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Brief Paper

Bifurcation test functions and surge control for axial flow compressors[1]

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Abstract

Surge control is investigated in conjunction with rotating stall control for axial flow compressors using a bifurcation approach. Test functions are developed to determine the existence and stability of the Hopf bifurcation associated with surge for closed-loop systems under linear state feedback. A control design method is proposed for the synthesis of linear feedback laws that eliminate surge, coupled with rotating stall for any given compact parameter set. Comparisons are made with existing results. Stabilization results are demonstrated with numerical simulations. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

It is well known that the compression system exhibits nonlinear hysteresis at the operating point of maximum pressure rise that is termed rotating stall, and nonlinear flow oscillation in the vicinity of the maximum pressure rise that is called surge. The nonlinear phenomena of rotating stall and surge are flow instabilities that effectively reduce the performance of aeroengines and limit further improvements on reliability and efficiency of future jet airplanes. Hence the suppression of rotating stall and surge is the key issue for compressor control.

This paper focuses on bifurcation control for axial flow engine compressors. The Moore–Greitzer model from McCaughan (1990) is adopted for analysis. As shown by McCaughan (1990), rotating stall is associated with transcritical and saddle-node bifurcations, and surge with a Hopf bifurcation. This motivated the bifurcation approach to compressor control in the past several years. Feedback control of the transcritical bifurcation at the axisymmetric equilibrium point has been studied by many authors. See for instance, Liaw and Abed (1996), Krener (1995), Eveker et al. (1995), Gu et al. (1996) and Kang (1995). There exists a family of state feedback laws which stabilize the nonaxisymmetric equilibria near the operating point and eliminate the hysteresis induced by rotating stall. However, the Hopf bifurcation associated with surge remains intact under these feedback laws. In this paper, we introduce test functions whose zeros are critical to the Hopf bifurcation for the closed-loop system. These test functions are given in compact form. A particular test function is developed to determine the stability of the periodic solutions born at a Hopf bifurcation. The analysis based on these test functions leads to a new method of feedback design for control of both stationary and Hopf bifurcations in axial flow compressors. In fact, using the techniques proposed in this paper, feedback controllers can be designed to meet several bifurcation control requirements, including elimination of surge, coupled with rotating stall.

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The paper is organized as follows. In Section 2, the three-state Moore–Greitzer model is introduced and the feedback control laws from Krener (1995) are presented that soften the transcritical bifurcation. However, the closed-loop system may still exhibit a Hopf bifurcation associated with surge. In Section 3, a test function is developed for the existence of Hopf bifurcation in the closed-loop compression system. The existence of a stationary bifurcation is studied in Section 4 where a different test function is derived. The results of these two sections show that, for any given compact set of parameters, there exists a linear feedback law such that the closed-loop system does not have a stationary bifurcation, nor a Hopf bifurcation along the curve of nonaxisymmetric equilibria. In Section 5, a particular test function is developed to determine the stability of the family of periodic solutions born at the critical point of a Hopf bifurcation. Finally in Section 6, a linear feedback law is designed for a compression system, and simulation results are used to show the stability of the closed-loop system, as well as feedback stabilization in the presence of both rotating stall and surge. Comparisons are made with existing results in Krstic et al. (1995) and Eveker et al. (1995).

2. Background

This section introduces the three-dimensional Moore–Greitzer model for axial flow compressors. Some key results of McCaughan (1990), Liaw and Abed (1996), and Krener (1995) are reviewed. The compression system in consideration is described by ordinary differential equations (ODEs):

$$\begin{aligned} \dot{R} &= \sigma R(1-\Phi^2-R), \\ \dot{\Phi} &= -\Psi+\Psi_c(\Phi)-3\Phi R, \\ \dot{\Psi} &= \frac{1}{\beta^2}(\Phi-\Phi_T(\Psi)), \end{aligned} \tag{2.1}$$

where the performance and throttle characteristics are of the form

$$\Psi_c(\phi)=\Psi_{c0}+1+\tfrac{3}{2}\phi-\tfrac{1}{2}\phi^3, \qquad \Phi_T=\sqrt{\gamma\Psi}-1. \tag{2.2}$$

All variables are nondimensionalized with Φ corresponding to annulus averaged mass flow rate, Ψ the pressure rise, and R the square amplitude of the rotating stall cell. The parameter Ψ_{c0} reflects the engineering nature of the compressor, and β combines a group of parameters including wheel speed, plenum volume as well as the cross-sectional area of the compressor. It is assumed that the throttle parameter γ can be decomposed into $\sqrt{\gamma}=u+\mu$, where u is proportional to the cross-sectional area of bleed valves and used as actuator, and μ is a parameter synthesized the effect of disturbances from inlet and combustion chamber. The compressor model described in Eqs. (2.1) and (2.2) can then be written into

$$\begin{bmatrix}\dot{R}\\ \dot{\Phi}\\ \dot{\Psi}\end{bmatrix}=f(R,\Phi,\Psi)+g(R,\Phi,\Psi)(u+\mu), \tag{2.3}$$

$$f(R,\Phi,\Psi)=\begin{bmatrix}\sigma R(1-\Phi^2-R)\\ -\Psi+\Psi_c(\Phi)-3\Phi R\\ \dfrac{1}{\beta^2}(\Phi+1)\end{bmatrix},$$

$$g(R,\Phi,\Psi)=\begin{bmatrix}0\\ 0\\ -\sqrt{\Psi}\end{bmatrix}. \tag{2.4}$$

Following the notation in Kang (1995), the equilibrium set of the control system is given by

$$E=\{(R,\Phi,\Psi,\mu)\,|\,\exists u_0 \text{ such that } f(R,\Phi,\Psi)+g(R,\Phi,\Psi)(u_0)=0\}. \tag{2.5}$$

Clearly $R<0$ has no physical meaning. Therefore, we only focus on the subset of E such that $R\geq 0$. It has two branches governed by

$$\begin{aligned} &\Psi_e=\Psi_c(\Phi_e), \quad R_e=0, \\ &(u+\mu)_e=\frac{\Phi_e+1}{\sqrt{\Psi_e}}, \quad \Phi_e\in\mathbb{R}. \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\Psi_e=\Psi_{c0}+1-\frac{3}{2}\Phi_e+\frac{5}{2}\Phi_e^3, \quad R_e=1-\Phi_e^2, \\ &(u+\mu)_e=\frac{\Phi_e+1}{\sqrt{\Psi_e}}, \quad -1<\Phi_e<1. \end{aligned} \tag{2.7}$$

The equilibrium subset for $R\geq 0$ in the $R\Phi\Psi$-space is shown in Fig. 1.

