NUMERICAL SOLUTION OF SOME STEADY STATE CONVECTION-DIFFUSION AND IMPACT PROBLEMS

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By

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DEDICATED TO

GODDESS BHADRAKALI

AND

ALL OF MY WELLWISHERS

CERTITIFICATE

This is to certify that the thesis entitled **"NUMERICAL SOLUTION OF SOME STEADY STATE CONVECTION DIFFUSION AND IMPACT PROBLEMS"** submitted to National Institute of Technology, Warangal, is the bonafide research **work** done by **Mr. K. SHARATH BABU under my supervision. The contents of this thesis have not been submitted elsewhere for the award of any Degree.**

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K. SHARATH BABU

NUMERICAL SOLUTION OF SOME STEADY STATE CONVECTION-DIFFUSION AND IMPACT PROBLEMS

ABSTRACT

In this thesis, we consider numerical solution of some steady state convection-diffusion and impact problems that deal with the fluid flow problems involving large Reynolds number and the non-linear wave prorogation in the case of impact problem. In the impact problem we have focused on longitudinal, one-dimensional wave propagation. Convection-diffusion problems and impact problem have their commonality in one aspect that both of them are convection-diffusion in nature. Convection diffusion problems form a class of Singular perturbation problems. The numerical treatment of these perturbation problems is far from trivial in view of the boundary layer behavior of the solutions. In a singular perturbation problem there arises a governing differential equation whose highest order derivative is multiplied by a perturbation parameter ε. The study of numerical solution of singular perturbation problems has attracted researchers in numerical analysis in view of the ever increasing efficiency of the high speed computers. The thesis is divided into five parts and consists of nine chapters.

Part-I consists of a single Chapter which is introductory in nature. In this Chapter we introduce the steady state convection-diffusion and impact problems and present a review of existing literature on the problems related to the thesis.

Part II deals with the steady state convection- diffusion problems which are solved by applying various numerical methods. It consists of four chapters 2, 3, 4 and 5. In Chapter 2, we present a computational method to solve steady state convection – diffusion problem. In Chapter 3, we deal with a uniformly convergent scheme for convection –diffusion problem. Chapter 4 is devoted to the application of finite element method to solve Singularly perturbed two point boundary value problems using cubic B-splines. Chapter-5 is devoted to study of a numerical integration method for solving general steady-state convection-diffusion problems.

Part-III deals with the Artificial-diffusion convection problem and two dimensional convection-diffusion problems. It consists of two chapters, chapter 6 and 7. Chapter 6 deal with a convection-diffusion problem in one-dimension with variable co-efficients wherein an artificial –diffusion term is present. In chapter-7 we present a numerical study of convection –diffusion problem in two- dimensional space**.**

Part IV consists of a single Chapter, Chapter 8. This chapter aims to study the numerical study of wave propagation in a non-linear medium due to impact. The problem studied in this part is analogous to those studied in the previous part. It reveals the non-linear wave propagation and possesses convection nature. Non-linear equation is reduced to linear by applying quasi-linearization technique.

In all the above problems, numerical methods are used and the analytical solutions are obtained wherever possible. In the numerical methods most of the part in the thesis finite difference methods are employed. In chapter-4 we employed finite element method to attain the reasonable accuracy. In a nut-shell the numerical methods presented in this thesis for solving convection-diffusion problems in differential equations have been shown to be accurate and efficient over the conventional methods. Above all, these methods are conceptually simple, easy to use and are readily adaptable for computer implementation with a modest amount of modeling the problem.

PART-I

INTRODUCTION

INTRODUCTION

The present thesis entitled **NUMERICAL SOLUTION OF SOME STEADY STATE CONVECTION- DIFFUSION AND IMPACT PROBLEMS** deals with the fluid flow problems involving large Reynolds number and the non-linear wave prorogation in the case of an impact problem. Convection-diffusion problems and impact problem have their commonality in one aspect that both of them are convection-diffusion in nature. Convection diffusion problems form a class of Singular perturbation problems. The numerical treatment of these perturbation problems is far from trivial in view of the boundary layer behavior of the solutions. In a singular perturbation problem there arises a governing differential equation whose highest derivative is multiplied by a perturbation parameter ε. The study of numerical solution of singular perturbation problems has attracted researchers in numerical analysis in view of the ever increasing efficiency of the high speed computers. The present study is motivated mainly by the study of Stynes [66] who dealt with problems of this nature and highlighted the notion of 'artificial diffusion'.

In the existing literature, this is one of the highly fertile fields which is receiving attention that it richly deserves. Perhaps the most common source of convection-diffusion problems is due to the Navier – Stokes equations which are highly nonlinear when Reynolds number is large. Morton, in his classic treatise [40], pointed out that this is by no means the only place where they arise and listed ten examples involving convection-diffusion equations starting from the drift-diffusion equations of semiconductor device modeling to the Black–Scholes equation that arises in financial modeling. He also observed that accurate modeling of the interaction between convective and diffusive processes is *'the most ubiquitous and challenging task'* in the numerical approximation of partial differential equations.

Convection-diffusion problems occur very frequently in the fields of science and engineering such as fluid dynamics, specifically the fluid flow problems involving large Reynolds number, problems in mass and heat transfer and problems dealing with chemical reactions.

A problem which we shall be referring to as impact problem is also discussed in the thesis in view of its commonality with the other problems studied in the thesis. In the impact problem, a non-linear convection-diffusion problem is studied. When two objects with distinct velocities come into contact with one another, an impact occurs and wave propagation occurs in the collided bodies. The nonlinear wave propagation that occurs as a result of the impact is modeled through nonlinear differential equation and this is studied by reducing it to linear equation by Quasi-linearization technique.

In the steady-state convection-diffusion problem there arises a governing differential equation in which the highest order derivative is multiplied with a perturbation or diffusion parameter. Convection-diffusion problems form a class of singular perturbation problems. In the impact problem the non-linear wave equation exhibits convection as well as diffusion nature. An introduction to the problem is presented and the methodology adopted is explained.

Convection is the process in which heat moves through a gas or a liquid as the hotter part rises and the cooler, heavier part sinks, where as in the diffusion a gas or liquid diffuses or is diffused in a substance, it becomes slowly mixed with that substance.

Singular perturbation problems occur very frequently in various fields of Science and Engineering such as Fluid Dynamics, specially the fluid flow problems involving large Reynolds number. In general, any differential equation in which the highest order derivative is multiplied by a small positive parameter ε $(o<\varepsilon<< 1)$ is called singular perturbation problem. In fact, any differential equation whose solution Changes rapidly in some parts of the interval are generally known as singular perturbation problem and also as boundary layer problem. A boundary layer by definition is a narrow region, where the solution of a differential equation changes rapidly. Further the thickness of the boundary layer tends to zero as $\varepsilon\rightarrow 0$.

Imagine a river flowing strongly and smoothly. Liquid pollution pours into the water at a certain point. What shape does the pollution stain form on the surface of the river? Two physical processes operate here: the pollution diffuses slowly through the water, but the dominant mechanism is the swift movement of the river, which rapidly convects the pollution downstream. Convection alone would carry the pollution along a one-dimensional curve on the surface; diffusion gradually spreads that curve, resulting in a long thin curved wedge shape. When convection and diffusion are both present in a linear differential equation and convection dominates, we have a convection-diffusion problem.

DEFINITION

In this section we give briefly the definition of singular perturbation problem in its simplest and most commonly used form. In general, any differential equation in which the highest order derivative is multiplied by a small positive parameter ε ($0<\varepsilon<< 1$) is called Singular perturbation Problem. Infact, any differential equation whose solution changes rapidly in some parts of the interval is generally known as Singular Perturbation problem and also called as Boundary Layer Problem. A Boundary Layer problem by definition is a narrow region, where the solution of a differential equation changes rapidly. In this region diffusion term dominates. Further the thickness of the boundary layer approaches to zero as $\varepsilon\rightarrow 0$.

MOTIVATION

Differential equations occur very frequently in the mathematical modeling of physical problems in Science and Engineering. Since exact solutions for most of these problems are not available, a resort to the approximation methods for getting the solution of such problems is unavoidable. The availability of high speed digital computers has made it possible to take such a task when the approximation method involves numerical computation. The most commonly employed approximate methods, for solving such type of problems are the finite difference method and the finite element method. Even though the finite element method is somewhat difficult than the finite difference method from the point of view of computer programming, it has certain inherent advantages, namely the approximation of solutions can be obtained easily in more complicated regions etc.

Convection-diffusion problems occur very frequently in the field of Fluid dynamics with Large Reynolds number, Heat and mass Transfer and Chemical Reaction problems. In a differential equation the highest order derivative multiplied with a perturbation parameter ε which is positive and very close to zero and the first order derivative terms serves as convective atmosphere. It means on most of the domain the solution has convection nature in the sense that solution behaves well but in the sub-domain near to the boundary layer region there exists a sub-region called narrow region where the gradient of the solution is large indicating that diffusion effects in this region stating that there is a boundary layer for specific values of the argument. The thickness of the boundary layer goes to zero as perturbation parameter approaches to zero. In this boundary layer region there are possible oscillations in the computed solution by employing numerical methods. The challenging task here is to apply suitable numerical methods like finite difference methods, finite element methods in order to get reasonable accuracy in the computed solution. We are selected most of the problems in this thesis which admits analytical solutions. The reason behind this choice is, we can compare the computed solution with the analytical solution.

We can see that the solution of convection-diffusion problem has a Convective nature on most of the domain of the problem, and the diffusive part of the differential operator is influential only in the certain narrow sub-domains. In the sub domain the gradient of the solution is large. This nature is described by stating that the solution has a boundary layer.
The interesting fact that the elliptic nature of the differential operator is disguised on most of the domain means that numerical methods designed for elliptic problems will not work satisfactorily. In practice they usually exhibit a certain degree of instability. The challenge then is to modify these methods into a stable form without neglecting their accuracy in numerical methods.

A problem which we shall be referring to as impact problem is also discussed in the thesis in view of its commonality with the other problems studied in the thesis. In the impact problem, a non-linear convection-diffusion problem is studied. When two objects with distinct velocities come into contact with one another, an impact occurs and wave propagation exists in the collided bodies. The nonlinear wave propagation that occurs as a result of the impact is modeled through nonlinear differential equation and this is studied by reducing it to linear equation by Quasi-linearization technique.

It is well known that differential equations occur very frequently in the mathematical modeling of physical problems in Science and Engineering. Since exact solutions for most of these problems are not available, approximation methods for obtain the solution of such problems is unavoidable.. The most commonly employed approximate methods, for solving such type of problems are the finite difference method and the finite element method. Even though the finite element method is somewhat more difficult than the finite difference method from the point of view of computer programming, it has certain natural advantages that the approximation of solutions can be obtained accurately even in more complicated regions.

In this thesis, we applied finite difference methods to compute the solutions of some such problems numerically. We observe that the solution of convection-diffusion problem has a convective nature on a larger part of the domain of the problem, and the diffusive part of the differential operator is influential only in a certain narrow sub-domain. In the sub domain the gradient of the solution is large. This nature is described by stating that the solution has a boundary layer. The numerical methods that are designed for solving such elliptic differential operators will not work satisfactorily through out the domain since the solution in general is well behaved in the convective region while it exhibits instability in the boundary layer region where the equation is influenced by diffusion. The challenge then is to modify these numerical methods into a form without neglecting the accuracy and obtain a well behaved solution through out the domain.

REVIEW OF LITERATURE:

It is well known that differential equations occur very frequently in the mathematical modeling of physical problems in Science and Engineering. Since exact solutions for most of these problems are not available, approximation methods for getting the solution of such problems is unavoidable. The availability of high speed digital computers has made it possible to take such a task when the approximation method involves numerical computation. The most commonly employed approximate methods, for solving such type of problems are the finite difference method and the finite element method. Even though the finite element method is somewhat more difficult than the finite difference method from the point of view of computer programming, it has certain inherent advantages that the approximation of solutions can be obtained accurately even in more complicated regions.

The study of the numerical solution of convection-diffusion problems goes back to 1950's. Allen and Southwell [4] in 1955 initiated the numerical studies dealing with singular perturbation problems while discussing the motion in two dimensions of a viscous fluid past a fixed cylinder. Only in 1970's, these studies acquired a research momentum that is continuing till now. The one-sided difference scheme has been described by Dorr [16] constructed a difference scheme which represents the rate of decay in the boundary layer correctly for the homogeneous singular perturbations problems. In 1972, Finlayson reviewed the method of weighted residuals and variational principles [20]. Hemker and Miller, in [24] made a detailed study of numerical analysis of singular perturbation problems. Eckhaus, in [19], exhaustively discussed the asymptotic analysis of singular perturbations. Keller [33] made a numerical solution of two point boundary value problems. Douglas and Dupont presented Galerkin methods for parabolic equations with nonlinear boundary conditions [17]. Doolan et al. [15] discussed some uniform numerical methods for problems with initial and boundary layers.

Osher in [47] considered some nonlinear singular perturbation problems and he discussed the one sided difference schemes. Ross in [57] derived the necessary convergence conditions for backward schemes in two dimensional case. Carey and Pardhanani has studied Multigrid Solution and Grid Redistribution for Convection Diffusion problem in [12]. Han and Kellogg studied the differentiability properties of solutions of the two dimensional convection diffusion equation in a square region in [23] in a two dimensional space. Brandt and Yavneh observed the inadequacy of first order upwind difference schemes with reference to certain recirculatory flows in [11]. Ross has presented ten ways to generate uniformly convergent numerical schemes to solve singular perturbation problems in [58]. Stynes and Tobiska derived necessary conditions for uniform convergence for difference schemes in two dimensional convection diffusion problems in [67]. Drofler in [18] obtained uniform a priori estimates for singularly perturbed elliptic equations in multidimensions. Shih and Elman developed some iterative methods for stabilized convection diffusion problems in [63]. In this context it is worth mentioning that the survey paper by Kadalbajoo and Reddy [30], gives an intellectually stimulating outline of the singular perturbation problems and of fluid dynamical boundary layers. This survey paper will remain as one of the most readable source on convection-diffusion (or singular perturbation) problems. In 2003, Kadalbajoo and Patidar made an exhaustive survey of singularly perturbed problems in partial differential equations in [31] and presented the then existing state of art. Another excellent survey article is due to Stynes [66], on steady state convection diffusion problems. Herein the author highlighted the notion of artificial diffusion.

