, $f_2(t)$, and $s(t)$ are arbitrary functions of t. In (3), the dot denotes derivative with respect to time.

It is the purpose of this communication to show that the plane squeeze film flow of a viscous fluid admits a similarity solution if the velocity of approach of the plates is proportional to $(2at + h)^{-1/2}$.

To do that, we need to find a particular subgroup of (3) that leaves the boundary conditions of the plane squeezing flow invariant. Now the velocity field at the upper plate ($y = H(t)$) must satisfy

$$u(x, y = H(t)) = \psi_y\,(x, y = H(t)) = 0 \tag{4}$$

$$v(x, y = H(t)) = -\psi_x\,(x, y = H(t)) = -\dot{H}(t) \tag{5}$$

Invariance of the boundary curve $y = H(t)$ implies

$$\tilde{y} = H(\tilde{t})$$

that is,

$$\rho(x, y = H(t)) = \dot{H}(t)\,\tau(x, y = H(t))$$

which requires that

$$-bx + aH(t) - ctx + f_2(t) + e = \dot{H}(t)(2at + h).$$

For this to be satisfied identically we need $b = 0 = c$ and

$$f_2(t) = \dot{H}(t)(2at + h) - aH(t) - e. \tag{6}$$

Next, invariance of (4) implies

$$\tilde{\psi}_{\tilde{y}}\,(x, y = H(t)) = 0,$$

which requires that

$$\frac{D\eta}{Dy} - \frac{D\xi}{Dy}\psi_x - \frac{D\rho}{Dy}\psi_y - \frac{D_\tau}{Dy}\psi_t = 0 \tag{7}$$

at $y = H(t)$. D/Dx_i is the total derivative (Bluman and Cole [2]) defined in the four-dimensional space (x, y, t, ψ) by

$$\frac{D}{Dx_i} \equiv \frac{\partial}{\partial x_i} + \psi_i\frac{\partial}{\partial\psi}, \quad x_i \equiv x, y, \text{ or } t.$$

Condition (7) requires that $\dot{f}_1 = 0$ or $f_1(t) =$ constant.

Invariance of the second boundary condition (5) requires that

$$\tilde{\psi}_{\tilde{x}}\,(x, y = H(t)) = \dot{H}(\tilde{t})$$

which implies

$$\frac{D\eta}{Dx} - \frac{D\xi}{DX}\psi_x - \frac{D\rho}{Dx}\psi_y - \frac{D\tau}{Dx}\psi_t = \ddot{H}(t)\tau \quad \text{at} \quad y = H(t),$$

that is,

$$\dot{f}_2(t) = -\ddot{H}(t)(2at + h) - a\dot{H}(t). \tag{8}$$

Compatibility between (8) and (5) dictates that the normal velocity of approach of the plates must satisfy

$$\ddot{H} + \frac{a}{2at + h}\dot{H} = 0,$$

that is,

$$\dot{H}(t) = q(2at + h)^{-1/2}, \quad H(t) = \frac{q}{a}(2at + h)^{1/2} \tag{9}$$

that is, if $\dot{H}(t)$ is given by (9), then the plane squeezing flow admits similarity solutions described by the following 6-parameter Lie group of transformations

$$\begin{cases} \xi = ax + \beta \\ \rho = ay \\ \tau = 2at + h \\ \eta = s(t) + p \end{cases} \tag{10}$$

For example, the case where $\beta = p = 0 \equiv s(t)$ admits the following similarity solution:

$$\frac{dx}{ax} = \frac{dy}{ay} = \frac{dt}{2at + h} = \frac{d\psi}{0} \tag{11}$$

that is,

$$\psi = F(\xi_1, \xi_2), \tag{12}$$

where

$$\xi_1 = \frac{x}{\sqrt{2at + h}} \quad \text{and} \quad \xi_2 = \frac{y}{\sqrt{2at + h}} \tag{13}$$

are two invariants of (11).

Substitution of (12) into (1) results in a reduction in the order of the p.d.e. Alternatively, the velocities can be scaled appropriately according to (12) and their substitution into the Navier-Stokes equations results in an ordinary differential equation. This equation has been studied extensively by Wang [3] who found that when H is proportional to $\sqrt{1 - \alpha t}$ a similarity solution for the plane (and circular) squeezing flow is possible. The flow is then described by a single parameter $S = \alpha R^2/\nu$, where R is a length scale. Among many other things reported in this paper, Wang has shown numerically that the squeezing force may not necessarily follow the direction of approach for certain values of S.