An *axisymmetric* equilibrium point is a point in E with $R=0$ (defined by Eq. (2.6)), and a *nonaxisymmetric* equilibrium point is a point in E with $R>0$ (defined by Eq. (2.7)). On the branch of axisymmetric equilibrium points, the maximum pressure rise occurs at the following critical operating point:

$$\Phi_0=1, \qquad \Psi_0=\Psi_{c0}+2, \qquad R_0=0. \tag{2.8}$$

The dotted curve in Fig. 1 consists of equilibrium points that are not stabilizable. The system has a bifurcation at point (2.8) where two branches of the equilibrium points meet. The bifurcations of system (2.1) have been thoroughly studied by McCaughan (1990), and are briefly summarized as follows:

(i) System (2.1) has a transcritical bifurcation at critical point (2.8).

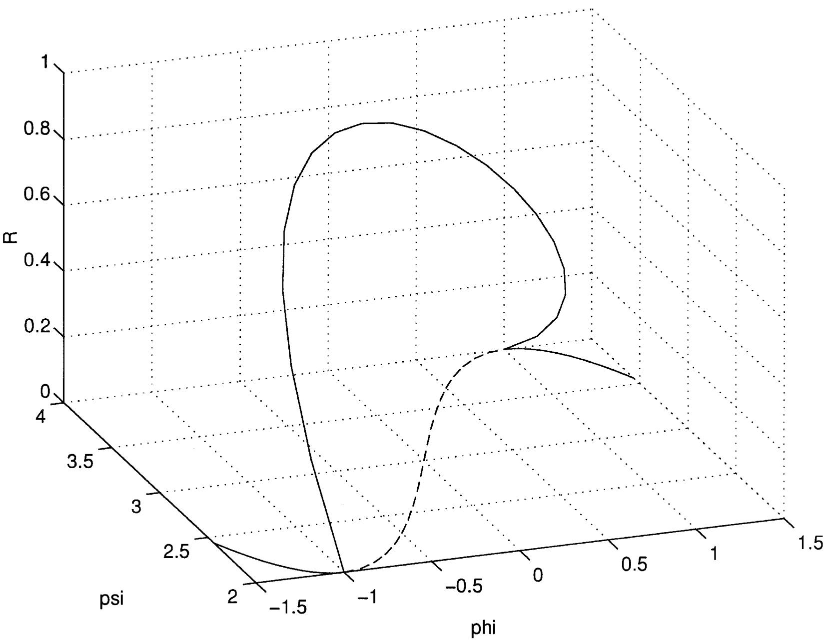

Fig. 1. The equilibrium set of E for the compressor system in $R\Phi\Psi$-space, $\Phi_{c0} = 2$.

(ii) If $\Psi_{c0} < 4$, there exists a point on the curve (2.7) at which Eq. (2.1) has a saddle-node bifurcation.

(iii) At some points of Eq. (2.7), the system exhibits Hopf bifurcation at critical values of β which are dependent on the equilibrium point.

Bifurcations (i) and (ii) cause the hysteresis on the plane of the compressor characteristic curve, which reduce the performance of the compressor. The periodic solutions due to the Hopf bifurcation (iii) reflect the surge with rotating stall, which can cause catastrophic consequences or severe damage to aeroengines. The feedback control of the bifurcation at point (2.8) has been studied by many authors. The goal is to find a feedback law such that it locally stabilizes the equilibrium points for the case $R > 0$. If such a feedback law exists, the transcritical bifurcation (i) is said to be *softened*. All linear state feedback laws that soften the bifurcation (i) are characterized by Krener (1995), and restated in Theorem 2.1.

This paper focuses on the detection and control of the bifurcations (i)–(iii). A fundamental difference between the transcritical bifurcation (i) and the bifurcations (ii) and (iii) is that the system is linearly uncontrollable at point (2.8), but linearly controllable at all points of Eq. (2.7). At a nonaxisymmetric equilibrium point, a state feedback law can remove or delay the bifurcation so that they do not occur for a given range of the parameters. The proposed multipurpose feedback design method leads to state feedback which fulfills several bifurcation control requirements. A feedback introduced in Section 4 is effective for the simultaneous control of the bifurcations (i)–(iii).

The feedback control law under investigation has the form

$$u = \alpha(R, \Phi, \Psi), \tag{2.9}$$

where α is a C^1 function. Its linear approximation at critical point (2.8) is given by

$$\alpha(R, \Phi, \Psi) = k_1 R + k_2(\Phi - \Phi_0) + k_3(\Psi - \Psi_0) + O(R, \Phi - \Phi_0, \Psi - \Psi_0)^2. \tag{2.10}$$

The state feedback law for control of the transcritical bifurcation (i) is summarized in the following theorem. Its proof can be found in Krener (1995). The feedback law in Theorem 2.1 stabilizes critical operating point (2.8), and the nonaxisymmetric equilibrium points locally. For this purpose, the nonlinear terms in the feedback do not matter. More general results on bifurcation stabilization can be found in Kang (1995) and Gu et al. (1997).

Theorem 2.1. *Suppose that the feedback law of Eqs.* (2.9) *and* (2.10) *satisfies*

$$k_2 < \Psi_0^{-1/2},\ k_3 > -\Psi_0^{-3/2},\ 2k_1 - k_2 - 6k_3 > \frac{6 - \Psi_0}{\Psi_0^{-3/2}}. \tag{2.11}$$

Then closed-loop system (2.3)–(2.9) *has a transcritical bifurcation at the operating point* (2.8). *Moreover, the system is locally asymptotically stable at the points of Eq.* (2.7) *in the vicinity of the critical point* (2.8), *and it is asymptotically stable at the points of Eq.* (2.6) *with* $\Phi > \Phi_0$.