While the numerical analysis of singularly perturbed convection-diffusion problems has received much attention in the recent years the main focus has been on the solution behavior in the boundary layer region. Roos et al. in the reference[59] have given interpretation about the nature of convection dominated flows with a physical interpretation.

Pearson [49] was perhaps the first to solve numerically linear convection-diffusion type problems using variable mesh size in the finite difference scheme. Pearson [50] also solved non-linear singular perturbation problems using variable mesh methods. Abrahamson et. al. [1] have described the refinement of upstream one-sided difference scheme. A modified upwind scheme for convective diffusion equations which combines the advantages of being stable of second order is described by Axelsson and Gustafsson [7].

Convection-diffusion problems are solved by many researchers for the past five decades which are the linearised equations from Navier –Stokes equations with a large Reynolds number. Due to the high speed computers computations are made simple for large amount of calculations. This survey cannot, for reasons of length, give a complete account of the many numerical methods used to solve steady-state convection-diffusion problems.

Multi Grid Adaptive techniques for solving convection-diffusion problem have been described by Brandt [11]. Multi-Grid adaptive technique is a general strategy of solving continuous problems by cycling between coarser and finer levels of discretization. It provides very fast general solvers together with nearly adaptive optimal discretization schemes. In the process, boundary layers are automatically either resolved or skipped, depending on a control function, which expresses the computational goal. The global error decreases exponentially as a function of the overall computational work, in a uniform rate independent of the magnitude of the singular perturbation terms. These methods are proved to be of high order and uniformly stable. These schemes are applicable for higher order dimensional problems. Hsiao, G.C. and Jordan, K.E [25] have studied the solutions to the difference equations of singular perturbation problems. In [9] Bellman and Kalaba solved non-linear singularly perturbed problem by applying quasi-linearization technique.

For a detailed theory and analytical discussion on singular Perturbation Problems one may refer to the treatises and high level monographs: O’Malley[44] , O’Malley[45], Nayfeh

[41], Nayfeh [42] , Nayfeh [43] , Kevorkian and Cole [35] Bender and Orszag[10] , Smith [64] , Meyer and Parter [37] , and Van Dyke [70].

For a detailed Numerical and Asymptotic discussion on Singular Perturbation Problems one may refer to the books and high level monographs: Hemker and Miller [24] , Miller[38] , Miller [39] , Axelsson et al.[8] Doolan et. al. [15].

The literature in numerical methods could not have been what it is but for the excellent monumental works of Meyer and Parter [37], Miller [38], Neyfeh [41,42], Protter and Weinberger [53], Reddy [55], Smith [64], O'Malley [45, 46], Il'in [27], Kevorkian and Cole [36], Samarskii [60], Verfurth [71], Shashkov[62], Wrigglers [72], Jain [28] and Quarteroni [54].

OUR PRESENT WORK:

Consider an elliptic operator in which the second order derivatives are multiplied by a parameter ε that is allowed to be close to zero. These derivatives model diffusion while the first– order derivatives are associated with the convective or transport processes. In classical problems where ε is not close to zero, diffusion is the dominant mechanism in the model and the first-order convective derivatives play a relatively minor role in the analysis. On the other hand, when ε is close to zero and the elliptic differential operator has convective terms, the convective terms have a significant influence on the theoretical and numerical solution of the problem and cannot be summarily dismissed as 'lower–order' terms. When ε is close to zero and the elliptic differential operator has convective terms, it is called a convection diffusion operator. The problems involving these operators are called convection-diffusion problems and these problems form a class of singular perturbation problems**.** In this thesis, we applied finite difference methods to compute the solutions of some such problems numerically**.** We observe that the solution of convection-diffusion problem has a convective nature on a larger part of the domain of the problem, and the diffusive part of the differential operator is influential only in a certain narrow sub-domain. In the sub domain the gradient of the solution is large. This nature is described by stating that the solution has a boundary layer. The numerical methods that are designed for solving such elliptic differential operators will not work satisfactorily through out the domain since the solution in general is well behaved in the convective region while it exhibits instability in the boundary layer region where the equation is influenced by diffusion.

A second- order differential operator in n- variables whose highest-order derivatives are

$$-\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \tag{1.1}$$

where the a_{ij} are constants, is said to be elliptic if

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq \sigma \sum_{i=1}^{n} \xi_i^2 \quad \text{for all} \quad \xi_i \; and \; \xi_j \tag{1.2}$$

where $\sigma > 0$ is called an ellipticity constant. Consider the second-order differential operator L in n- variables defined on some bounded domain Ω with open connected set D by

$$Lu(x) = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + g(x)\, u(x) \tag{1.3}$$

where a_{ij} are constants. We assume that L is elliptic in the sense of (1.2). Denote the closure of D by $\hat{D}$ and its boundary by ∂D, and let $C_k(s)$ denote the space of functions that are defined on a set S and k-times differentiable on S.

In the differential operators in convection-diffusion problems the ellipticity constant σ can be close to zero. If the value of σ is near zero, then the convergence of the computed solution by employing numerical scheme is a challenging job. Taking this into account, we employ a numerical method so that its solution is stable and appropriate. In this thesis an attempt is made to solve convection-diffusion problem numerically to attain reasonable accuracy for the solution near the boundary layer. Numerically computed solution is compared with the analytical solution and found that the diffusion coefficient is significant especially in the boundary layer region.

While using a uniformly convergent scheme for a convection– diffusion problem we considered a general convection-diffusion equation

$$L\,U(x) = -\varepsilon U''(x) + a(x)U'(x) + b(x)\,U(x) = f(x) \;\; \text{for } 0 < x < 1 \tag{1.4}$$

with the Dirichlet's boundary conditions $\qquad U(0) = U(1) = 0$

where $0 < \varepsilon << 1$, $a(x) > \alpha > 0$ and $b(x) \geq 0$ on [0,1] and presume that $a(x) \leq 1$. Here L is the differential operator. The above problem is solved by the method proposed by the Il'in – Allen-Southwell which is uniformly convergent method. The convergence criteria are realized through computation, based on Roos et al.[59] for most of the values of the diffusion coefficient. In this method Green's function operator is used to find the new finite difference scheme.

We have employed finite element method in this thesis to solve singularly perturbed two point boundary value problems using cubic B splines. The finite element method involves variational methods like Rayleigh-Ritz method, Least squares method, Petrov-Galerkin method, Galerkin method, Collocation method etc. In finite element method, approximate solution of a given differential equation is a linear combination of a set of basis functions which constitutes a basis for the approximation space under consideration. We have employed Galerkin method for solving certain class of singularly perturbed two point boundary value problems with cubic B-splines as basis functions. The basis functions have been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions is applied. A finer mesh has been taken near and around a parameter δ close to zero where the left boundary layer is located. Several examples including linear and nonlinear cases have been considered for testing the efficiency of the proposed method. The solution for a nonlinear problem is obtained as the limit of the solution of a sequence of linear problems generated by quasi-linearization technique due to Bellman and Kalaba [9]. The solutions obtained, by the method developed for the considered examples have been compared with the exact solutions. We observed that the approximate solutions obtained by the developed method are in good agreement with the exact solutions of some known problems available in the existing literature.

Consider the following linear singular perturbed two-point boundary value problem

$$\varepsilon y''(x) + a(x) y'(x) + b(x)y(x) = c(x);\ 0 < x < 1 \qquad (1.5)$$

with $y(0) = y_0$ and $y(1) = y_1$

where ε is a small positive parameter ($0< \varepsilon << 1$) and y_0 , y_1 are given constants and a(x) , b(x) and c(x) are assumed to be continuously differentiable functions in [0,1]. Further, we assume that $a(x) \geq M>0$ throughout the interval [0, 1] where M is some positive constant. This assumption purely implies that the boundary layer will be in the neighborhood of x=0. Existing numerical methods produce good results only when we take step length of interval $h \leq \varepsilon$. This is very costly and time consuming process. Hence the researchers are concentrating on developing the methods, which can work with reasonable step length h. For this, nowadays researchers are adopting one of the following methods.

(i) The interval is subdivided into two regions $[0,\delta]$ and $[\delta, 1]$, where δ is the point near which the boundary layer is located. The region $[0,\delta]$ is called inner region and the region $[\delta,1]$ is called outer region. The problem in the inner region is treated as an initial value problem and the problem in the outer region is treated as a boundary value problem. The initial value problem in the inner region problem is solved and terminal boundary condition is obtained. Using this terminal boundary condition, the boundary value problem in the outer region problem is solved.

(ii) Using the variable mesh, one can take finer mesh around and near the point where the boundary layer is located.

Since the finite element method can be easily adaptable with variable mesh, we intend to use finite element method to solve the given singular perturbation problem.

For the case of single differential equation, it is shown in Douglas and Dupont [17] that the cubic B- splines yield 4th order accurate results. Accordingly, B-splines as basis functions have been used by us in our work.

The existence of the cubic Spline interpolate S(x) to a function f(x) in closed interval [0,1] for spaced knots $0 = x_0 <x_1 < x_2 < x_3 < \ldots < X_{n-2} < x_{n-1} < x_n =1$ is established by constructing it. The construction of S(x) is done with the help of cubic B-Splines. Introduce six additional knots x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2} and x_{n+3} such that

$x_{-3} < x_{-2} < x_{-1} < x_0$ and $x_{n+3} > x_{n+2} > x_{n+1} > x_n$.

Now the cubic B-splines Bi (x) , given in [13] , are defined by

$$\left.\begin{aligned}
B_i(x) &= \sum_{r=i-2}^{r=i+2} \frac{(x_r - x)^3}{\prod'(x_r)}, \quad x \in [x_{i-2}, x_{i+2}] \\
&= 0, \quad \text{otherwise} \\
\text{where} \\
(x_r - x)^3 &= (x_r - x)^3, \text{ if } x_r \geq x \\
&= 0, \text{ if } x_r \leq x \\
\text{and } \prod(x) &= (x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})(x - x_{i+2})
\end{aligned}\right\} \quad (1.6)$$

It can be shown that the set $\{B_{-1}(x), B_0(x), B_1(x), B_2(x), B_n(x), B_{n+1}(x)\}$ forms a basis for the space $S_3(\pi)$ of cubic polynomial splines [52]. Schoenberg [61] has proved that the cubic B-splines are the unique non-zero splines of smallest compact support with knots at

$x_{-3} < x_{-2} < x_{-1} < x_0 < x_n < x_{n+1} < x_{n+2} < x_{n+3}$.

Any cubic spline defined with a unique set of given knots [3] can be uniquely expressed as a linear combination of B-spline basis set $\{B_{-1}(x), B_0(x), B_1(x), B_2(x), B_n(x), B_{n+1}(x)\}$

We develop a method based on Galerkin method with B-spines as basis functions for solving a general linear singularly perturbed two point boundary value problem with left boundary layer.

We discussed a numerical integration method in this thesis. This method reduces a second order differential equation into a first order differential equation with a small deviating argument. To set the stage for the numerical integration method, we consider the following governing linear Convection-diffusion equation.

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \; 0 \leq x \leq 1 \qquad (1.6(a))$$

with $y(0) = \alpha$ and $y(1) = \beta$

where ε is a small positive parameter called diffusion parameter which lies in the interval $0 < \varepsilon << 1$; α and β are given constants; a(x), b(x) and f(x) considered to be sufficiently continuously differentiable functions in [0,1]. Furthermore, we assume that $a(x) \geq M > 0$ throughout the interval [0,1] in equation (1.6(a)), where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of x=0.

Let δ be a small positive deviating argument (0< δ ≤ 1). By applying Taylor series expansions in the neighborhood of the point x, we have

$$y(x-\delta) \cong y(x) - \delta\, y'(x) + \frac{\delta^2}{2} y''(x) \qquad (1.7)$$

Consequently applying equation (1.7) in Eq. (1.5) the second order derivative is replaced by the first-order derivative with a small deviating argument δ. The resultant first order differential equation is numerically solved by applying Simpson's one-third rule to get the three term recurrence relation. The three term recurrence relation is solved by Thomas algorithm. The main advantage of this method is that it does not require very fine mesh size.

In this thesis we have considered a convection-diffusion problem in one-dimension with variable coefficient wherein an artificial –diffusion term [66] is present. As a closed form solution, in general, is not possible the classical Frobenious method of series solution was used to solve the governing differential equation. Further the problem is also solved by making use of a central difference scheme. The Frobenious series solution is numerically computed and the results are compared with those obtained by central difference scheme. The results are depicted through graphs and the results obtained by both the methods seem to be in good agreement. It is observed that the artificial diffusion term plays a significant role in the behaviour of the solution.

