Studies in Cascade Reliability—I

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Abstract—An n-Cascade system is defined as a special type of standby system with n components. A component fails if the stress on it is not less than its strength. When a component in cascade fails, the next in standby is activated and will take on the stress. However, the stress on this component will be a multiple k times the stress that acted on its predecessor. The system fails if due to an initial stress, each of the components in succession fails.

The stress is random and the component strengths are independent and identically distributed variates, with specified probability functions; k is constant.

Expressions for system reliability are obtained when the stress and strength distributions are exponential. Reliability values for a 2-cascade system with Gamma and Normal stress and strength distributions are computed, some of which are presented graphically.

Reader Aids:
Purpose: Widen state of the art
Special math needed for explanations: Probability
Special math needed for results: Same
Results useful to: Theoreticians

1. INTRODUCTION

The strength of a component or system is the minimum stress required to cause the component or system to fail. If the stress equals or exceeds the strength of the component, it fails; otherwise it works. In practical situations, the magnitude of the stress is random, with considerable variations. Imperfections in the manufacture and nonuniformity in the materials give a random character to the component's strength, which is thus also a random variable.

The reliability of the component is $R_c = \Pr\{Y < X\}$ where X and Y are the component strength and the stress applied to it respectively.

Problems of reliability as defined above are the subject of quite a few investigations [1-5]. All of these deal with a single component, i.e., no redundancy. We shall consider here the Reliability of Cascaded Systems. By a cascaded system we mean a standby redundant system where a standby component, taking the place of the failed component, is subjected to a changed stress. We shall assume that this changed stress is k times the preceding value.

Roberts [6] has given an expression for component reliability when stress Y and strength X are randomly distributed. He points out that more overlap between the frequency curves of X and Y reduces reliability. This observation, however, needs to be qualified; the probability of failure is not that simple. Shooman [7] has considered a Poisson process for the occurrence of stresses. Leve [4] and Shooman [7] have considered the possibility of time-deterioration of strength.

In section 2, we formulate a general problem of Reliability of an n-Cascade system. Section 3 deals with exponential stress and strength distributions. Explicit expressions for n-Cascade system reliability are obtained, a general recursive rule is indicated for obtaining these expressions and for $k = 1$, a finite series expansion is obtained. Sections 4 and 5 deal with the 2-cascade system where X and Y both follow the Gamma and the Normal distributions respectively.

2. THE n-CASCADE MODEL

The following assumptions are made in the n-Cascade model.

1. The system consists of n components whose strengths are s-independently and identically distributed (i.i.d.) random variables.
2. Only one component is active, the others are standbys. The active component is subjected to a stress following a specified distribution.
3. If the active component fails, another from among the standby components takes its place. However, this newly active component will face not the previous stress but a stress k times the previous stress, where k is a constant.
4. This cascading process continues until either

a) the current stress after s stages will be less than the strength of the component which becomes active at stage $s + 1$, or
b) the stress in stage n equals or exceeds the strength of the last component.

In (a) the system survives the impact of the stress, though with a loss of s components. In (b) the system fails to survive the impact.

Let $x_1, x_2, ..., x_n$ be the strengths of the n components in the order of activation. Let y_1 be the stress on the first component. The x_r's are iid random vairables with pdf $f(x)$ while y_1 is a random variable s-independent of the x's with pdf $g(y_1)$.

If $y_1 < x_1$, the first component and hence the system survives. Otherwise, the component fails and the second component in line, with a strength x_2 takes its place. However, the stress on this component will not be y_1, but will be $y_2 = ky_1$, a deterministic multiple of the first-stage stress.

If $y_2 < x_2$, the second component and hence the system survives the stress (though the system has suffered the loss of one component). Otherwise, the second component also fails and the third component, of strength x_3, gets activated and is to withstand a stress $y_3 = ky_2 = k^2y_1$. In general, with an initial stress y_1, if component i fails, component $i+1$ with strength x_{i+1} gets activated and is subjected to a stress of $y_{i+1} = k^i y_1$.

The system survives, with a loss of s components, if $y_i \geqslant x_i$, $i = 1, ..., s$ and $y_{s+1} < x_{s+1}$; it fails only if all components fail, i.e., if $x_i \leqslant y_i$, $i = 1, ..., n$.