3. Test function for the existence of Hopf bifurcations

In this section, a test function is sought for the existence of Hopf bifurcations at points (2.7). It is known that a system has a Hopf bifurcation if its linearization has a pair of imaginary eigenvalues. The following gives a sufficient condition for the existence of imaginary eigenvalues in a 3×3 real matrix. Its algebraic proof is omitted.

Lemma 3.1. *Given a 3×3 real matrix A. Suppose that $\det(A-\mathrm{trace}\{A\}I)\neq 0$, where I denotes a 3×3 identity matrix. Then the matrix A has no eigenvalues on the imaginary axis.*

Define the function F by

$$F(R,\Phi,\Psi,\mu):=\frac{1}{\beta^2}(\Phi-\sqrt{\Psi}(\alpha(R,\Phi,\Psi)+\mu)+1). \quad (3.1)$$

Denote F_R, F_Φ, and F_Ψ the partial derivatives of F with respect to R, Ψ, and Ψ, respectively. Substituting feedback law (2.9) into the system (2.1), and linearizing the closed-loop system at the equilibrium points (2.7) give an A matrix of the form

$$A=\begin{bmatrix} -\sigma R_e & -2\sigma R_e\Phi_e & 0 \\ -3\Phi_e & -\frac{3}{2}R_e & -1 \\ F_R & F_\Phi & F_\Psi \end{bmatrix}. \quad (3.2)$$

Elementary row operations transform the matrix $A-\mathrm{trace}\{A\}I$ into

$$\begin{bmatrix} \frac{3}{2}R_e-F_\Psi & -2\sigma R_e\Phi_e & 0 \\ -3\Phi_e & \sigma R_e-F_\Psi & -1 \\ F_R-3\Phi_e(\sigma R_e+\frac{3}{2}R_e) & F_\Phi+(\sigma R_e-F_\Psi)(\sigma R_e+\frac{3}{2}R_e) & 0 \end{bmatrix}.$$

Thus the determinant of $A-\mathrm{trace}\{A\}I$ is obtained as

$$F_\Phi(\tfrac{3}{2}R_e-F_\Psi)+(\tfrac{3}{2}R_e+\sigma R_e)(\sigma R_e-F_\Psi)(\tfrac{3}{2}R_e-F_\Psi)$$
$$+2\sigma R_e\Phi_eF_R-6\sigma R_e\Phi_e^2(\tfrac{3}{2}R_e+\sigma R_e).$$

Multiplying the above by β^4 and making substitution:

$$F_R(R_e,\Phi_e,\Psi_e)=-\beta^{-2}\sqrt{\Psi_e}\alpha_R,$$

$$F_\Phi(R_e,\Phi_e,\Psi_e)=\beta^{-2}(1-\sqrt{\Psi_e}\alpha_\Phi),$$

$$F_\Psi(R_e,\Phi_e,\Psi_e)=-\beta^{-2}\left(\frac{\alpha(R_e,\Phi_e,\Psi_e)+\mu_e}{2\sqrt{\Psi_e}}+\sqrt{\Psi_e}\alpha_\Psi\right),$$

$$\alpha(R_e,\Phi_e,\Psi_e)+\mu_e=\frac{\Phi_e+1}{\sqrt{\Phi_e}}, \quad (3.3)$$

we obtain the test function

$$\begin{aligned}T_{\mathrm{Hopf}}(\Phi_e,\beta)&=(1-\sqrt{\Psi_e}\alpha_\Phi)(\tfrac{3}{2}R_e\beta^2+\frac{\Phi_e+1}{2\Psi_e}\\&\quad+\sqrt{\Psi_e}\alpha_\Psi)-2\beta^2\sigma R_e\Phi_e\sqrt{\Psi_e}\alpha_R\\&\quad-6\beta^4\sigma R_e^2\Phi_e^2(\tfrac{3}{2}+\sigma)\\&\quad+R_e(\tfrac{3}{2}+\sigma)(\sigma R_e\beta^2+\frac{\Phi_e+1}{2\Psi_e}\\&\quad+\sqrt{\Psi_e}\alpha_\Psi)\left(\frac{2}{3}R_e\beta^2+\frac{\Phi_e+1}{2\Psi_e}+\sqrt{\Psi_e}\alpha_\Psi\right).\end{aligned} \quad (3.4)$$

Since Ψ_e and R_e are functions of Φ_e by Eq. (2.7), the function T_{Hopf} has only two independent variables Φ_e and β. The following result follows from Lemma 3.1.

Theorem 3.1. *Suppose that*

$$T_{\mathrm{Hopf}}(\Phi_e,\beta)\neq 0,\quad \mathrm{for}(\Phi_e,\beta)\in(\Phi_{e1},\Phi_{e2})\times(0,\beta_1]. \quad (3.5)$$

where $-1\le\Phi_{e1}<\Phi_{e2}\le 1$ and $\beta_1>0$. Then closed-loop system (2.3)–(2.9) *has no Hopf bifurcation at the points* (2.7) *for $\beta\in(0,\beta_1]$ and $\Phi_e\in(\Phi_{e1},\Phi_{e2})$.*

Remark. The condition in Theorem 3.1 is sufficient. However, it becomes necessary if no stationary bifurcations exist at points of Eq. (2.7), and if the feedback softens the transcritical bifurcation at point (2.8). More specifically, if the feedback satisfies Eqs. (2.11) and (4.3) in the next section, then condition (3.5) holds if and only if the system (2.3)–(2.9) has no Hopf bifurcation at the points of Eq. (2.7) for the given interval of β.

Remark. Consider a linear feedback

$$\alpha(R,\Phi,\Psi)=k_1R+k_2(\Phi-\Phi_0)+k_3(\Psi-\Psi_0). \quad (3.6)$$

Suppose that $k_3>0$. Then $\frac{3}{2}R_e\beta^2+(\Phi_e+1)/2\Psi_e+\sqrt{\Psi_e}\alpha_\Psi>0$. Therefore for any fixed β_1, a large value of $-k_2$ guarantees that $T_{\mathrm{Hopf}}(\Phi_e,\beta)\neq 0$. This argument proves the fact that, for any given domain $D=(\Phi_{e1},\Phi_{e2})\times(0,\beta_1]$, there always exists a state feedback under which the system does not have Hopf bifurcation point in D.