The governing equation of artificial diffusion-convection problem in one-dimension is

$$-\left(\varepsilon + \frac{hx}{2}\right)\frac{d^2u}{dx^2} + x\frac{du}{dx} + u = 1 \qquad \text{with } u(0)=0,\ u(1)=0 \qquad (1.8)$$

Let $p(x) = -\dfrac{x}{(\varepsilon + \frac{h\,x}{2})}$, $q(x) = -\dfrac{1}{(\varepsilon + \frac{h\,x}{2})}$, $r(x) = -\dfrac{1}{(\varepsilon + \frac{h\,x}{2})}$.

we can bring (1.8) to the standard form:

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)\,u = r(x) \qquad \text{with } u(0)=0,\ u(1)=0 \qquad (1.9)$$

The differential equation (1.9) is linear with variable coefficients. Closed form solution for this equation seems to be out of reach. Hence we propose to solve by applying series solution method due to Frobenious with x =0 as an ordinary point of (1.9). Method of series solution was used to solve the governing differential equation (1.9). Further the problem is also solved by making use of a central difference scheme. The Frobenious series solution is numerically computed and the results are compared with those obtained by central difference scheme. The results are depicted through graphs and the results obtained by both the methods seem to be in good agreement. It is observed that the artificial diffusion term plays a significant role in the behaviour of the solution. The results are compared with the results in Chapter-2. We have observed that artificial diffusion plays a dominant role in the boundary layer region.

The convection-diffusion problem is extended to two-dimensional space. In two-dimensional space the proposed problem is solved on a unit square mesh with the prescribed boundary conditions by finite difference method where in central difference scheme is employed. In the process finite difference scheme of Standard five point formula was employed. Initial approximations to temperature distribution function were given on the basis suitable to physical nature of the problem.

Here the governing differential equation in two dimensions is

$$-\varepsilon\, \Delta u(x,y) + a(x,y)\ \nabla u\ (x,y) + b(x,y)\ u(x\ ,y) = f(x,y)$$

$$\text{on} \quad \Omega \subset R^2 \text{ with } u(x,y) = g(x,y) \text{ on } \partial\Omega \tag{1.10}$$

where $0 < \varepsilon << 1$, and the functions a, b and f which are assumed to be Holder continuous on $\overline{\Omega}$, the closure of Ω. Here we also assume that $b \geq 0$ on $\overline{\Omega}$. Here Ω is any bounded domain in R^2 with a piecewise Lipschitz–continuous boundary $\partial\Omega$. Let us suppose that g is continuous except perhaps for a jump discontinuity at a single point. The results thus obtained are plotted through graphs and the physical nature of the problem is discussed. It is observed that there is a boundary layer at some specific values of arguments.

In this thesis a problem which is related to wave propagation is also studied. This problem is taken up for study in view of its analogy with convection – diffusion problem.

When two bodies which have distinct velocities come into contact, an impact occurs. The impact force is a function of time ‘t’ which is acting like a compression force. The impact time is very short and the stresses generated are high. We have studied non-linear wave-propagation after impact that occurs in the bodies after impact. The governing equation proposed by Gol’dberg has Non-linear convection-diffusion nature which is analogous to the nature of differential equations studied in the earlier chapters. The wave propagation in the bodies is naturally dependent on the material of the bodies with which they are composed. Here we considered two materials of same physical nature. Nonlinearity is studied after impact. This chapter presents a numerical study of propagating pulses and harmonic waves in nonlinear media using a Finite difference scheme. This study focuses on longitudinal, one-dimensional wave propagation. In the finite difference scheme, non-linear system is reduced to a linear system by applying Quasi-linearization method.

The governing non-linear wave equation which is developed by Gol'dberg (1961) is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(1 - \gamma \frac{\partial u}{\partial x}\right) \frac{\partial^2 u}{\partial x^2} \tag{1.11}$$

with the prescribed physical conditions. This non-linear differential equation is made linear by applying quasi-linearization method. The resultant linear equation is solved by applying central differencing scheme. The numerically computed results reveal the material nature.

STRUCTURE OF THE THESIS:

The thesis consists of five parts

Part-I consists of a single chapter, Chapter- 1 which is introductory in nature and gives an introduction to the steady-state convection-diffusion problems and impact problem. In the steady-state convection-diffusion problem there arises a governing differential equation in which the highest order derivative is multiplied with a perturbation or diffusion parameter. In the impact problem the non-linear wave equation exhibits convection as well as diffusion nature. An introduction to the problems is presented and the methodology adopted is explained.

Part II consists of Chapters 2-5.

In Chapter 2, we present a computational method to solve steady state convection – diffusion problem. In convection-diffusion problem, in a larger part of the domain, transport processes dominate where as diffusion effects restrict only to a relatively small portion of the domain. This state of affairs means that one cannot depend on the elliptic nature of the differential operator to ensure the convergence of standard numerical algorithms. In this chapter, the asymptotic nature of solution to stationary convection-diffusion problem is considered and a numerical technique to control the oscillatory behavior of the computed solution in a boundary layer region at the specific value of the argument is proposed. This is achieved through a stretched variable transformation.

We have solved the problem on steady state convection-diffusion by Finite difference method where in a central difference scheme is employed. The same problem is also studied by asymptotic expansions method. We observed that there is a right-boundary layer near specific value of the argument. In this chapter the diffusion coefficient ε is a small positive parameter and coefficient of convection C is a parameter independent of ε. Here C takes values according to the choices of different mesh sizes.

In Chapter 3, we deal with a uniformly convergent scheme for Convection –Diffusion problem. The above problem is solved by the method proposed by the Il'in –Allen, which is a uniformly convergent method [26]. The convergence criteria is realized through computation and based on the axioms proposed by Roos et al.[59], for lower values of the diffusion coefficient. Under a certain condition, the solution is seen to be uniformly convergent for any choice of the diffusion parameter. The study provides a first- order uniformly convergent method with discrete maximum norm. It was observed that the error decreases as step size h gets smaller for smaller or larger values of the perturbation parameter where as for the mid range values of the perturbation parameter the trend are reversed. An analysis is carried out to check the validity of the solution with some existing analytical solutions available. The uniformly convergent method gives better results than the finite difference methods. The computed and plotted solutions of this method are in good agreement with the exact solution available.

Chapter 4 is devoted to the application of finite element method to solve singularly perturbed two point boundary value problems using cubic B- splines. The finite element method involves variational methods like Rayleigh-Ritz method, Least squares method, Petrov-Galerkin method, Galerkin method, Collocation method etc. In finite element method, approximate solution of a given differential equation is a linear combination of a set of basis functions which constitutes a basis for the approximation space under consideration. In this chapter we have employed Galerkin method for solving certain class of singularly perturbed two point boundary value problems with cubic B-splines as basis functions. The basis functions have been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary condition is defined. A finer mesh has been taken near and around a parameter δ close to zero where the left boundary layer is located. Several examples including linear and nonlinear cases have been considered for testing the efficiency of the proposed method. The solution for a nonlinear problem is obtained as the limit of the solution of a sequence of linear problems generated by quasi-linearization technique due to Bellman and Kalaba [9]. The solutions obtained, by the method developed for the considered examples have been compared with the exact solutions. We observed that the approximate solutions obtained by the developed method are in good agreement with the exact solutions of some known problems available in the existing literature.

Chapter-5 is devoted to the study of a numerical integration method for solving general steady-state convection-diffusion problems. In the fifth chapter the Numerical Integration method is developed by introducing the deviating argument. In this process, Simpson rule is applied to calculate the quadrature. The results thus obtained show good agreement between the exact solution and the computed solution.

In this Chapter the governing second-order differential equation is replaced by an approximate first-order differential equation with a small deviating argument. Then, Simpson one-third formula is used to obtain the three term recurrence relationship. Thomas Algorithm is applied to solve the resulting tri-diagonal algebraic system of equations. The proposed method is iterative on the deviating argument. The method is to be repeated for different choices of the deviating argument until the solution profile stabilizes. The main advantage of this method is that it does not require a very fine mesh size. To examine the applicability of the method employed, we have solved several linear model problems with left-end boundary layer or right –end boundary layer or an internal layer and presented the numerical results. It is observed that the numerical integration method approximates the exact solution extremely well.

In this context it is worth mentioning that the survey paper by Kadalbajoo [30], gives an erudite exposition of the singular perturbation problems and their treatment on fluid dynamical boundary layers. This survey paper will remain as one of the most readable source on convection-diffusion (or singular perturbation) problems.

Part III consists of Chapters 6 and 7. Chapter 6 deals with a convection-diffusion problem in one-dimension with variable coefficient wherein an artificial –diffusion term is present. Stynes [66] introduced the notion of artificial diffusion with respect to a general convection-diffusion problem to get a reasonably accurate solution in the boundary layer region. The numerical artificial-diffusion controls the oscillations near the boundary layer region.

As a closed form solution, in general, is not possible, the classical Frobenious method of series solution was used to solve the governing differential equation in this chapter. Further, the problem is also solved by making use of a central difference scheme. The Frobenious series solution is numerically computed and the results are compared with those obtained by central difference scheme. The results are depicted through graphs and the results obtained by both the methods seem to be in good agreement. It is observed that the artificial diffusion term plays a significant role in the behavior of the solution and reduces the oscillations in the computed solution.

In Chapter-7 we present a numerical study of convection –diffusion problem in two-dimensional space**.** It is solved on a unit square mesh with the prescribed boundary conditions by finite difference method wherein central difference scheme is employed. In the process, finite difference scheme of standard five point formula was employed. Initial approximations to temperature distribution function were given motivated by the physical nature of the problem by intuition. The results thus obtained are plotted through graphs and the physical nature of the problem is discussed. It is observed that there is a boundary layer at the specific values of arguments.

Part IV consists of a single Chapter, Chapter 8. This chapter aims to study the Numerical study of wave propagation in a non-linear medium due to impact.

When two objects which distinct velocities have come into contact, an impact occurs. The impact force is a function of time‘t’ which is acting like a compression force. The impact time is very short and the stresses generated are high. We have studied non-linear wave-propagation after impact that occurs in the bodies after impact. The governing equation proposed by Gol’dberg has non-linear convection-diffusion nature which is analogous to the nature of differential equations studied in the earlier chapters. The wave propagation in the bodies is naturally dependent on the material of the bodies with which they are composed.

Here we considered two materials of same physical nature. Nonlinearity is studied after impact. This chapter presents a numerical study of propagating pulses and harmonic waves in nonlinear media using a Finite difference scheme. This study focuses on longitudinal, one-dimensional wave propagation. In the finite difference scheme, non-linear system is reduced to a linear system by applying Quasi-linearization method in which iteration-across the time step concept is used. The results numerically obtained reveal the material nature.

In this chapter we solved non-linear convection-diffusion type wave equation by applying quasi-linearization technique. In this technique the successive approximation values are calculated by iteration across the time step. The Governing equation (1.11) is the non-linear wave - equation developed by Gol’dberg (1961). It can be easily noticed that this is the well known one dimensional wave equation when $\gamma = 0$.

Part V consists of a single chapter, Chapter 9.

This chapter is devoted to present the main conclusions of the Thesis. We also present some problems which deserve to be studied as a sequel to the present work.

Some of the work is published in standard journals and most of the work is presented in various conferences and symposia, the details of which are presented below:
The references are given at the end of the thesis in alphabetical order.

PAPERS PUBLISHED / COMMUNICATED / PRESENTED:

A part of the work presented in the thesis is published and most of the work is presented in various conferences and symposia, the details of which are presented below:

PAPERS PUBLISHED:

1) "Computational method to solve steady-state convection-diffusion problem" International Journal of Mathematics, Computer Sciences and informational Technology Vol.1, No.1-2 January-December 2008, PP.245-254, ISSN 0974-5580 Serial publications.

2) "Numerical study of one-dimensional contact problem" International eJournal of Mathematics and Engineering (2010) 104-114 ISSN 0976-1411.

3) " Numerical study of convection-diffusion problem in two-dimensional space" International Journal of research and reviews in applied science".Vol.5, Issue-2, November-2010

PAPERS COMMUNICATED FOR PUBLICATION:

1) "Uniformly convergent scheme for Convection-Diffusion problem ", International journal of applied mathematics and computation.

2) "Artificial diffusion-Convection problem in one-dimensional space ". Applied Mathematics and computation, Elsevier Publications.

PAPERS PRESENTED AT CONFERENCES

1) Presented the paper entitled "COMPUTATIONAL METHOD TO SOLVE STEADY STATE CONVECTION-DIFFUSION PROBLEM at 18th Congress of Andhra Pradesh Society for Mathematical Sciences (APSMS) AT **AVN COLLEGE VISHAKAPATNAM, DECEMBER -2009.**
2) Presented the paper Entitled "NUMERICAL STUDY OF ONE-DIMENSIONAL CONTACT PROBLEM "at **INDIAN INSTITUTE OF TECHNOLOGY ROORKEE, December- 2009**.

3) Presented the paper entitled "Artificial diffusion-convection in one dimension a computational approach" at **INTERNATIONAL CONGRESS OF MATHEMATICS, (ICM 2010) Hyderabad, India August 17-24, 2010**

4) Presented the paper entitled " NUMERICAL STUDY OF CONVECTION-DIFFUSION PROBLEM IN TWO –DIMENSIONAL SPACE" at 19TH Congress of Andhra Pradesh Society for Mathematical Sciences (**APSMS**) AT Jyothishmathi Institute of Technology and Science Karimnager 12-14 NOVEMBER , 2010
5) Presented the paper entitled "Numerical study of wave propagation in a Non-linear medium due to impact" at **55th ISTAM, NIT, HAMIRPUR, 17-21 December, 2010.**

PART-II

CHAPTER-2

COMPUTATIONAL METHOD TO SOLVE STEADY STATE CONVECTION – DIFFUSION PROBLEM

INTRODUCTION:

In convection-diffusion problem, transport processes dominate where as diffusion effects restrict to a relatively small portion of the domain. This state of affairs means that one cannot depend on the ellipticity nature of the differential operator to ensure the convergence of standard numerical algorithms. In this chapter, the asymptotic nature of solution to stationary convection-diffusion problem is considered and a numerical technique to control the oscillatory behavior of the computed solution at the specific value of argument is developed.