Denoting by $R(s)$ the probability of system survival after $s-1$ components fail, we can write

$$R(s) = \Pr\left\{\left[\bigcap_{i=1}^{s-1} (x_i \leqslant y_i)\right] \cap (x_s > y_s)\right\}$$

$$= \int_{y_1=0}^{\infty} g(y_1)\{\int_{x_1=0}^{y_1} f(x_1)[\int_{x_2=0}^{y_2=ky_1} f(x_2) \ldots$$

$$\{\int_{x_{s-1}=0}^{k^{s-2}y_1} f(x_{s-1}) \int_{x_s=y_s=k^{s-1}y_1}^{\infty} f(x_s)\, dx_s$$

$$\cdot\, dx_{s-1}\} \ldots dx_2]\, dx_1\}\, dy_1 \quad (1)$$

The n-cascade system's reliability is

$$R_n = \sum_{s=1}^{n} R(s) \quad (2)$$

Clearly, $R(2), R(3)$ ... are the increases in system reliability due to successive additions of the second, third, ... components in the cascade system.

When $k = 1$, the system is not analogous to the 'simple' standby system; as can easily be checked, $R_2 \neq 1 - (1 - R_1)^2$.

Extensions of this model in many directions are obvious. For instance, one could assume that $y_{i+1} = k_i y_i$ where the k_i need not be equal; or that the k_i are random variables. In the present study we are not investigating these extensions.

3. EXPONENTIAL STRESS AND STRENGTH DISTRIBUTIONS

$$g(y_1) = \mu \exp(-\mu y_1), y_1 > 0, \mu > 0$$

$$f(x) = \rho\mu \exp(-\rho\mu x), x > 0, \rho > 0.$$

$$R_1 = R(1) = \Pr\{x_1 > y_1\}$$

$$= \int_0^{\infty} g(y_1)\{\int_{y_1}^{\infty} f(x_1)\, dx_1\}\, dy_1 = \frac{1}{1+\rho}$$

$$R(2) = \Pr\{(x_1 \leqslant y_1) \cap (x_2 > ky_1)\}$$

$$= \int_0^{\infty} \{g(y_1)\{\int_0^{y_1} f(x_1) \int_{ky_1}^{\infty} f(x_2)\, dx_2\, dx_1\}\, dy_1\}$$

$$= \frac{1}{1+\rho k} - \frac{1}{1+\rho+\rho k}$$

For this exponential case, the marginal reliability increment in a $(n-1)$–cascade system, can be expressed as the sum of 2^{n-1} terms of the form:

$$\frac{\alpha_{n,l}}{1+\rho g_{n,l}(k)}$$ where the $g_{n,l}(k)$ can be expressed as $\sum_{r=0}^{n-1} a_r k^r$ where $a_0 = 1$ or $1+\rho$, $a_{n-1} = \rho$, $a_r = 0$ or ρ otherwise; $\alpha_{n,l} = +1$ if the number of nonzero terms in $g_{n,l}(k)$ is odd and -1 otherwise.

Thus, for instance

$$R(3) = \frac{1}{1+\rho k^2} - \frac{1}{1+\rho+\rho k^2} - \frac{1}{1+k\rho+\rho k^2} + \frac{1}{1+\rho+k\rho+k^2\rho}$$

In particular, taking $k = 1$ (each component faces the same stress when it becomes active), we get

$$R(s+1) = \sum_{r=0}^{s} (-1)^r \binom{n}{r} \frac{1}{1+(r+1)\rho}$$

Values of $R(n)$ and R_n for different values of n, ρ and k have been computed. Figure 1 graphically presents some of these values.

4. GAMMA DISTRIBUTIONS

Let x and y_1 have the following Gamma pdf's:

$$f(x) = \frac{1}{\Gamma(m)} x^{m-1} \exp(-x), x > 0, m > 0$$

$$g(y_1) = \frac{1}{\Gamma(l)} y_1^{\,l-1} \exp(-y_1), y_1 > 0, \quad l > 0.$$

For integer m, $R(1)$ and $R(2)$ are finite weighted sums of Gamma functions.

Tables of $R(1)$, $R(2)$ and R_2 have been computed for some values of l and m, part of which are presented in Figure 2.