4. Test function for the existence of stationary bifurcations

In this section, a function of Φ_e is obtained to test the existence of stationary bifurcations at nonaxisymmetric equilibrium points given by Eq. (2.7). Based on the test function, sufficient conditions are given under which the feedback removes the saddle-node bifurcation (ii) from the closed-loop system.

In this section, we still focus on the nonaxisymmetric equilibrium points. Consider matrix A in Eq. (3.2).

Substituting Eq. (3.3) into det(A) yields

$$\det(A) = \frac{\sigma R_e \sqrt{\Psi_e}}{\beta^2}\Bigg(-2\Phi_e \alpha_R + \left(\frac{\Phi_e + 1}{2\Psi_e^{3/2}} + \alpha_\Psi\right)\left(-\frac{3}{2} + \frac{15}{2}\Phi_e^2\right) + \frac{1}{\sqrt{\Psi_e}} + \alpha_\Phi\Bigg). \tag{4.1}$$

The following function

$$T_{\text{stationary}}(\Phi_e) = -2\Phi_e \alpha_R + \left(\frac{\Phi_e + 1}{2\Psi_e^{3/2}} + \alpha_\Psi\right) \times \left(-\tfrac{3}{2} + \tfrac{15}{2}\Phi_e^2\right) + \frac{1}{\sqrt{\Psi_e}} + \alpha_\Phi \tag{4.2}$$

is called the test function for stationary bifurcations along curve (2.7). Since $R_e \neq 0$ and $\Psi_e \neq 0$ at the points of Eq. (2.7), det(A) $= 0$ if and only if $T_{\text{stationary}} = 0$. It is known that the closed-loop system (2.3)–(2.9) has no stationary bifurcation if the "A" matrix of linearization has full rank. This is equivalent to the fact that $T_{\text{stationary}} \neq 0$. Therefore the following result holds.

Theorem 4.1. *On the branch of nonaxisymmetric equilibrium points* (2.7), *closed-loop system* (2.3)–(2.9) *has no stationary bifurcation for* $\Phi_e \in (\Phi_{e1}, \Phi_{e2})$ *if*

$$T_{\text{stationary}}(\Phi_e) \neq 0, \quad \Phi_{e1} < \Phi_e < \Phi_{e2}. \tag{4.3}$$

Remark. It is easy to check that

$$\det(A) = \sigma R_e\left(-F_\Phi - F_R \frac{dR_e}{d\Phi_e} - F_\Psi \frac{d\Psi_e}{d\Phi_e}\right),$$

$$F_\Phi + F_R \frac{dR_e}{d\Phi_e} + F_\Psi \frac{d\Psi_e}{d\Phi_e} + F_\mu \frac{d\mu}{d\Phi_e} = 0.$$

Therefore $T_{\text{stationary}}(\Phi_e) = (\beta^2/\sqrt{\Psi_e})F_\mu(d\mu_e/d\Phi_e)$. This implies that the system can have stationary bifurcation at those points where μ has a local extreme value. For example, it is shown in McCaughan (1990) that system (2.1) has a saddle-node bifurcation at the operating point where γ has a local maximum value.

If the linear feedback law (3.6) is used, the test function for stationary bifurcation is

$$T_{\text{stationary}}(\Phi_e) = -2\Phi_e k_1 + \left(\frac{\Phi_e + 1}{2\Psi_e^{3/2}} + k_3\right) \times \left(-\tfrac{3}{2} + \tfrac{15}{2}\Phi_e^2\right) + \frac{1}{\sqrt{\Psi_e}} + k_2. \tag{4.4}$$

Remark. Since the values of Φ_e and Ψ_e are bounded on curve (2.7), the function as in Eq. (4.4) is always less than zero if the value of $-k_2$ is large enough for any fixed values of k_1 and k_3.

It is known in Section 3 that condition (3.5) holds if $-k_2$ is sufficiently large and $k_3 > 0$. A positive k_3 and a large value of $-k_2$ can also soften the transcritical bifurcation at the point (2.8) (Theorem 2.1). The remark above shows that a large value for $-k_2$ guarantees inequality (4.3). The combination of these results implies the existence of a state feedback law satisfying Eq. (2.11) such that the closed-loop system has no stationary nor Hopf bifurcation points of Eq. (2.7) in $(\Phi_{e1}, \Phi_{e2}) \times [0, \beta_1]$. In summary, we have the following result on the existence of a feedback law which treats the three bifurcations (i)–(iii) simultaneously.

Theorem 4.2. *Given* $-1 < \Phi_{e1} < \Phi_{e2} < 1$ *and* $\beta_1 > 0$, *there always exists a* (*linear*) *state feedback law* (2.9) *satisfying Eqs.* (2.11), (3.5) *and* (4.3) *in* $(\Phi_e, \beta) \in (\Phi_{e1}, \Phi_{e2}) \times (0, \beta_1]$. *Under this feedback, the closed-loop system* (2.3)–(2.9) *meets the following requirements*:

(a) The state feedback softens the transcritical bifurcation at point (2.8).
(b) There is no stationary bifurcation for the closed-loop system at the nonaxisymmetric equilibrium points (2.7), provided that $\Phi_e \in (\Phi_{e1}, \Phi_{e2})$.
(c) The closed-loop system has no Hopf bifurcation point along curve (2.7) if $(\Phi_e, \beta) \in (\Phi_{e1}, \Phi_{e2}) \times (0, \beta_1]$.

Remark. The conclusion (c) does not necessarily imply the vanish of periodic solution. It is possible that, a Hopf bifurcation occurs at a point of Eq. (2.7) with $\beta_{HB} > \beta_1$. The periodic solution may bifurcate into the region of $\beta < \beta_{HB}$. This implies the existence of periodic solutions for $\beta \in (0, \beta_1]$. However, the conclusions (b) and (c) together imply that the periodic solution must be unstable if it exists for some $\beta \in (0, \beta_1]$, and its amplitude decays to zero eventually. Practically, the closed-loop system does not exhibit surge with rotating stall under the state feedback given in Theorem 4.2. Furthermore, the example in Section 6 shows that the closed-loop system can have a periodic solution only around (R_e, Φ_e, Ψ_e) which is not close to (R_0, Φ_0, Ψ_0) if $\beta \in (0, \beta_1]$.