Consider the elliptic operator whose second-order derivative is multiplied by a parameter ε that is allowed to be close to zero. These derivatives model diffusion while the first – order derivatives are associated with the convective or transport processes. In classical problems where ε is not close to zero, diffusion is the dominant mechanism in the model and the first-order convective derivatives play a relatively minor role in the analysis. On the other side, when ε is near zero and the elliptic differential operator has convective terms, it is called a convection-diffusion operator. The convective terms have a significant influence on the theoretical and numerical solution of the problem and cannot be summarily dismissed as 'lower–order' terms. The Convection-diffusion problems form a class of singular perturbation problems. Here we applied finite difference method to compute the solution numerically.

We can see that the solution of convection-diffusion problem has Convective nature on most of the domain of the problem, and the diffusive part of the differential operator is influential only in the certain narrow sub-domain. In the sub domain the gradient of the solution is large. This nature is described by stating that the solution has a boundary layer.

The interesting fact that the elliptic nature of the differential operator is disguised on most of the domain means that numerical methods designed for elliptic problems will not work satisfactorily. In practice they usually exhibit a certain degree of instability. The challenge then is to modify these methods into a stable form without neglecting their accuracy in numerical methods.

A second- order differential operator in n- variables whose highest-order derivatives are

$$-\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \qquad (2.1)$$

Where the a_{ij}'s are the constants, is said to be elliptic if

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq \sigma \sum_{i=1}^{n} \xi_i^2 \quad \text{for all } \xi_i \text{ and } \xi_j \qquad (2.2)$$

where $\sigma > 0$, called the ellipticity constant. The differential operators in convection-diffusion problems stretch these ellipticity constants close to zero. If the value of σ is near zero, then the convergence of the computed solution by employing numerical scheme is a challenging job. Taking this into account, we develop a numerical method so that its solution is stable and appropriate. This motivates us to solve convection-diffusion problem numerically to attain reasonable accuracy for the solution near the boundary layer.

To solve convection-diffusion problem one has to understand the concepts about Maximum principles and asymptotic expansions. To carry out any numerical analysis we want a priori knowledge of some bounds on the derivatives of the solution of this problem.

MOTIVATION

The numerical solution of convection-diffusion problems goes back to the 1950s Allen and Southwell 1955[4] but only in the 1970s did it acquire a research momentum that has continued to this day. In the literature this field is still very active and as we shall see much effort can be put in. Perhaps the most common source of convection-diffusion problem is the Navier–Stokes equations having nonlinear terms with large Reynolds number. Morton in his classic treatise [40] pointed out that this is by no means the only place where they arise. He listed ten examples involving convection-diffusion equations that include the drift-diffusion equations of semiconductor device modeling and the Black–Scholes equation from financial modeling. He also observed that accurate modeling of the interaction between convective and diffusive processes is the most ubiquitous and challenging task in the numerical approximation of partial differential equations.

In this chapter, the diffusion coefficient ε is small positive parameter and coefficient of convection C will denote a generic constant that is independent of ε. Here C takes values according to different mesh sizes.

ANALYTICAL TOOLS

Consider the second-order differential operator L in n- variables defined on some bounded domain Ω with open connected set D by

$$Lu(x) = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + g(x)\, u(x) \qquad (2.3)$$

where a_{ij}'s are constants. We assume that L is elliptic in the sense of (2.1). Denote the closure of D by $\hat{D}$ and its boundary by ∂D, and let $C_k(s)$ denote the space of functions that are defined on a set S and k-times differentiable on S.

MAXIMUM PRINCIPLE

Let $u \in C^0(\hat{D}) \cap C^2(D)$ satisfy the differential inequality $Lu \geq 0$ on D. Suppose that functions b_i and g are bounded on D and $g \geq 0$ on D. Suppose also that $u \geq 0$ on ∂D then $u \geq 0$ on $\hat{D}$.

FORMULATION OF THE PROBLEM

The two-point boundary value problem

$Lu(x) = -\varepsilon u''(x) + u'(x) = f(x)$ for $0 < x < 1$ (2.4)

with $u(0)=0$, $u(1)=0$

defines convection – diffusion problem. Here ε is a very small positive parameter and f is continuously differentiable in the closed interval [0,1]. The coefficient of the first-order derivative is much larger in magnitude than the coefficient of the second-order derivative i.e. Diffusion is the dominant mechanism in the model and the first-order convective derivatives play a relatively minor role in the analysis. If we set ε=0 then (2.4) becomes a first order differential equation by bringing a phenomenal change. So we expect that this problem is Singularly Perturbed. i.e. in a Singularly Perturbed problem, for $x \in [0,1]$ near the boundary layer $\hat{x} = 1$ we have

$$\lim_{\varepsilon \to 0} \lim_{x \to \hat{x}} u(x) \neq \lim_{x \to \hat{x}} \lim_{\varepsilon \to 0} u(x) \qquad (2.5)$$

To get some immediate insight into the solution of (2.4), we select a simple case with f(x)=1. Then the closed form solution of (2.4) takes the form

$$u(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} \text{ for } 0 \le x \le 1 \qquad (2.6)$$

The graph of the equation (2.6) at some selected value of the parameter ε is displayed below.

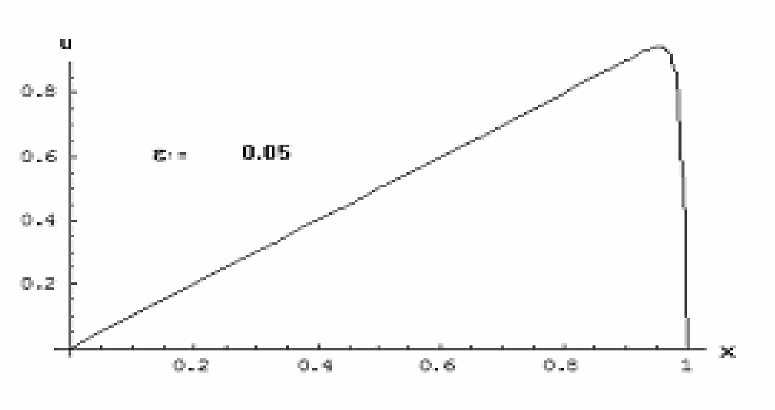


Figure 2.1.

It is clear that (2.5) holds well for the boundary layer at $\hat{x} = 1$. This is a narrow region where u is bounded and is independent of ε but its derivative increases as $\varepsilon \rightarrow 0$. The asymptotic nature of solution to convection-diffusion problem will provide useful information about boundary layer. The behavior of the derivatives of the solution of (2.4) is critical for the numerical computing.

With certain exceptional combinations of the boundary conditions and the force function f the problem (2.4) fails to be singularly perturbed. For instance consider f(x) =1 and the boundary conditions are changed to u(0)=0, u(1) =1, then the solution of (2.4) becomes the well-behaved function u(x)=x and condition (2.5) need not be taken. i.e with this modified boundary conditions, the equation (2.4) becomes a regular perturbation problem.

Consider an asymptotic expansion

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\, \varepsilon^n \tag{2.7}$$

for the solution u(x) of a boundary-value problem (2.4).

Substituting this in (2.4) we get

$$-\varepsilon \sum_{n=0}^{\infty} u_n''(x)\varepsilon^n + \sum_{n=0}^{\infty} u_n'(x) = f(x) \tag{2.8}$$

Comparing the coefficients of powers of ε, we get

$$u_0'(x) = f(x), u_1'(x) = u_0''(x), u_2'(x) = u_1''(x)) \tag{2.9}$$

Here (2.9) consists of first-order differential equations with two boundary conditions $u_n(0)=0$, $u_n(1)=0$ $\forall$ n. If we consider the condition $u_n(0)=0$ and discard the condition $u_n(1)=0$, then we may be able to construct an asymptotic expansion. i.e., in forming the asymptotic expansion, we can discard one boundary condition where a boundary layer occurs. Now solve the equations (2.9) for u(x)

$$u_0(x) = \int_0^x f(t)dt, \quad u_1(x) = f(x) - f(0), \quad u_2(x) = f'(x) - f'(0)$$

Then (2.7) becomes $u(x)=\sum_{n=0}^{\infty}(F^n(x)-F^n(0))\,\varepsilon^n$ (2.10)

Where

$$F(x)=\int_0^x f(t)dt$$

It is easily shown that

$$u(x)=\sum_{n=0}^{\infty}(F^n(x)-F^n(0))\,\varepsilon^n+0(\varepsilon^k) \tag{2.11}$$

for each $k \geq 0$. This function u(x) increases monotonically up to certain state $0 \leq x \leq \delta$, where δ is a constant in (0,1). for a choice of ε and falls steeply before x=1 such that u(1)=0 in the interval (δ,1]. We say that at x=1, u(x) has a boundary layer.

In solution (2.6) the term $e^{-1/\varepsilon}$ is very small and can be ignored. To account the term $e^{-(1-x)/\varepsilon}$ of (2.6), the asymptotic expansion of (2.10) when f(x) =1 should contain a function of the variable $\frac{(1-x)}{\varepsilon}$ to control the oscillations. Hence in the boundary layer equation, we define the stretched variable $\rho = \frac{1-x}{\varepsilon}$ and rewrite the differential equation as a function of ρ instead of a function of x.

Thus set u (ρ)=u(x) for $0 < \rho < \frac{1}{\varepsilon}$ which corresponds to 0<x<1. Now the differential equation takes the form

$$-\varepsilon u'' + u' = \frac{-1}{\varepsilon}(u_{\rho\rho} + u_{\rho}) = Lu$$

the original asymptotic expansion $\sum_{n=0}^{\infty} u_n(x)\varepsilon^n$ in (2.7) is satisfied

i.e., $L[\sum_{n=0}^{\infty} u_n(x)\varepsilon^n] = f$.

So the correction v(*ρ*) that is to be added to this expansion must satisfy Lv=0 i.e. $v_{\rho\rho} + v_{\rho} = 0$. This second order differential equation needs boundary conditions on v(*ρ*) at both *ρ*=0 (at x=1) and *ρ*=1/ε (at x=0). We can now finally apply the original boundary condition u(1)=0, requiring that modified asymptotic expansion satisfies this condition i.e

$$\sum_{n=0}^{\infty} u_n(1)\,\varepsilon^n + v(0) = 0$$

The two point boundary value problem that defines v is now completely specified and can be solved explicitly.

$$V(\rho) = e^{-\rho}\, v(0) \quad e^{-\frac{(1-x)}{\varepsilon}} \sum_{n=0}^{\infty} u_n(1)\,\varepsilon^n$$

$$= \quad -e^{-\frac{(1-x)}{\varepsilon}} \sum_{n=0}^{\infty} (F^n(1) - F^n(0))\,\varepsilon^n$$

Adding this term to (2.10) the new proposed expansion is

$$U_{Asy}(x) = \sum_{n=0}^{\infty} (F^n(x) - F^n(0))\,\varepsilon^n - e^{-\frac{(1-x)}{\varepsilon}} \sum_{n=0}^{\infty} F^n(1) - F^n(0))\,\varepsilon^n \tag{2.12}$$

This is indeed a valid asymptotic expansion. i.e. $u(x) \sim u_{Asy.}(x)$

Thus equation (2.12) is an asymptotic expansion of u(x) that is valid for $0 \le x \le 1$.

FINITE DIFFERENENCE METHOD

Consider the steady-state convection-diffusion problem

$$Lu(x) = -\varepsilon u''(x) + u'(x) = f(x) \quad \text{for} \quad 0<x<1 \tag{2.13}$$

with $u(0) = 0$, $u(1)=0$

where $0 < \varepsilon << 1$, $a(x) > 0$ assumed to be in $C^{\infty}[0,1]$.

Divide the interval [0, 1] into N the equidistant mesh points $x_i = ih$ for i=0, 1,2…..N where h=1/N. Our aim is to compute approximate solution of (2.13) by introducing the finite difference methods. The central and forward difference schemes of first and second order derivatives of u are defined by

$$u'(x)=\frac{u_{i+1}-u_{i-1}}{2h} \text{ and } u''(x)=\frac{u_{i-1}-2u_i+u_{i+1}}{h^2} \quad (2.14)$$

Where $u_i = u(x_i)$. Using (2.14) in (2.13) for f(x) =1, we get

$$-\varepsilon\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}+\frac{u_{i+1}-u_{i-1}}{2h}=1 \; ; \qquad \text{for } 0<x<1$$

The final difference scheme takes the form

$$u_{i+1}+a\,u_i-b\,u_{i-1}=c \quad (2.15)$$

Where

$$a=\frac{4\varepsilon}{h-2\varepsilon},\; b=\frac{h+2\varepsilon}{h-2\varepsilon} \text{ and } c=\frac{2h^2}{h-2\varepsilon} \quad (2.16)$$

The boundary conditions u (0) = u (1) =0 are represented by $u_0=0$ and $u_N=0$. Equation (2.15) represents a Tri-diagonal matrix of the form

$$A\vec{u}=\vec{D} \quad (2.17)$$

where the coefficient matrix A is of order (n-1). The Non-Homogeneous linear system (2.17) is solved by applying Thomas Algorithm. The main idea here is to select suitable values of a, b so that the coefficient Matrix A is a Monotonic -Matrix.

The properties of Monotonic -Matrix are stated below.

1) All the off-diagonal elements must be either zeros or negative i.e. $a_{ij}\le 0$ for $i\ne j$.

2) The coefficient matrix must be a diagonally dominant Matrix.

If these two conditions are satisfied then our numerical method is stable and consistent. The concept incorporated in this problem reduces the oscillations in the computed solution.