5. NORMAL DISTRIBUTIONS

If many different factors additively fix the component strength or stress, we can expect that the central limit theorem will hold and hence that the component strength and stress distributions will be s-normal.

Let x and y_1 be s-normally distributed. Without loss of generality we can shift the origin to the mean of y_1 and change the scale of measurement to make the standard deviation of y_1 be unity. In these new units, let μ and σ be the mean and standard deviation of strengths. That is, we assume that the variability in strength as measured by its standard deviation is σ times the variability in stress and that the mean strength is μ units higher than the mean stress.

Notation:

gau base name implying the Gaussian (s-normal) distribution
gaud pdf of the Gaussian distribution
gauf Cdf of the Gaussian distribution

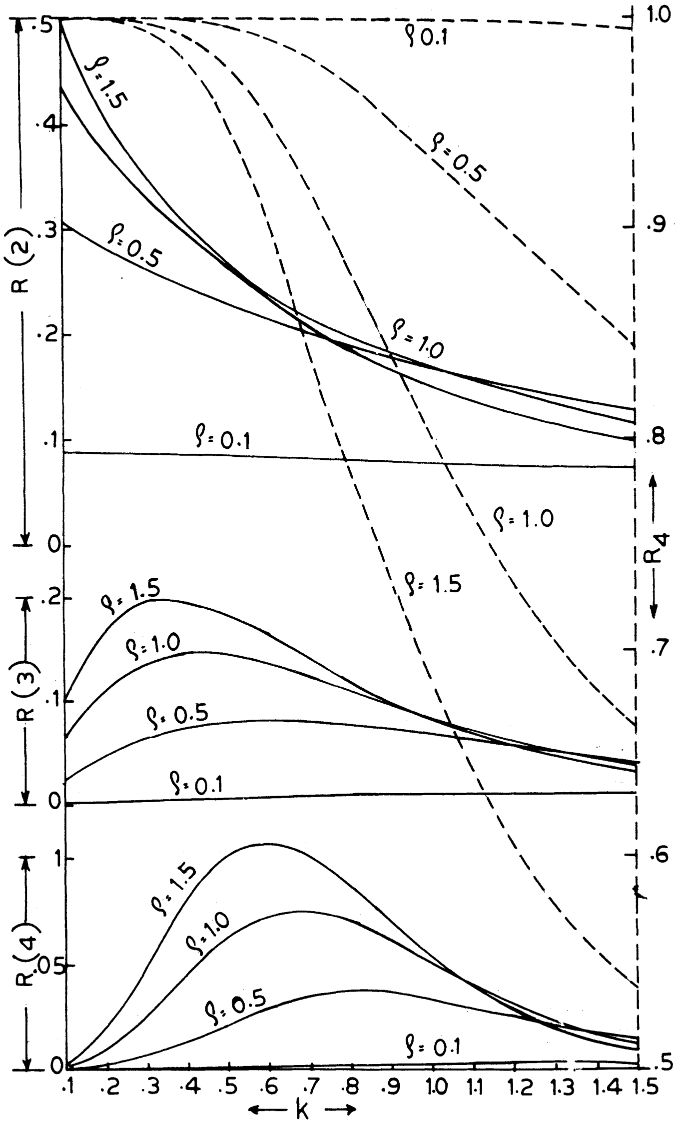

Fig. 1. Reliability of n cascade components $R(n)$ vs stress-ratio k. Stress and strength have the exponential distribution with parameters $\rho\mu$ and μ respectively.