5. Stability of the periodic solution around a Hopf bifurcation point

In this section, a method is proposed to determine the stability of the periodic solutions in the presence of Hopf bifurcations. The Hopf bifurcation is called supercritical if the periodic solutions are stable, and called subcritical if the periodic solutions are unstable.

Suppose that the feedback satisfies the conditions in Theorem 4.2. The linearization of the closed-loop system at a point of Eq. (2.7) does not have pure imaginary eigenvalues for $\beta \in (0, \beta_1]$. However, it is possible to have a Hopf bifurcation inception point β_{HB} which is larger than β_1. In other words, the feedback "delays" the Hopf bifurcation, but does not eliminate it. In this case, the stability of the periodic solutions needs be determined.

Suppose that the linearization matrix A of closed-loop system (2.3)–(2.9) has a pair of imaginary eigenvalues $\pm\omega\sqrt{-1}$ at a point (R_e, Φ_e, Ψ_e) for $\beta = \beta_{HB}$. Then, there exists a nonsingular matrix T satisfying

$$T^{-1}AT = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

The linear change of coordinates

$$[z_1 \quad z_2 \quad z_3]^{\mathrm{T}} = T[R - R_e \quad \Phi - \Phi_e \quad \Psi - \Psi_e]^{\mathrm{T}} \tag{5.1}$$

transforms Eqs. (2.3)–(2.9) into the following form:

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} f_1^{[2]}(z_1, z_2, z_3) \\ f_2^{[2]}(z_1, z_2, z_3) \\ f_3^{[2]}(z_1, z_2, z_3) \end{bmatrix} + O(z)^3. \tag{5.2}$$

The center manifold of Eq. (5.2) has the form $z_3 = \pi(z_1, z_2)$ such that the first-order derivatives of $\pi(z_1, z_2)$ equal zero at (0, 0). Furthermore, it satisfies the following equation:

$$\begin{aligned} \lambda\pi + f_3^{[2]}(z_1, z_2, \pi) + \cdots &= \frac{\partial\pi}{\partial z_1}(-\omega z_2 + f_1^{[2]}(z_1, z_2, \pi) + \cdots) \\ &+ \frac{\partial\pi}{\partial z_2}(\omega z_1 + f_2^{[2]}(z_1, z_2, \pi) + \cdots). \end{aligned} \tag{5.3}$$

From the results in Carr (1981), the center manifold is approximated to the quadratic degree by solving Eq. (5.3) to the second degree. By comparing the quadratic terms in Eq. (5.3), we have the following.

Corollary 5.1. *Let* $\pi(z_1, z_2) = az_1^2 + bz_1z_2 + cz_2^2 + O(z_1, z_2)^3$, *then the coefficients a, b and c are given by*

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \lambda & -\omega & 0 \\ 2\omega & \lambda & -2\omega \\ 0 & \omega & \lambda \end{bmatrix}^{-1} \begin{bmatrix} \gamma_{z_1z_1} \\ \gamma_{z_1z_2} \\ \gamma_{z_2z_2} \end{bmatrix}, \tag{5.4}$$

where $\gamma_{z_iz_j}$ *is the coefficient of* z_iz_j *in* $f_3^{[2]}(z_1, z_2, 0)$.

The reduced dynamic system on the center manifold is obtained as

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} f^{[2]}(z_1, z_2, \pi) \\ f_2^{[2]}(z_1, z_2, \pi) \end{bmatrix} + \cdots = \begin{bmatrix} \bar f_1(z_1, z_2) \\ \bar f_2(z_1, z_2) \end{bmatrix}. \tag{5.5}$$

Then define

$$\begin{aligned} \bar T_{\mathrm{Hopf}} = {} & \frac{1}{16}(\bar f_{1z_1z_1z_1} + \bar f_{2z_1z_1z_2} + \bar f_{1z_1z_2z_2} + \bar f_{2z_2z_2z_2}) \\ & + \frac{1}{16\omega}(\bar f_{1z_1z_2}(\bar f_{1z_1z_1} + \bar f_{1z_2z_2}) - \bar f_{2z_1z_2}(\bar f_{2z_1z_1} \\ & + \bar f_{2z_2z_2}) - \bar f_{1z_1z_1}\bar f_{2z_1z_1} + \bar f_{1z_2z_2}\bar f_{2z_2z_2}), \end{aligned} \tag{5.6}$$

where all the partial derivatives are evaluated at the equilibrium point $(z_1, z_2, z_3) = (0, 0, 0)$. Since points (2.7) are determined by the value of Φ_e, $\bar T_{\mathrm{Hopf}}$ can be considered as a function of Φ_e. It is proved in Glendinning (1994) that the Hopf bifurcation is supercritical if $\bar T_{\mathrm{Hopf}} < 0$, and is subcritical if $\bar T_{\mathrm{Hopf}} > 0$. A drawback of this test function is that a compact formula for $\bar T_{\mathrm{Hopf}}$ is not available because it involves the eigenvalues and eigenvectors. This made the theoretical analysis difficult. However, given any equilibrium point of Eq. (2.7) and any Hopf bifurcation inception point $\beta = \beta_{HB}$, the value of $\bar T_{\mathrm{Hopf}}$ can be computed rather easily. The computation of T is coded in MAPLE, which is used in the example and the simulation in the next section.