If $d=\frac{-2h^2}{h-2\varepsilon}$, $\overline{D}$ =d[1,1,1, …, 1]^T.

The solution matrix $\overline{u}=[u_1, u_2, \ldots, u_{N-1}]^T$. Hence we have

$$\begin{bmatrix} -a & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & -a & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & -a & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & -a & -1 & 0 & 0 & 0 \\ & & & & & .... & & \\ & & & & & .... & & \\ 0 & 0 & 0 & 0 & 0 & b & -a & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & -a \end{bmatrix} \begin{bmatrix} u1 \\ u2 \\ u3 \\ u4 \\ u5 \\ ..... \\ ...... \\ uN-1 \end{bmatrix} = \begin{bmatrix} d \\ d \\ d \\ d \\ ... \\ .... \\ .... \\ d \end{bmatrix}$$

To achieve the condition of M -matrix, the value of h is restricted as below

$$h < 2\varepsilon \quad ,$$

Since a and b are to be negative. This is the condition for convergence.

Let us consider ε =0.01, 0.05 with h=0.01. Then the above inequality $h < 2\varepsilon$ is satisfied. Here x takes the values from 0 to 1 with step size 0.01. By using these values in Thomas Algorithm we can get the computed values as presented in the Table 2.1and 2.2 below and compared with the exact values with a specified parameter value of ϵ.

Case 1: ε = 10 $^{-2}$ and h = 0.01

Sl.No.	Exact value (u_i)	Computed value(U_i)	Sl.No	Exact value (u_i)	Computed value(U_i)
1	0.01000	0.010000	51	0.510000	0.510000
2	0.020000	0.020000	52	0.520000	0.520000
3	0.030000	0.030000	53	0.530000	0.530000
4	0.040000	0.040000	54	0.540000	0.540000
5	0.050000	0.050000	55	0.550000	0.550000
6	0.060000	0.060000	56	0.560000	0.560000
7	0.070000	0.070000	57	0.570000	0.570000
8	0.080000	0.080000	58	0.580000	0.580000
9	0.090000	0.090000	59	0.590000	0.590000
10	0.10000	0.10000	60	0.600000	0.600000
11	0.110000	0.110000	61	0.610000	0.610000
12	0.120000	0.120000	62	0.620000	0.620000
13	0.130000	0.130000	63	0.630000	0.630000
14	0.140000	0.140000	64	0.640000	0.640000
15	0.150000	0.150000	65	0.650000	0.650000
16	0.160000	0.160000	66	0.660000	0.660000
17	0.170000	0.170000	67	0.670000	0.670000
18	0.180000	0.180000	68	0.680000	0.680000
19	0.190000	0.190000	69	0.690000	0.690000
20	0.200000	0.200000	70	0.700000	0.700000
21	0.210000	0.210000	71	0.710000	0.710000
22	0.220000	0.220000	72	0.719999	0.720000
23	0.230000	0.230000	73	0.729999	0.730000
24	0.240000	0.240000	74	0.739999	0.740000
25	0.250000	0.250000	75	0.749999	0.750000
26	0.260000	0.260000	76	0.759999	0.760000
27	0.270000	0.270000	77	0.769999	0.770000

28	0.280000	0.280000	78	0.779999	0.780000
29	0.290000	0.290000	79	0.789999	0.790000
30	0.300000	0.300000	80	0.799999	0.800000
31	0.310000	0.310000	81	0.809999	0.810000
32	0.320000	0.320000	82	0.819999	0.820000
33	0.330000	0.330000	83	0.829999	0.830000
34	0.340000	0.340000	84	0.839999	0.840000
35	0.350000	0.350000	85	0.849999	0.850000
36	0.360000	0.360000	86	0.859999	0.859999
37	0.370000	0.370000	87	0.869999	0.869999
38	0.380000	0.380000	88	0.879999	0.879994
39	0.390000	0.390000	89	0.889997	0.889983
40	0.400000	0.400000	90	0.899994	0.899955
41	0.410000	0.410000	91	0.909982	0.909877
42	0.420000	0.420000	92	0.919948	0.919665
43	0.430000	0.430000	93	0.929845	0.929088
44	0.440000	0.440000	94	0.939537	0.937521
45	0.450000	0.450000	95	0.948614	0.943262
46	0.460000	0.460000	96	0.955843	0.941684
47	0.470000	0.470000	97	0.957530	0.9202130
48	0.480000	0.480000	98	0.942592	0.844665
49	0.490000	0.490000	99	0.877777	0.622121
50	0.500000	0.500000	100	0	0

Table. 2.1

Case ii: ε = 0.05 and h =0.01

Sl.No.	Exact value (u_i)	Computed value(U_i)	Sl.No	Exact value (u_i)	Computed value(U_i)
1	0.01000	0.010000	51	0.509945	0.509900
2	0.020000	0.020000	52	0.519932	0.519900
3	0.030000	0.030000	53	0.529917	0.529900
4	0.040000	0.040000	54	0.539899	0.539900
5	0.050000	0.050000	55	0.549877	0.549900
6	0.060000	0.060000	56	0.559849	0.559800
7	0.070000	0.070000	57	0.569816	0.569800
8	0.080000	0.080000	58	0.579775	0.579800
9	0.090000	0.090000	59	0.589725	0.589700
10	0.10000	0.100000	60	0.599665	0.599700
11	0.110000	0.110000	61	0.609590	0.609600
12	0.120000	0.120000	62	0.619500	0.619500
13	0.130000	0.130000	63	0.629389	0.629400
14	0.140000	0.140000	64	0.639253	0.639300
15	0.150000	0.150000	65	0.649088	0.649100
16	0.160000	0.160000	66	0.658886	0.658900
17	0.170000	0.170000	67	0.668640	0.668700
18	0.180000	0.180000	68	0.678338	0.678400
19	0.190000	0.190000	69	0.687971	0.688000
20	0.200000	0.200000	70	0.697521	0.697600
21	0.210000	0.210000	71	0.706972	0.707000
22	0.220000	0.220000	72	0.716302	0.716400
23	0.230000	0.230000	73	0.725483	0.725600
24	0.240000	0.240000	74	0.734483	0.734600
25	0.250000	0.250000	75	0.743262	0.743400
26	0.260000	0.260000	76	0.751770	0.751900
27	0.270000	0.270000	77	0.759948	0.760100

28	0.279999	0.280000	78	0.767723	0.767900
29	0.289999	0.290000	79	0.775004	0.775200
30	0.299999	0.300000	80	0.781684	0.781900
31	0.309999	0.310000	81	0.787629	0.787900
32	0.319999	0.320000	82	0.792676	0.793000
33	0.329998	0.330000	83	0.796627	0.797000
34	0.339998	0.340000	84	0.799238	0.799700
35	0.349998	0.350000	85	0.800213	0.800700
36	0.359997	0.360000	86	0.799190	0.799800
37	0.369997	0.370000	87	0.795726	0.796400
38	0.379997	0.380000	88	0.789282	0.790100
39	0.389995	0.390000	89	0.779197	0.780100
40	0.3999994	0.400000	90	0.764665	0.765700
41	0.409992	0.410000	91	0.744701	0.745800
42	0.419991	0.420000	92	0.718103	0.719400
43	0.429989	0.430000	93	0.683403	0.684800
44	0.439986	0.440000	94	0.638806	0.640300
45	0.449983	0.450000	95	0.582121	0.583700
46	0.45998	0.460000	96	0.510671	0.512300
47	0.4699750	0.470000	97	0.421188	0.422800
48	0.47997	0.480000	98	0.309680	0.311200
49	0.489963	0.490000	99	0.171269	0.172600
50	0.499955	0.499900	100	0	0

Table. 2.2

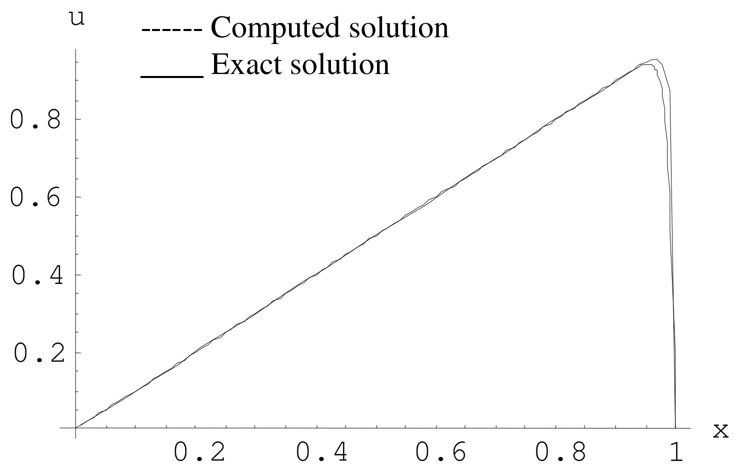


Figure. 2.2

RESULTS AND DISCUSSION

The computed solution is fairly close to the exact solution in the interval (0,1) and in the neighborhood of x=1 the computed solution slightly deviates from the Exact solution because at x=1 there is a boundary layer. The solution may be termed as smooth solution in the interval $(0,\delta)$ where the exact and the computed values of u are very close to each other and the remaining part of solution is called asymptotic solution. Geometrically near at x=1 there is a chaotic behavior. It means we can observe finitely many oscillations near x=1 but by M-Matrix criteria minimizes these *un-even* oscillations. Here boundary layer dies off rapidly as h becomes small. u(x) can be written as the sum of a well–behaved term and a boundary layer term. This decomposition of u is visible in the Tables 2.1, 2.2. The solution has certain lower–order derivatives bounded independent of the perturbation parameter. If other finite difference schemes are taken, we can observe many oscillations in the solution which are not expected in the exact solution.

This above numerical method indicates if there is too little diffusion then the computed solution is often oscillatory, while if there is too much diffusion, then the computed layers are smeared.

CONCLUSIONS

Steady state convection –diffusion problem is solved in the one-dimensional space by using Finite difference method. The solution of the problem is also compared with the exact solution. For convergence criteria there is a condition established by the help of Monotonic Matrix. The monotonic matrix is the coefficient matrix of the system of equations (2.15) appeared after discretization. The numerical results are very fair upto the reasonable accuracy in the smooth region. In the boundary layer region also we can observe that analytical solution very fairly close to the computed solution. For very lower values of the mesh we can get a stable and convergent solution. Asymptotic analysis is also made to test the nature of the problem (2.1) and noticed that there is a right boundary layer near the argument x=1. By enlarge we observed the equation (2.1) is class of singularly perturbed problem so that in the inner region there are some possible oscillations in the computed solution.

CHAPTER-3

UNIFORMLY CONVERGENT SCHEME FOR CONVECTION –DIFFUSION PROBLEM

In this chapter a study of uniformly convergent method proposed by Il'in –Allen-Southwell scheme was made. A condition was contemplated for uniform convergence in the specified domain. The scheme developed is uniformly convergent for any choice of the diffusion parameter. The method provides a first- order uniformly convergent method with discrete maximum norm. Then an analysis carried out by [58] was employed to check the validity of solution with respect to physical aspect and it was in agreement with the analytical solution. The uniformly convergent method gives better results than the finite difference methods. The computed and plotted solutions of this method are in good agreement with the exact solution.

INTRODUCTION

Consider the elliptic operator whose second order derivative is multiplied by a parameter ε that is close to zero. These derivatives model diffusion while first-order derivatives are associated with the convective or transport process. In classical problems ε is not close to zero. To summarize when a standard numerical method is applied to a convection-diffusion problem, when there is too little diffusion then the computed solution is often oscillatory. There is a lot of work in literature dealing with the numerical solution of singularly perturbed problems, showing the interest in this nature of problems in Kellog and Tsan [10].

We can see that the solution of this problem has a convective nature on most of the domain of the problem, and the diffusive part of the differential operator is influential only in the certain narrow sub-domain. In this region the gradient of the solution is large. This nature is described by stating that the solution has a boundary layer.

MOTIVATION

The numerical solution of convection-diffusion problems dates back to the 1950's [4] but only in the 1970s it did acquire a research momentum that has continued to this day. In the literature this field is still very active and as we shall see more effort can be put in. Perhaps the most common source of convection-diffusion problem is the Navier – Stokes equation having nonlinear terms with large Reynolds number.

In this chapter, the diffusion coefficient ε is a small positive parameter and coefficient of convection a(x) is continusely differentiable function. Under these assumptions, Consider the convection –diffusion problem

$L\,u(x) = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x)$ for $0 < x < 1$ With the boundary conditions $u(0) = u(1) = 0$ (3.1)

where $0 < \varepsilon << 1$, $a(x) > \alpha > 0$ and $b(x) \geq 0$ on [0, 1], we also assume that $a(x) \leq 1$ for stable solution in the computation. The above problem is solved by the method proposed by the Il'in –Allen uniformly convergent method. The convergence criteria are realized through computation, based on explanation given by Roos et. al. [59] for lower values of the diffusion coefficient. The reciprocal of the diffusion coefficient is called the Peclet number. For a finite Peclet number the solution patterns matches with the exact solution.