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Torsional Vibrations of Poroelastic Cylinders

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Introduction

The study of torsional vibrations is of importance, both from theoretical and practical considerations. Such vibrations, for example,

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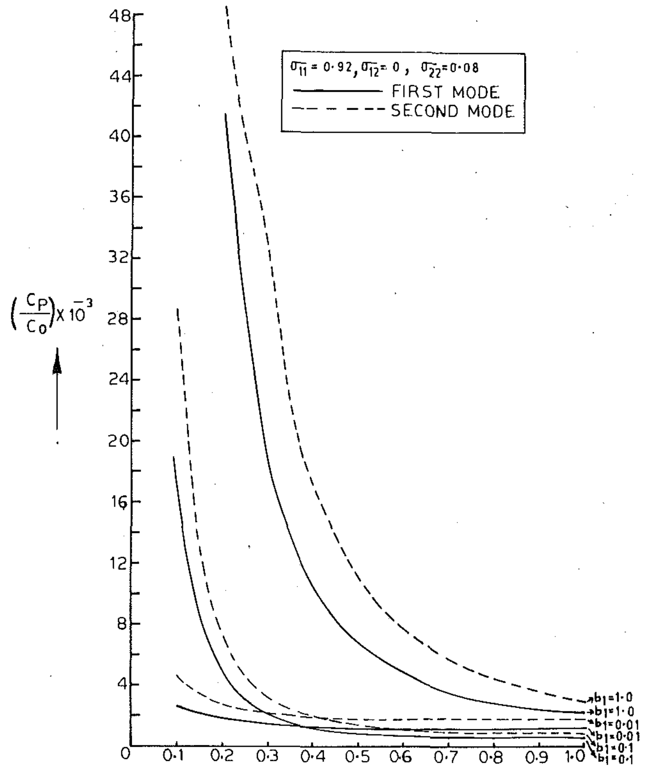

Fig. 1

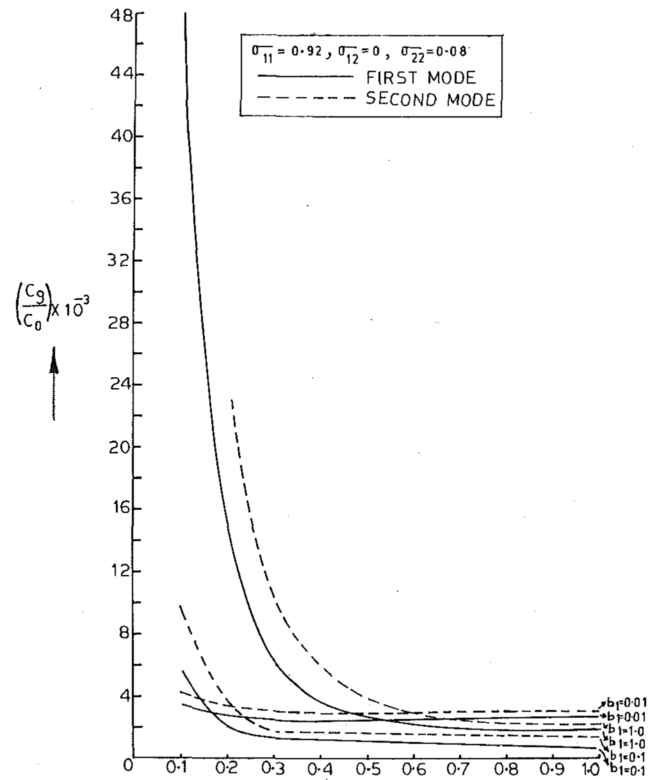

Fig. 2

are used in delay lines. Further, based on the reflections and refractions during the propagation of a pulse, imperfections can be identified. Still another use of torsional vibrations is the measurement of the shear modulus of a crystal.

In this Note, torsional vibrations of an infinite, isotropic, homogeneous poroelastic cylinder are studied. Plots of nondimensional phase velocity, group velocity, and wavelength as a function of nondimensional frequency are presented.