6. A design example and discussions

In this section, Theorem 4.2 is applied to system (2.1) and a feedback law is designed which removes the hysteresis and guarantees elimination of surge with rotating stall in the system performance for a given range of β and Φ_e. The data for our simulation is from the MG3 compressor of Caltech:

$$\lambda = 1.256, \quad l_c = 21.67, \quad B = 0.2, \quad a = 0.1,$$

$$H = 0.0616, \quad W = 0.1341, \quad \Psi_{c0} = 0.1469.$$

The values of σ and β are

$$\sigma = \frac{3al_c}{1 + a\lambda} = 5.7756, \qquad \beta = \frac{2BH}{W} = 0.1837.$$

Suppose that the parameter β is varying in the finite interval (0, 0.8). There exists a linear feedback law as in Eq. (3.7) satisfying the conditions in Theorem 4.2, that in turn is equivalent to

$$k_2 < 0.6825, \quad k_3 > -0.3179,$$

$$2k_1 - k_2 - 6k_3 > 1.2249. \tag{6.1}$$

We choose $k_1 = 0$. Therefore, it is not necessary to measure the rotating stall. It is known from the previous sections that one method of finding a feedback satisfying Eq. (3.5) and (4.3) is to fix $k_3 > 0$, and then choose a value for $-k_2$ which is large enough. We take $k_2 = -4.7, k_3 = 0.5$. The plots of the functions $T_{\mathrm{stationary}}$ and T_{Hopf} (see Figs. 2 and 3) show that these two functions do not equal zero for $\Phi_e \in (-1, 1)$ and

$\beta \in (0, 0.8)$. In other words, conditions (4.3) and (3.5) are satisfied.

Therefore, the closed-loop system under the linear feedback

$$u = -4.7(\Phi - \Phi_0) + 0.5(\Psi - \Psi_0) \tag{6.2}$$

meets the requirements (a)–(c) in Theorem 4.2. More specifically, the closed-loop system has the following properties: (a) the feedback softens the transcritical bifurcation at point (2.8); (b) the closed-loop system does not have any stationary bifurcation around the points in the set of Eq. (2.7); (c) there is no Hopf bifurcation point for $(\Phi_e, \beta) \in (-1, 1) \times (0, 0.8)$; (d) the closed-loop system is locally asymptotically stable at all the nonaxisymmetric equilibrium points (2.7) if $0 < \beta \leq 0.8$.

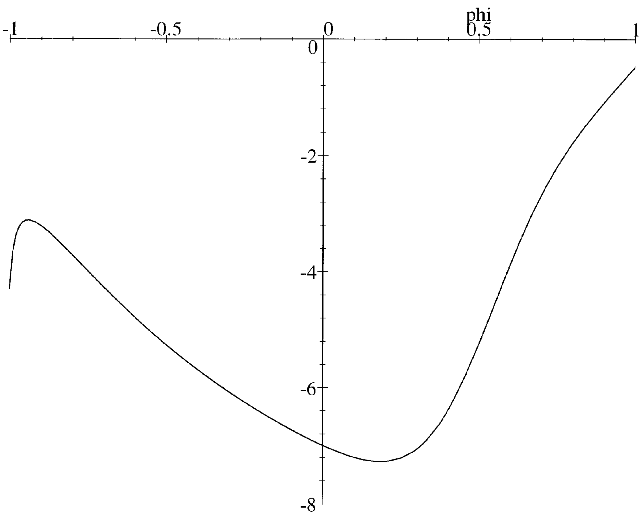

Fig. 2. The graph of $T_{\text{stationary}}(\Phi_e)$.

Simulations are carried out to test the performance of the designed feedback. The stability of the system is illustrated for $\beta = 0.18$ with the responses of the state variables $(x_1, x_2, x_3) = (R - R_e, \Phi - \Phi_e, \Psi - \Psi_e)$ versus time variable ξ as shown in Fig. 4. The initial values are $R = 0.45$, $\Phi = -0.7$ and $\Psi = 1.58$ with $\Phi_e = -0.5$ for Fig. 4a, and $R = 0.45$, $\Phi = 0.7$ and $\Psi = 0.4$ with $\Phi_e = 0.5$ for Fig. 4b.

Fig. 5 shows the trajectory of a different simulation, in which the feedback is used to control surge in the presence of rotating stall. As pointed out in McCaughan (1990), stable periodic solutions exist near the nonaxisymmetric equilibria if $\Psi_{c0} > \sim 2.07$. In this simulation, the parameters are chosen as $\Psi_{c0} = 2.5$ and $\beta = 0.6698$. At time $\xi = 0$, the system has no feedback control, and rotating stall cells coupled with surge appear at $\mu = 0.9029$, which is the periodic curve shown in solid line in Fig. 5a. At time $\xi = 50$, feedback (6.2) is applied to the system, and the state variables are driven to the operating point (2.8). See the dotted curve in Fig. 5a. Fig. 5b shows the response of the state variables $(x_1, x_2, x_3) = (R - R_0, \Phi - \Phi_0, \Psi - \Psi_0)$ versus time ξ. In the simulation, the parameter μ in the closed-loop system assumes its nominal value $\mu = 0.9428$. Otherwise, the states may converge to a different point nearby. The system can have both supercritical and subcritical Hopf bifurcations at $\beta_{\text{HB}} > 0.8$. The plot of $\bar{T}_{\text{Hopf}}(\Phi_e)$ is shown in Fig. 6. For $|\Phi_e| < \sim 0.61$, the Hopf bifurcation is subcritical because $\bar{T}_{\text{Hopf}} > 0$. For $|\Phi_e| > \sim 0.61$, $\bar{T}_{\text{Hopf}}(\Phi_e) < 0$ and the Hopf bifurcation at β_{HB} is supercritical. At a point of Eq. (2.7) near $\Phi_0 = 1$, the Hopf bifurcation is supercritical and the system is stable for $\beta \in (0, 0.8)$. Therefore, there are no periodic solutions around a point of Eq. (2.7) near Eq. (2.8) if $0 < \beta < 0.8$. However, if $\beta > 0.8$, it is possible to have stable periodic solutions.