Construction of a Uniformly Convergent Method

We describe a way of construction of uniformly convergent difference scheme. We start with the standard derivation of an exact scheme for the convection-diffusion problem (3.1).Introduce the formal adjoint operator L^* of L and for the sake of convenience select b=0 in (3.1)

Let g_i be local Green's function of L^* with respective to the argument x_i ; i.e.

$$L^* g_i = -\varepsilon g_i'' - a g_i' = 0 \text{ in } (x_{i-1}, x_i) \cup (x_i, x_{i+1}) \tag{3.2}$$

Let us impose boundary conditions

$$g_i(x_{i-1}) = g_i(x_{i+1}) = 0 \tag{3.2 (a)}$$

and impose additional conditions

$$\varepsilon\,(g_i'(x_i^-) - g_i'(x_i^+)) = 1$$

Equation (3.1) is multiplied by g_i integrated with respective to x between the limits

x_{i-1} to x_{i+1} to get $\int_{x_{i-1}}^{x_{i+1}} (Lu)\, g_i\, dx = \int_{x_{i-1}}^{x_{i+1}} f\, g_i\, dx$

$$\int_{x_{i-1}}^{x_{i+1}} (-\varepsilon u''(x) + a u'(x))\, g_i\, dx = \int_{x_{i-1}}^{x_{i+1}} f\, g_i\, dx \qquad (3.\,2(b))$$

Now L.H.S of (3.2(b)):

$$= \int_{x_{i-1}}^{x_i} (-\varepsilon u''(x) + a u'(x))\, g_i\, dx + \int_{x_i}^{x_{i+1}} (-\varepsilon u''(x) + a u'(x)\, g_i\, dx$$

$$= \left[(-\varepsilon u' + a u)\, g_i(x)\right]_{x_{i-1}}^{x_i} + \left[(-\varepsilon u' + a u)\, g_i(x)\right]_{x_i}^{x_{i+1}}$$

$$- \int_{x_{i-1}}^{x_i} (-\varepsilon u' + a u)\, g_i'\, dx - \int_{x_i}^{x_{i+1}} (-\varepsilon u' + a u)\, g_i'\, dx$$

$$= \left[-\varepsilon u'(x_i^-) + a u(x_i)\right] g_i(x_i) - \left[-\varepsilon u'(x_{i-1}) + a\, u(x_{i-1})\right] g_i(x_{i-1})$$

$$+ \left[-\varepsilon u'(x_{i+1}) + a u(x_{i+1})\right] g_i(x_{i+1}) - \left(-\varepsilon u'(x_i^+) + a u(x_i)\right) g_i(x_i)\Big]$$

$$- \int_{x_{i-1}}^{x_i} (a u)\, g_i'\, dx - \int_{x_i}^{x_{i+1}} (a u)\, g_i'\, dx + \int_{x_{i-1}}^{x_i} (\varepsilon u')\, g_i'\, dx + \int_{x_i}^{x_{i+1}} (\varepsilon u')\, g_i'\, dx$$

$$-(-\varepsilon u'(x_i^+) + a u(x_i))\, g_i(x_i)] - \int_{x_{i-1}}^{x_i} (au)\, g_i'\, dx - \int_{x_i}^{x_{i+1}} (au)\, g_i'\, dx + \int_{x_{i-1}}^{x_i} (\varepsilon u')'\, g_i'\, dx + \int_{x_i}^{x_{i+1}} (\varepsilon u')'\, g_i'\, dx$$

$$= -\varepsilon u'(x_i^-)\, g_i(x_i) + \varepsilon u'(x_i^+)\, g_i(x_i) + [\varepsilon u(x)\, g_i'(x)]_{x_{i-1}}^{x_i} + [\varepsilon u(x)\, g_i'(x)]_{x_i}^{x_{i+1}}$$

$$+ \int_{x_{i-1}}^{x_i} (-\varepsilon g_i'' - a g_i')\, u\, dx + \int_{x_i}^{x_{i+1}} (-\varepsilon g_i'' - a g_i')\, u\, dx$$

Since u' is continuous on (x_{i-1}, x_{i+1}), we have

$$=[\varepsilon\, u(x_i) g_i'(x_i^-) - \varepsilon\, u(x_{i-1}) g_i'(x_{i-1}^+)] + [\varepsilon\, u(x_{i+1}) g_i'(x_{i+1}^-) - \varepsilon u\,(x_i) g_i'(x_i^+)]$$

$$= -\varepsilon g_i'(x_{i-1})\, u_{i-1} + u_i + \varepsilon g_i'(x_{i+1})\, u_{i+1} = f \int_{x_{i-1}}^{x_{i+1}} g_i \; dx \qquad (3.3)$$

The difference scheme of equation (3.2) is exact. We can evaluate each g_i' exactly

The solution of the equation (3.2) is given by

$$g_i(x^-) = c_1 + c_2 \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax}{\varepsilon}} \quad \text{on } (x_{i-1}, x_{i+1}) \qquad (3.4(a))$$

$$g_i(x^+) = c_1' + c_2' \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax}{\varepsilon}} \quad \text{on } (x_{i-1}, x_{i+1}) \qquad (3.4(b))$$

Here there are 4 unknowns c_1, c_2, c_1', c_2' requiring 4 equations

$$g_i(x_{i-1}) = 0 \qquad (3.5)$$

$$g_i(x_{i+1}) = 0 \qquad (3.6)$$

$$\varepsilon\,(g_i'(x_i^-) - g_i'(x_i^+)) = 1 \qquad (3.7)$$

and, from continuity of g_i at $x=x_i$

$$g_i(x_i^-) = g_i(x_i^+). \qquad (3.8)$$

On imposing boundary conditions (3.5) and (3.6) on (3.4(a)), (3.4(b)) it can be seen

$$g_i(x_{i-1}) = c_1 + c_2 \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax_{i-1}}{\varepsilon}} = 0 \qquad (3.9)$$

$$g_i(x_{i+1}) = c_1' + c_2' \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax_{i+1}}{\varepsilon}} = 0 \qquad (3.10)$$

On differentiation of equations (3.4(a)), (3.4(b))

$$g_i'(x_i^-) = c_2 \left(-\frac{\varepsilon}{a}\right)\left(\frac{-a}{\varepsilon}\right) e^{\frac{-ax_i}{\varepsilon}}, \quad g_i'(x_i^+) = c_2'\left(-\frac{\varepsilon}{a}\right)\left(\frac{-a}{\varepsilon}\right) e^{\frac{-ax_i}{\varepsilon}}$$

Then the equation (3.7) can be written in the following form

$$\varepsilon\,(c_2 e^{\frac{-a x_i}{\varepsilon}} - c_2^{'} e^{\frac{-a x_i}{\varepsilon}}) = 1 \;\Rightarrow c_2 - c_2^{'} = \frac{1}{\varepsilon} e^{\frac{-a x_i}{\varepsilon}} \qquad (3.11)$$

Using the fact $g_i(x_i^-) = g_i(x_i^+)$ at x= x_i in (3.9), (3.10) it follows

$$\Rightarrow \quad c_1 + c_2 (\frac{-\varepsilon}{a})\, e^{\frac{-a x_i}{\varepsilon}} - [c_1^{'} + c_2^{'} (\frac{-\varepsilon}{a}) e^{\frac{-a x_i}{\varepsilon}}] = 0 \qquad (3.12)$$

On assumption that $\alpha_i = \frac{a x_i}{\varepsilon}, \rho_i = \frac{ah}{\varepsilon}$, above equations may be rewritten as

$$e^{\frac{a x_{i+1}}{\varepsilon}} = e^{\frac{a(x_i + h)}{\varepsilon}}$$

$$e^{\frac{a x_{i+1}}{\varepsilon}} = e^{\frac{a(x_i + h)}{\varepsilon}} = e^{\alpha_i + \rho_i},\; e^{\frac{a x_{i-1}}{\varepsilon}} = e^{\alpha_i - \rho_i}$$

Hence on transformation of the equations (3.9) to (3.12) in to the equations (3.13) to (3.16)

$$c_1 + c_2 (\frac{-\varepsilon}{a}) e^{-\alpha_i + \rho_i} = 0 \qquad (3.13)$$

$$c_1^{'} + c_2^{'} (\frac{-\varepsilon}{a})\, e^{-(\alpha_i + \rho_i)} = 0 \qquad (3.14)$$

$$c_2 - c_2^{'} = \frac{1}{\varepsilon} e^{\alpha_i} \qquad (3.15)$$

$$(c_1 - c_1^{'}) + (c_2 - c_2^{'})(-\frac{\varepsilon}{a})\, e^{-\alpha_i} = 0 \qquad (3.16)$$

On insertion of (3.15) into the equation (3.16) we can get

$$(c_1 - c_1^{'}) + \frac{1}{\varepsilon} e^{\alpha_i} (\frac{-\varepsilon}{a})\, e^{-\alpha_i} = 0$$

$$(c_1 - c_1^{'}) = \frac{1}{a} \qquad (3.17)$$

Subtracting the equation (3.14) from the equation (3.13), then by using equations (3.15) & (3.17) it may be obtained

$$(c_1 - c_1') + (c_2 e^{-\alpha_i + \rho_i} - c_2' e^{-\alpha_i - \rho_i})\left(\frac{-\varepsilon}{a}\right) = 0$$

$$\frac{1}{a} + (c_2 e^{-\alpha_i + \rho_i} - (c_2 - \frac{1}{\varepsilon} e^{\alpha_i}) e^{-(\alpha_i + \rho_i)})\left(\frac{-\varepsilon}{a}\right) = 0$$

$$\frac{1}{a} + (c_2 e^{-\alpha_i + \rho_i} - c_2 e^{-(\alpha_i + \rho_i)} + \frac{1}{\varepsilon} e^{\alpha_i} e^{-\alpha_i - \rho_i})\left(\frac{-\varepsilon}{a}\right) = 0 \tag{3.18}$$

From (3.18) it follows

$$c_2 = \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \tag{3.19}$$

To find c_2' the value of c_2 is substituted in (3.15) , to get

$$c_2' = \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{\rho_i})}{(e^{\rho_i} - e^{-\rho_i})} \tag{3.20}$$

Again employing the value of c_2 in (3.13) the value of c_1 can be obtained as

$$c_1 = \frac{1}{a} \frac{e^{\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} \tag{3.21}$$

Next the value of c_1 is used in (3.17) to obtain c_1'

$$c_1' = \frac{1}{a} \frac{e^{-\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} \tag{3.22}$$

Now on imposition of equations (3.19) to (3.22), on (3.4(a)), (3.4(b)) they may be rewritten as

$$g_i(x^-) = \frac{1}{a} \frac{e^{\rho_i} - 1}{(e^{\rho_i} - e^{-\rho_i})} + \frac{e^{\alpha_i}}{\varepsilon} \frac{(1 - e^{-\rho_i})}{(e^{\rho_i} - e^{-\rho_i})}\left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax}{\varepsilon}} \tag{3. 23(a)}$$

$$g_i(x^+) = \frac{1}{a}\,\frac{e^{-\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})} + \frac{e^{\alpha_i}}{\varepsilon}\,\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}\left(\frac{-\varepsilon}{a}\right)e^{\frac{-ax}{\varepsilon}} \qquad (3.23(b))$$

The derivatives of equations (3.23(a)) , (3.23(b)) are

$$g'_i(x^-) = \frac{1}{\varepsilon}\,e^{\frac{-ax}{\varepsilon}}\,e^{\frac{ax_i}{\varepsilon}}\,\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})} \qquad (3.23(c))$$

$$g'_i(x^+) = \frac{1}{\varepsilon}\,e^{\frac{-ax}{\varepsilon}}\,e^{\frac{ax_i}{\varepsilon}}\,\frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})} \qquad (3.23(d))$$

Now from (3.23(c)) , (3.23(d)) and (3.7) it follows.

$$g'_i(x^-_{i-1}) = \frac{1}{\varepsilon}\,e^{\frac{ah}{\varepsilon}}\,\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}.$$

i.e.,
$$g'_i(x^-_{i-1}) = \frac{1}{\varepsilon}\,\frac{(e^{\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})} \qquad (3.24(a))$$

$$g'_i(x^+_{i+1}) = \frac{1}{\varepsilon}\,\frac{(e^{-\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})} \qquad (3.24(b))$$

Now by inserting values of g_i^+ and g_i^- from (3.24(a)), (3.24(b)) in (3.2(a)) & (3.2(b)) it may be obtained as

$$f\int_{x_{i-1}}^{x_{i+1}} g_i\,dx = f\left[\int_{x_{i-1}}^{x_i} g_i^-\,dx + \int_{x_i}^{x_{i+1}} g_i^+\,dx\right] \text{ where } \rho_i = \frac{ah}{\varepsilon},\ \alpha_i = \frac{ax_i}{\varepsilon}$$

$$= \int_{x_{i-1}}^{x_i}\left[\frac{1}{a}\,\frac{e^{\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})} + \frac{e^{\alpha_i}}{\varepsilon}\,\frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}\left(\frac{-\varepsilon}{a}\right)e^{\frac{-ax}{\varepsilon}}\right]dx \quad +$$

$$\int_{x_i}^{x_{i+1}} \left[\frac{1}{a} \frac{e^{-\rho_i}-1}{(e^{\rho_i}-e^{-\rho_i})} + \frac{e^{\alpha_i}}{\varepsilon} \frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})} \left(\frac{-\varepsilon}{a}\right) e^{\frac{-ax}{\varepsilon}} \right] dx$$

$$= \left[\frac{h}{a}\frac{(e^{\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})}\right] + \left[\frac{\varepsilon}{a^2} e^{\alpha_i} e^{\frac{-ax_i}{\varepsilon}} \frac{(1-e^{-\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}\left(1-e^{\frac{ah}{\varepsilon}}\right)\right]$$

$$+ \left[\frac{h}{a}\frac{(e^{-\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})}\right] + \left[\frac{\varepsilon}{a^2} e^{\alpha_i} e^{\frac{-ax_i}{\varepsilon}} \frac{(1-e^{\rho_i})}{(e^{\rho_i}-e^{-\rho_i})}\left(e^{\frac{-ah}{\varepsilon}}-1\right)\right]$$

$$= \frac{h}{a}\frac{(e^{\rho_i}+e^{-\rho_i}-2)}{(e^{\rho_i}-e^{-\rho_i})} + \left[\frac{\varepsilon}{a^2} e^{\alpha_i} e^{\frac{-ax_i}{\varepsilon}} \left(\frac{(1-e^{-\rho_i})(1-e^{\rho_i}) + (1-e^{\rho_i})(e^{-\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})}\right)\right]$$

$$= \frac{h}{a}\frac{\left(e^{\frac{\rho_i}{2}}-e^{\frac{-\rho_i}{2}}\right)^2}{\left(e^{\frac{\rho_i}{2}}-e^{\frac{-\rho_i}{2}}\right)\left(e^{\frac{\rho_i}{2}}+e^{\frac{-\rho_i}{2}}\right)} = \frac{h}{a}\frac{(e^{\rho_i}-1)}{(e^{\rho_i}+1)}$$

Finally, it can be represented as follows

$f\int_{x_{i-1}}^{x_{i+1}} g_i\,dx = f\,\frac{h}{a}\frac{(e^{\rho_i}-1)}{(e^{\rho_i}+1)}$ This gives the final scheme as

$$-\frac{(e^{\rho_i}-1)}{(e^{\rho_i}-e^{-\rho_i})}u_{i-1} + u_i - \frac{1-e^{-\rho_i}}{(e^{\rho_i}-e^{-\rho_i})}u_{i+1} = f\,\frac{h}{a}\frac{(e^{\rho_i}-1)}{(e^{\rho_i}+1)} \qquad (3.25)$$

here $\rho_i = \frac{ah}{\varepsilon}$.