Solution of the Problem

Let r, θ, z be cylindrical polar coordinates with z-axis along the axis of the cylinder. The nonzero displacement component of solid u_θ and liquid U_θ are to be determined from

$$N(\nabla^2 - r^{-2})u_\theta = \frac{\partial^2}{\partial t^2}(\rho_{11}u_\theta + \rho_{12}U_\theta) + b\frac{\partial}{\partial t}(u_\theta - U_\theta)$$

$$0 = \frac{\partial^2}{\partial t^2}(\rho_{12}u_\theta + \rho_{22}U_\theta) - b\frac{\partial}{\partial t}(u_\theta - U_\theta). \quad (1)$$

Here ρ_{11}, ρ_{12}, and ρ_{22} are mass densities as introduced in [1], N is a shear modulus, b is a dissipation coefficient, and ∇^2 is the Laplacian operator. From the conditions of stress-free curved surface, the frequency equation of torsional vibrations of a circular poroelastic cylinder of radius a is

$$J_2(R) = 0, \quad (2)$$

where J_2 is the Bessel function of first kind and of order two.

The propagation mode shapes are given by

$$u_\theta = \begin{cases} C_1 J_1(k_n r)\exp[i(\alpha z + pt)] & \text{when} \quad k_n \neq 0 \\ C_1 r \exp[i(\alpha z + pt)] & \text{when} \quad k_n = 0, \end{cases} \quad (3)$$

where α is the wave number, p is the frequency, and J_1 is the Bessel function of first kind and of order one. In these equations, R_n is the nth nonzero root of equation (2) and

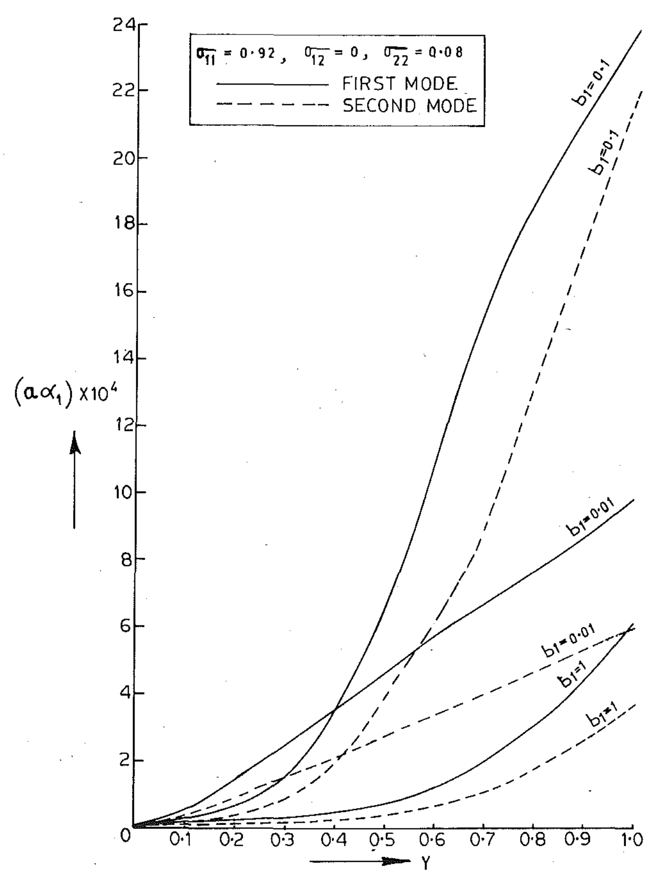

Fig. 3

$$R_n{}^2 = k_n{}^2 a^2, \quad k_n{}^2 = \frac{p^2(\tau_{11}\tau_{22} - \tau_{12}{}^2)}{N\tau_{22}} - \alpha^2, \tag{4}$$

where

$$\tau_{11} = \rho_{11} - \frac{ib}{p}, \quad \tau_{12} = \rho_{12} + \frac{ib}{p}, \quad \tau_{22} = \rho_{22} - \frac{ib}{p}.$$

The roots of equation (2) are well known.

On combining and rearranging equations (4), we can write

$$\frac{N(R_n{}^2 + a^2\alpha^2)}{a^2p^2\rho} = E_r - iE_i,$$

where

$$E_r = \frac{y^2\sigma_{22}(\sigma_{11}\sigma_{22} - \sigma_{12}{}^2) + b_1{}^2}{y^2\sigma_{22}{}^2 + b_1{}^2},$$

$$E_i = \frac{yb_1(\sigma_{12} + \sigma_{22})^2}{y^2\sigma_{22}{}^2 + b_1{}^2}. \tag{5}$$

b_1, y, σ_{ij} are nondimensional dissipation coefficient, frequency, and mass densities, respectively, defined by

$$b_1 = \frac{ab}{\rho c_0}, \quad y_1 = \frac{ap}{c_0}, \quad \sigma_{ij} = \frac{\rho_{ij}}{\rho}, \quad \rho = \rho_{11} + 2\rho_2 + \rho_{22}, \quad c_0{}^2 = \frac{N}{\rho}.$$