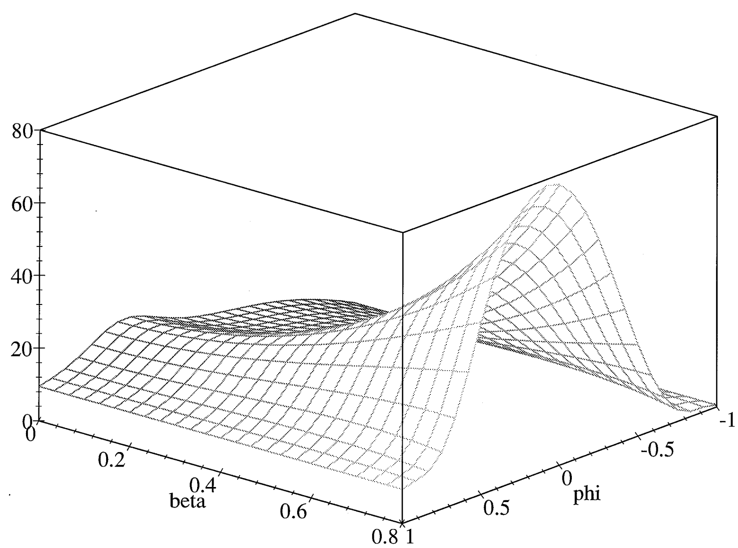

Fig. 3. The graph of $T_{\text{Hopf}}(\Phi_e, \beta)$.

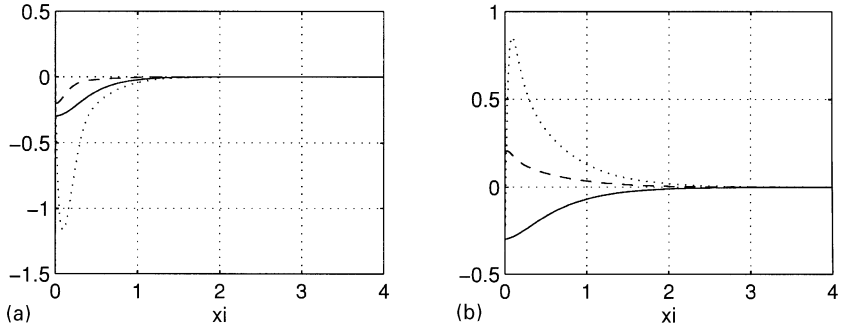

Fig. 4. Time response with solid line for x_1, dashed line for x_2, and dotted line for x_3.

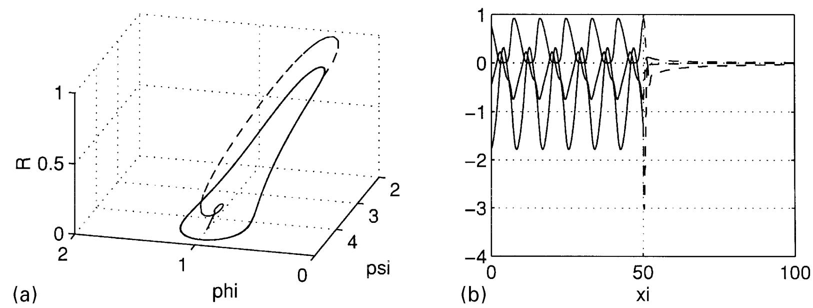

Fig. 5. Periodic trajectory in 5(a) and state time response in 5(b).

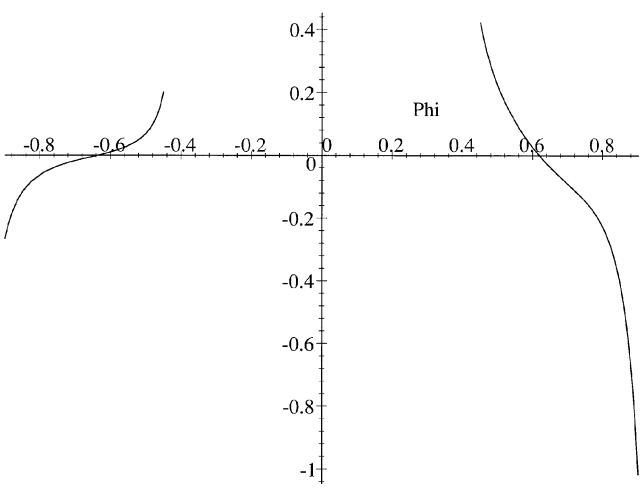

Fig. 6. The graph of $\bar{T}_{\mathrm{Hopf}}(\Phi_e)$.

Fig. 7 is the graph of a periodic solution at $\mu = 0.7801$, $\beta = 1.3476$.

It is known that the value of β is dependent on the value of B, which combines wheel speed, volume of plenum and cross-sectional area of the compressor duct. Among other factors, a plenum with larger size can increase the value of β. In order to control the surge coupled with rotating stall for larger values of β, it is necessary to use larger gain in the feedback. Numerical experiments are carried out to obtain the feedback law for $\beta \in (0, 1.8)$. In this interval of β, the upper limit is almost ten times as large as the value of β introduced at the beginning of this section. Since the system is not likely to run very close to the equilibrium with $\Phi_e = -1$, we focus on the interval $\Phi_e \in (-0.7, 1)$. The gains in the state feedback law are

$$k_1 = 0, \qquad k_2 = -16.5, \qquad k_3 = 2.3.$$

The graph of $T_{\mathrm{stationary}}$ and T_{Hopf} are shown in Figs. 8 and 9, respectively. The test functions do not equal zero for

$$(\Phi_e, \beta) \in [-0.7, 1) \times (0, 1.8). \tag{6.3}$$

Therefore, the closed-loop system meets the requirements (a)–(c) in Theorem 4.2. In other words, the closed-loop system is stabilized at the equilibria with $\Phi_e \in [-0.7, 1)$ and there is no Hopf bifurcation point on Eq. (2.7) if (Φ_e, β) lies in domain (6.3).