The equation (3.25) is the Il'in-Allen-Southwell scheme.

This method is tested for a linear problem by applying a mixture of perturbation parameter values with in the defined range. It is observed from the numerical results that Il'in-Allen scheme is converging uniformly in the defined domain. Especially in the boundary layer region, it is appreciable thing that the scheme is uniformly converging to the exact solution.

For testing the algorithm outlined above we are considered the two-point boundary value problem $-\varepsilon u''(x) + u'(x) = 2x$ with u(0) = u(1) = 0 (3.26)

is considered with $0 < a(x) \le 1$ so that a right-boundary layer exists.

The analytical solution of (3.26) is

$$u(x) = \frac{(1+2\varepsilon)}{(e^{\frac{1}{\varepsilon}} - 1)} - \frac{(1+2\varepsilon)}{(e^{\frac{1}{\varepsilon}} - 1)} e^{\frac{x}{\varepsilon}} + x^2 + 2\varepsilon x, \; 0 < \varepsilon << 1 \tag{3.27}$$

The computational method is executed with various choices of the diffusion coefficient by applying forward difference method, upwind method, central difference method and the Il'in-Allen scheme. The results obtained are presented in the table **3.1(a)** to **3.1(d).**

Case 1:

ε=0.05

X	Forward scheme	Backward scheme	Central Scheme	Allen-Il'in scheme	Exact solution
0	0	0	0	0	0
0.01	0.001000	0.001200	0.001100	0.001103	0.0010999
0.02	0.002200	0.002600	0.002400	0.002407	0.0023999
0.03	0.003600	0.004200	0.003900	0.005613	0.0038999
0.04	0.005200	0.006000	0.005600	0.005613	0.005599
0.05	0.007000	0.008000	0.007500	0.007517	0.0074999
0.06	0.009000	0.010200	0.009600	0.009620	0.0095999
0.07	0.011200	0.012600	0.011900	0.011923	0.0118999
0.08	0.013600	0.015200	0.014400	0.014427	0.0143999
0.09	0.016200	0.018000	0.017100	0.017130	0.0170999
0.1	0.019000	0.02100	0.020000	0.020033	0.019999
0.2	0.058000	0.062000	0.060000	0.060067	0.06509985
0.3	0.123999	0.122998	0.119999	0.120100	0.127098
0.4	0.204997	0.203985	0.19995	0.200129	0.209091
0.5	0.305981	0.304899	0.299960	0.300127	0.2999500
0.6	0.426848	0.425363	0.419700	0.419895	0.41963099
0.7	0.566617	0.563036	0.557771	0.557961	0.5572733
0.8	0.716180	0.703420	0.703422	0.703453	0.6998527
0.9	0.790694	0.756763	0.776690	0.756044	0.75113119
0.91	0.780071	0.745514	0.768388	0.767636	0.737271
0.92	0.762193	0.728374	0.754197	0.753337	0.7463138
0.93	0.735194	0.704127	0.732763	0.731799	0.7166433
0.94	0.696747	0.671311	0.702433	0.701373	0.706286
0.95	0.643937	0.628171	0.661185	0.660049	0.6866133
0.96	0.573125	0.572603	0.606549	0.605369	0.646286
0.97	0.479760	0.502081	0.535505	0.534331	0.592832
0.98	0.358154	0.413575	0.444363	0.44321	0.4342072
0.99	0.20119	0.167335	0.182736	0.182179	0.17849617
1	0	0	0	0	0

Table 3.1 (a)

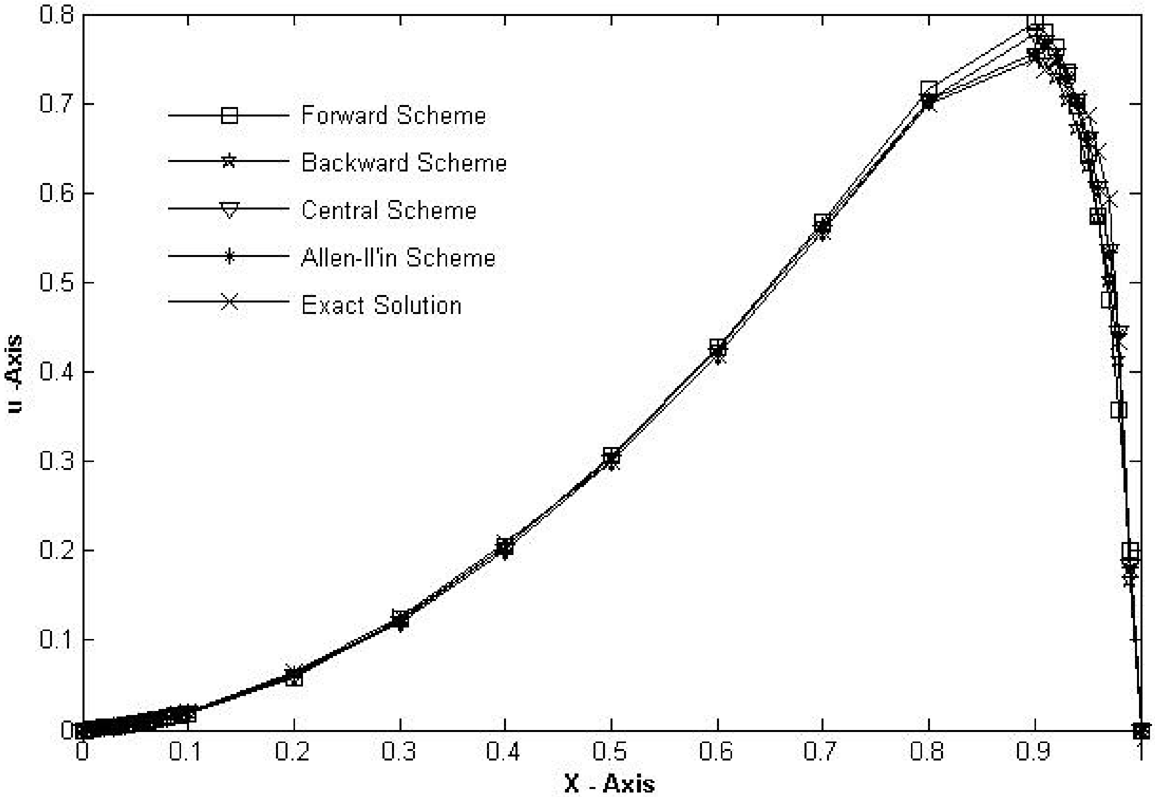

0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
u -Axis
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
X - Axis
Forward Scheme
Backward Scheme
Central Scheme
Allen-Il'in Scheme
Exact Solution

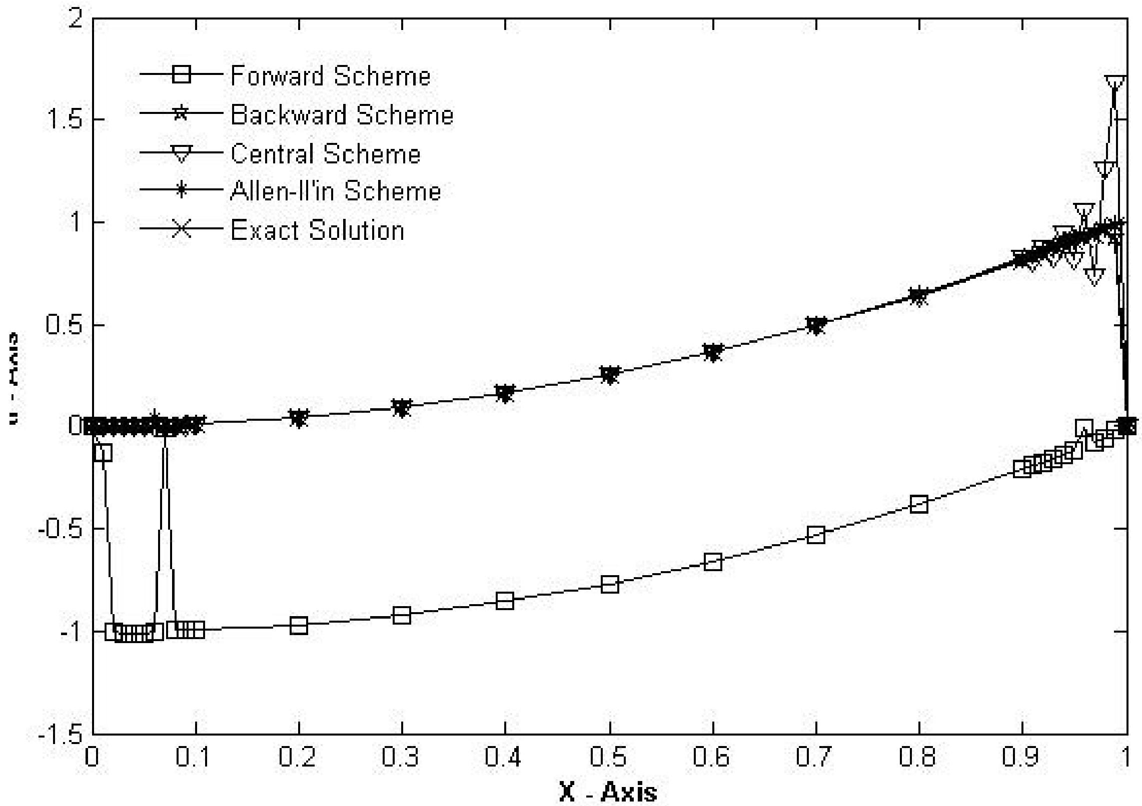
Figure. 3.1(a)

Case: 2

When $\varepsilon = 10^{-3}$:

x	Forward scheme	Backward scheme	Central Scheme	Allen-Il'in scheme	Exact solution
0	0	0	0	0	0
0.01	-0.12447	0.000220	0.000120	0.00020	0.00012
0.02	-0.99928	0.000640	0.000440	0.000600	0.00044
0.03	-1.01274	0.001260	0.000960	0.001200	0.00096
0.04	-1.01058	0.002080	0.001680	0.002	0.00168
0.05	-1.00993	0.003100	0.002600	0.0030	0.002600
0.06	-1.00080	0.004320	0.003720	0.04200	0.0037199
0.07	-0.00858	0.005740	0.005040	0.005600	0.005040
0.08	-0.99616	0.007360	0.006560	0.007200	0.006560
0.09	-0.99354	0.009180	0.008280	0.009000	0.00828
0.1	-0.99072	0.011200	0.010200	0.01100	0.01020
0.2	-0.97362	0.042400	0.040400	0.042000	0.04040
0.3	-0.92442	0.093600	0.090600	0.093000	0.0906
0.4	-0.85522	0.164800	0.160800	0.164000	0.160800
0.5	-0.76602	0.256000	0.251000	0.255000	0.251000
0.6	-0.65682	0.367200	0.361200	0.366001	0.3611999
0.7	-0.52762	0.498400	0.491404	0.497001	0.49140
0.8	-0.37842	0.649600	0.641805	0.648001	0.641600
0.9	-0.20922	0.820800	0.823617	0.819001	0.81180
0.91	-0.19120	0.839020	0.812195	0.837201	0.829920
0.92	-0.17298	0.857440	0.874828	0.855601	0.848240
0.93	-0.15456	0.876060	0.826879	0.874201	0.866760
0.94	-0.13594	0.894880	0.945302	0.893001	0.885480
0.95	-0.11712	0.913899	0.814667	0.912001	0.904400
0.96	-0.00981	0.933114	1.058120	0.931201	0.92352
0.97	-0.07888	0.952469	0.740940	0.950601	0.9428399
0.98	-0.05946	0.971384	1.265210	0.970201	0.9623599
0.99	-0.02002	0.91816	1.683413	0.990001	0.9820345
1	0	0	0	0	0

Table.3.1 (b)