Because of the dissipative nature of the medium, in general, the wave number α is complex [1]. Letting

$$\alpha = \alpha_r + i\alpha_i,$$

then phase velocity $c_p (= p/|\alpha_r|)$ is given by

$$c_p/c_0 = 2^{1/2}y(B_1 + B_2)^{-1/2}. \tag{6}$$

The group velocity is

$$c_g/c_0 = 2^{3/2}B_3{}^{-1}(B_1 + B_2)^{1/2}. \tag{7}$$

The attenuation x_a $(= 1/|\alpha_i|)$ is

$$x_a/a = 2^{1/2}(B_1 - B_2)^{-1/2}, \tag{8}$$

where

$$B_1 = \{y^4(E_r{}^2 + E_i{}^2) - 2y^2E_rR_n{}^2 + R_n{}^4\}^{1/2}, \quad B_2 = y^2E_r - R_n{}^2,$$

$$B_3 = y^2G_1(1 + y^2E_rB_1{}^{-1} - R_n{}^2B_1{}^{-1}) + 2yE_r(1 - R_n{}^2B_1{}^{-1}) + y^3B_1{}^{-1}(yE_iG_2 + 2E_r{}^2 + 2E_i{}^2),$$

$$G_1 = \frac{2b_1{}^2(E_r - 1)}{y(y^2\sigma_{22}{}^2 + b_1{}^2)}, \quad G_2 = \frac{(b_1{}^2 - y^2\sigma_{22}{}^2)E_i}{y(y^2\sigma_{22}{}^2 + b_1{}^2)}. \tag{9}$$

It is observed that the square of the wave number is the average of B_1 and B_2.

Discussions

In the general case, even the least mode is observed to be dispersive where as it is nondispersive in the absence of dissipation. Consequently, the least mode can be used in delay lines [2]. In higher modes vibrations are dispersive. Phase velocity, group velocity, wavelength are calculated for different values of frequency for a cylindrical bone whose parameters are given in [3] and are presented graphically. From Fig. 1, it is observed that when dissipative coefficient increases from 0.01 to 0.10, the phase velocity curves of first and second modes intersect around the wavelength (= y) is equal to 0.4. For wavelength greater than 0.4 phase velocity is decreasing in both the modes and when dissipative force is equal to 1, the wave velocity is higher than in all other cases. The group velocity and wavelength are given in Figs. 2 and 3. When the values of dissipative force are small, the graphs for wavelength are straight lines and their slope increases with increasing b_1.

In absence of dissipative force vibrations are not attenuated and the same conclusions as that of classical theory are valid.

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A Condition of Bending-Free Torsion to Define the Center of Twist

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Introduction

A beam subjected to terminal tractions and displacement restraints will in general experience direct and shearing stresses dependent on the magnitude and distribution of the stresses over the end surface rather than the types of forces and couples producing these terminal stresses. However, it is convenient for the engineer to be able to identify the stress resultants in terms of the applied loads. Thus a cantilevered beam subjected to a terminal shearing force [1] will experience direct and shearing stresses and it is convenient to differentiate between direct stresses due to bending and warping restraints, and shearing stresses due to shear and torsion. Toward this end the center of flexure is defined as that point through which the terminal shearing force must pass in order to produce "torsion-free bending," a state usually defined [2] by zero overall local rotation of the section which is mathematically equivalent to zero rotation of the centroid of the section, vanishing of shearing stresses due to torsion and hence zero torsional stress resultant.

The center of twist is usually defined [3] according to a minimum potential energy of warping, which can easily be shown to correspond mathematically to rotation about an axis such that the warping integral will be a minimum. The relationship between the two centers as defined previously is shown in [4]. The exact solution for torsion with restrained warping is not known, but since restraint gives rise to axial direct stresses it seems natural to investigate the condition under which these stresses do not constitute a resultant bending moment, equivalent to "bending-free torsion," and it is shown that this leads to coordinates of the center of twist agreeing with those obtained on the basis of minimum warping energy.

Theory

If x and y are the principal axes and z coincides with the axis of centroids then for a uniform isotropic rod the most general form of the displacements during twist are [3]

$$u = \frac{d\theta}{dz}(-zy + a + qz - ry) \tag{1a}$$

$$v = \frac{d\theta}{dz}(zx + b + rx - pz) \tag{1b}$$

$$w = \frac{d\theta}{dz}(\phi(x,y) + c + py - qx) \tag{1c}$$

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