Before concluding this section, it is necessary to compare our proposed design method with the existing ones as in Eveker et al. (1995) and Krstic et al. (1995). Both investigated rotating stall and surge control with different feedback laws. In Eveker et al. (1995) the feedback law is a linear combination of rotating stall control law in Liaw and Abed (1996) and surge control law of Badmus et al. (1996). Specifically the control input is given by

$$u = K_{\mathrm{rs}} R + K_{\mathrm{sur}} \dot{\Phi}.$$

It was shown in Eveker et al. (1995) that such a control law is effective for simultaneous control of rotating stall

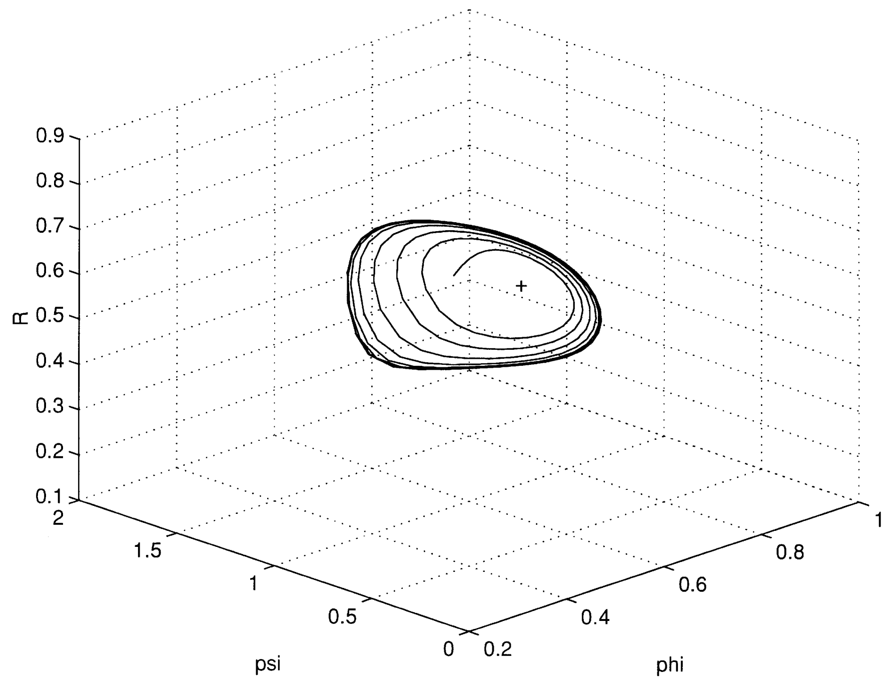

Fig. 7. The graph of $\bar{T}_{\mathrm{Hopf}}(\Phi_e)$.

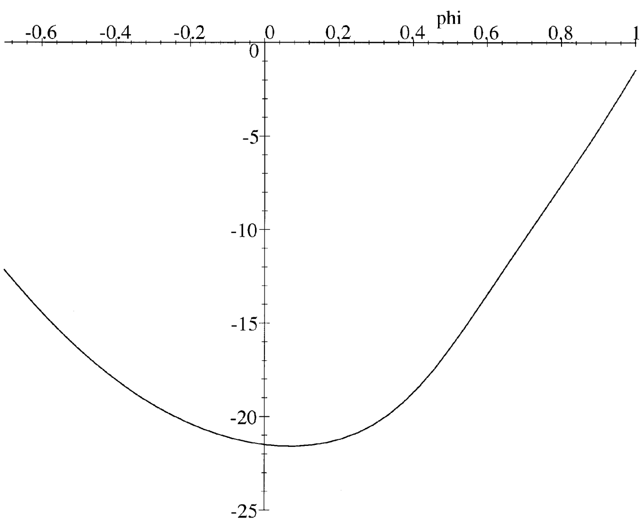

Fig. 8. The graph of $T_{\mathrm{stationary}}(\Phi_e)$.

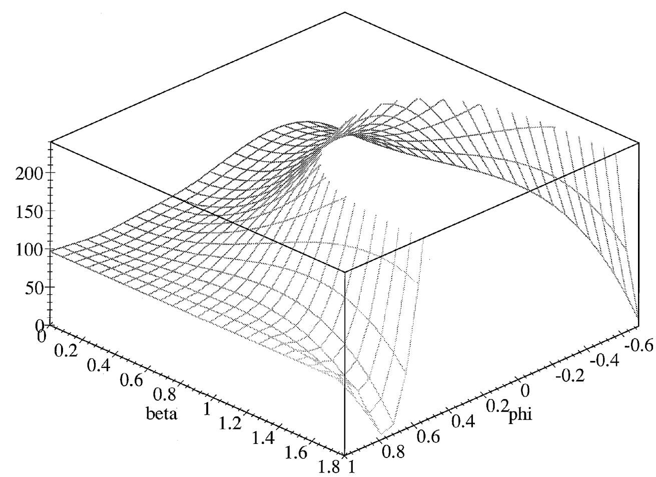

Fig. 9. The graph of $T_{\mathrm{Hopf}}(\Phi_e, \beta)$.

and surge. Moreover, the above control law is experimentally validated. Our control method is apparently different from that of Eveker et al. (1995). We have employed the same feedback control law for both rotating stall and surge controls, and the use of derivative of flow rate Φ is avoided. More importantly the surge control law of Badmus et al. (1996) is derived from nonlinear model-based Luenberger-type observer, and our surge control method is based on Hopf bifurcation theory. Hence our results provide an alternative for the compressor control law proposed in Eveker et al. (1995). Regarding the control law in Krstic et al. (1995), the global stability of the peak operating point was shown using backstepping method. While global stability is important and impressing, its robustness is not clear especially with respect to the throttle and B parameters at critical operating points of Eq. (2.7) due to the possible Hopf bifurcation associated with surge. Our results are local (that is also the case for the work of Eveker et al. (1995). However, the stability of the equilibrium trajectory achieved using both our method and the method of (Eveker et al., 1995) admit certain robustness with respect to the throttle and B parameters.

7. Conclusion

In summary, the contributions of this paper are (a) test functions are obtained for closed-loop systems under any nonlinear state feedbacks, which can be used to test the existence of stationary or Hopf bifurcations and to test the stability of periodic solutions; (b) feedback design method is given which can meet several bifurcation control requirements, mainly to soften the transcritical bifurcation and to remove stable periodic solutions around nonaxisymmetric equilibria within a given range of parameters. Comparisons are made with existing results for rotating stall and surge control investigated in Krstic et al. (1995) and Eveker and et al. (1995). Stabilization results are demonstrated with numerical simulations.

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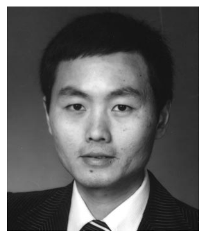

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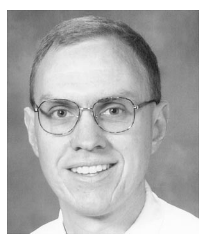

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