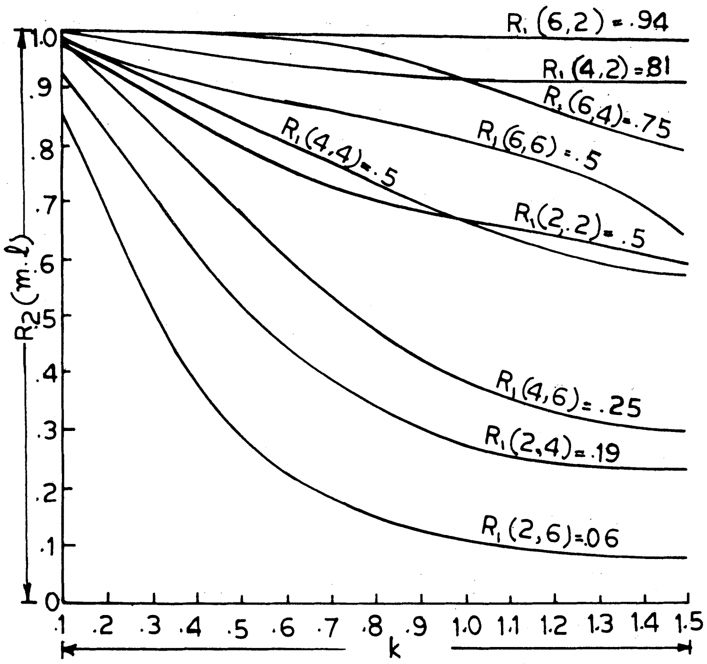

Fig. 2. Reliability of n cascade components R_n vs stress-ratio k. Stress and strength have the gamma distribution with unity scale parameters and shape parameters of m and l respectively.

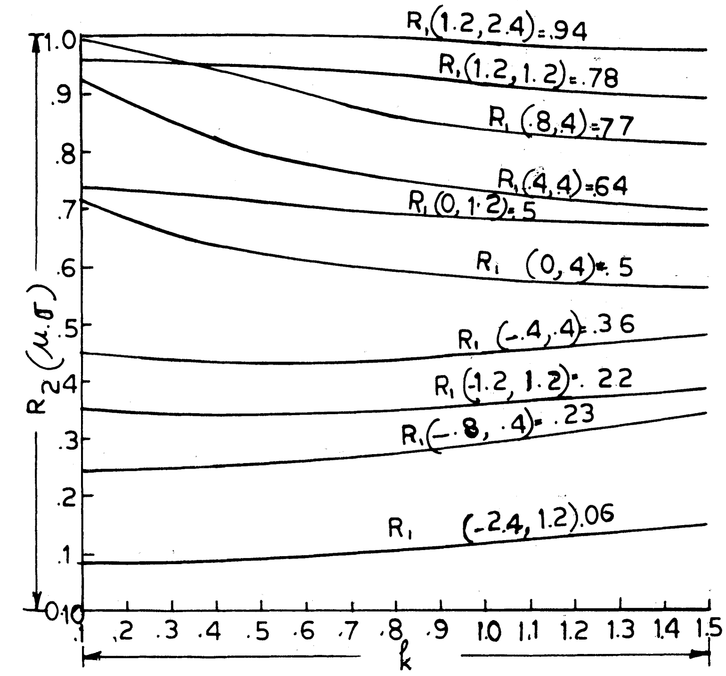

Fig. 3. Reliability of n cascade components R_n vs stress-ratio k. Stress and strength have the Gaussian distribution (see text, Sec. 5).

It is true that sensible ranges for strength and stress will be only from a natural zero to infinity. However, in practice, the mean values, in terms of the corresponding standard deviations, will be considerably above the natural zero value and hence the nonnegativity of stress and strength will not be practically incompatible with the 2-sided infinite range required by the assumption of s- normality.

$$R_1 = \text{gauf}\left(\frac{\mu}{1+\sigma^2}\right)$$

$$R(2) = 1 - R_1 - \int_{-\infty}^{\infty} \text{gaud}\,(y_1)\,\text{gauf}\left(\frac{ky_1 - \mu}{\sigma}\right)$$

$$\text{gauf}\left(\frac{y_1 - \mu}{\sigma}\right)\, dy_1$$

Using the Gauss-Hermite quadrature formulae we have evaluated the 2-cascade system reliabilities R_2 for different values of μ and σ. Some of these results are graphically presented in Figure 3.

Two points emerge from these results:

1) Somewhat against intuition, it transpires that when μ is negative, i.e.; when the mean stress is more than the mean strength, reliability decreases with k (but again, for some values of k, it starts increasing with decreasing k). We may cite as example the R_2 values for the cases when $\mu = \pm 1$ and $\sigma = 1$.

2) For the same k and σ, $R(2)$ decreases with an increase in $|\mu|$, i.e., the more the two means are separated, the less is the increase in reliability due to the second component.

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Manuscripts Received

If you wish more information, write to the author at the address listed; do NOT write to the Editor.

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