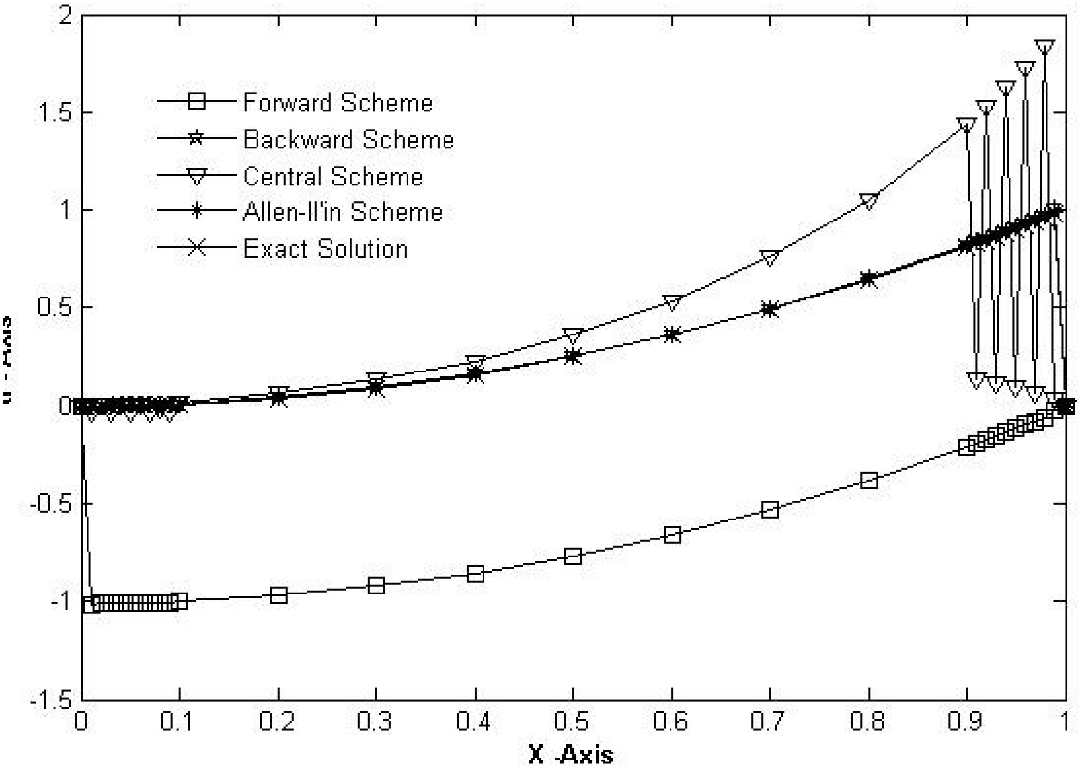
Forward Scheme
Backward Scheme
Central Scheme
Allen-Il'in Scheme
Exact Solution
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
X - Axis

Figure. 3.1(b)

When $\varepsilon = 10^{-4}$

x	Forward scheme	Backward scheme	Central Scheme	Allen-Il'in scheme	Exact solution
0	0	0	0	0	0
0.01	-1.020404	0.000202	-0.03588	0.000200	0.000102
0.02	-1.009895	0.000604	0.00187	0.000600	0.000404
0.03	-1.009597	0.001206	-0.03661	0.001200	0.000906
0.04	-1.008994	0.002008	0.00466	0.002000	0.001608
0.05	-1.008192	0.003010	-0.03666	0.003000	0.0025100
0.06	-1.00719	0.004212	0.00839	0.004200	0.0036199
0.07	-1.005988	0.005614	-0.03605	0.007200	0.0049140
0.08	-1.004586	0.007216	-0.01306	0.007200	0.006416
0.09	-1.002984	0.009018	-0.03479	0.009000	0.00818
0.1	-1.001182	0.011020	0.01869	0.011000	0.01002
0.2	-0.972162	0.042040	0.06165	0.042000	0.040040
0.3	-0.923142	0.093060	0.13098	0.093000	0.09006
0.4	-0.854122	0.164080	0.22981	0.164000	0.16008
0.5	-0.765102	0.255100	0.36280	0.255000	0.250100
0.6	-0.656082	0.366120	0.53694	0.366000	0.360120
0.7	-0.527062	0.497140	0.76261	0.497000	0.490140
0.8	-0.378042	0.648100	1.05533	0.648000	0.640160
0.9	-0.209022	0.819180	1.43826	0.819000	0.81018
0.91	-0.191020	0.837382	0.13857	0.837200	0.828282
0.92	-0.172818	0.855784	1.52845	0.855600	0.846584
0.93	-0.154416	0.874386	0.11939	0.874200	0.865086
0.94	-0.135814	0.893188	1.62392	0.893000	0.883788
0.95	-0.117012	0.912190	0.09635	0.912000	0.902690
0.96	-0.098010	0.931216	1.72504	0.931200	0.921792
0.97	-0.078808	0.950616	0.06906	0.950600	0.941094
0.98	-0.059406	0.970216	1.83223	0.970200	0.9605959
0.99	-0.020002	1.008987	0.03709	0.990000	0.980298
1	0	0	0	0	0

Table. 3.1 (c)



Forward Scheme
Backward Scheme
Central Scheme
Allen-Il'in Scheme
Exact Solution
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
X -Axis

Figure. 3.1(c)

Case: 4:

When $\varepsilon = 10^{-5}$

x	Forward scheme	Backward scheme	Central Scheme	Allen-Il'in scheme	Exact solution
0	0	0	0	0	0
0.01	-1.011037	0.000200	-0.818348	0.00200	0.000100
0.02	-1.009825	0.000600	0.003681	0.000600	0.0121022
0.03	-1.009426	0.001201	-0.820841	0.001200	0.0144024
0.04	-1.008825	0.002001	0.008188	0.002000	0.0169026
0.05	-1.008025	0.003001	-0.822561	0.003000	0.0196028
0.06	-1.007025	0.004201	0.013522	0.004200	0.022503
0.07	-1.005825	0.005601	-0.082350	0.005600	0.0256032
0.08	-1.004424	0.007202	0.019682	0.007200	0.0289034
0.09	-1.002824	0.009002	-0.823680	0.009000	0.0324036
0.1	-1.001024	0.011002	0.026669	0.01100	0.0361028
0.2	-0.972021	0.042004	0.074019	0.042000	0.0841058
0.3	-0.923018	0.093006	0.142077	0.193000	0.1521078
0.4	-0.854015	0.164008	0.230871	0.264000	0.2401097
0.5	-0.765012	0.255010	0.340432	0.355000	0.3481117
0.6	-0.656009	0.366011	0.470792	0.466000	0.4761138
0.7	-0.527007	0.497013	0.621983	0.697000	0.6241158
0.8	-0.378005	0.648014	0.794038	0.74800	0.7921178
0.9	-0.209002	0.819016	0.986993	0.819000	0.81001
0.91	-0.191002	0.837216	-0.168016	0.837200	0.827004
0.92	-0.172802	0.855616	1.028095	0.855600	0.848282
0.93	-0.154402	0.874216	-0.135937	0.874200	0.8650680
0.94	-0.135801	0.89016	1.070035	0.893000	0.873788
0.95	-0.117001	0.912016	-0.103095	0.912000	0.900691
0.96	-0.098001	0.931216	1.112814	0.931200	0.920006
0.97	-0.078801	0.950616	-0.069491	0.950600	0.940002
0.98	-0.059401	0.970216	1.156430	0.970200	0.9600231
0.99	-0.039800	1.008987	-0.035126	0.99000	0.980098
1	0	0	0	0	0

Table. 3.1(d)

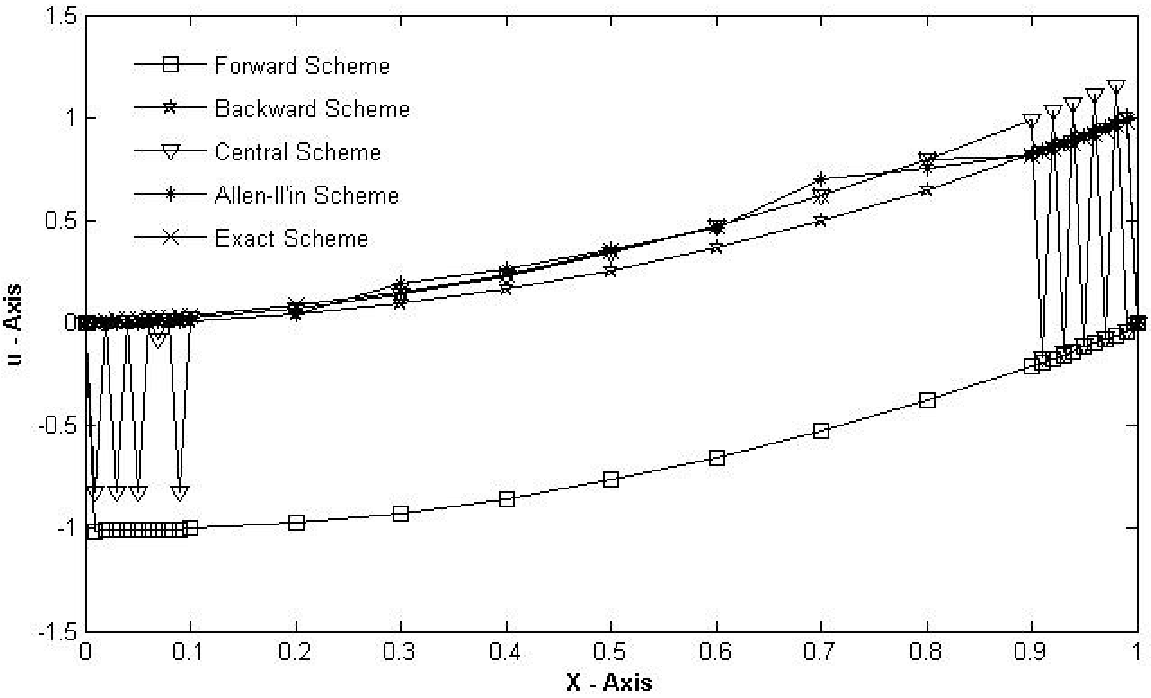


Figure. 3.1 (d)

Error Analysis:

The present scheme is first-order uniformly convergent with the discrete maximum norm.

$$\max_{i} \left| u(x_i) - u_i \right| \le C\,h$$

The region of the solution u is divided into two parts

1) Smooth region with bounded derivatives.

2) Boundary layer region with chaotic behavior where in u = v+z , v is a boundary layer function and z is the smooth function. The bound on the smooth function $\left| z^{j} \right|$ has a factor ε^{1-j} .

The calculation of $\left|z(x_i) - z_i\right|$ is now considered. The corresponding consistency error $\left|\tau_i\right|$ is estimated with the help of Taylor series, proposed by H.G. Roos et. al. [59] which gives the inequality

$$\left|\tau_i\right| \le C \int_{x_{i-1}}^{x_{i+1}} (\varepsilon\left|z^3(t)\right| + a\left|z''(t)\right|)\,dt$$

$$\le Ch + C\varepsilon^{-1} \int_{x_{i-1}}^{x_{i+1}} \exp(-a_0 \frac{1-t}{\varepsilon})dt$$

$$\le Ch + C \sinh(\frac{a_0 h}{\varepsilon}) \exp(-a_0 \frac{1-x_i}{\varepsilon})$$

An application of the discrete comparison principle indicates the increase of power of ε

i.e., $\left|z(x_i) - z_i\right| \le Ch + C \sinh(\frac{a_0 h}{\varepsilon}) \exp(-a_0 \frac{1-x_i}{\varepsilon})$ for i= 1,2 ,3,……n

for $\varepsilon \le h$ that can be easily obtained

$$\left| z(x_i) - z_i \right| \le Ch.$$

In the second case $h \le \varepsilon$, using the inequality $1-e^{-t} \le ct$ for $t > 0$ the desired estimate can be put as $\left| z(x_i) - z_i \right| \le Ch$

Similarly $\left| v(x_i) - v_i \right| \le C\frac{h^2}{h+\varepsilon} \le Ch$ as proposed by Kellogg [32, 34]

This shows that Il'in-Allen scheme is uniformly convergent of first order.

In the above scheme the value of a(x) the convection coefficient is less than or equal to unity, then the scheme converges faster to the exact solution.

RESULT ANALYSIS

We have solved the proposed convection-diffusion problem which is linear and has a right boundary layer region by using forward difference scheme, upwind scheme, central difference scheme and Il'in- Allen scheme by selecting the fine mesh size h = 0.01 and allowed the diffusion coefficient to take different values. We have selected ε = 0.05, 0.001, 0.0001, 0.00001.

1) For ε =0.05 all the schemes behave similarly in the smooth region as well as in the boundary layer region.

2) For ε = 0.001 forward scheme is not matching with the exact solution , upwind scheme converging to exact solution well and the central difference scheme converges in the smooth region and oscillates in the boundary layer. where as Il'in scheme converges uniformly in the entire region.

3) For ε = 0.0001, 0.00001 forward scheme diverges, central scheme oscillates. Upwind scheme has produced good numeric results in the specified domain. But at the boundary i.e. near to the point x=1 the upwind scheme is not matching with the exact solution. The solution of the upwind scheme is not uniformly convergent with the discrete maximum norm, where as the proposed scheme is uniformly convergent of first order even for lower values of ε through out the domain.

4) For finite value of the Peclet number Il'in-Allen scheme behaves well with the exact solution in the region [0,1].

5) The standard finite difference scheme of upwind and central scheme on equally spaced mesh does not converge uniformly. Because, the point wise error is not necessarily reduced by successive uniform improvement of the mesh in contrast to solving unperturbed problems. The standard central difference scheme is of order $O(h^2)$. It is numerically unstable in the boundary layer region and gives oscillatory solutions unless the mesh width is small comparatively with the diffusion coefficient but it is practically not possible as diffusion coefficient is very small.

6) For any value of x in [0,1] , $a(x) \leq 1$ Il'in- Allen scheme converges uniformly. This has been thoroughly verified through computation.

CONCLUSIONS

In this chapter a method developed by Il'in- Allen-Southwell scheme is applied to a convection-diffusion problem which is linear in nature. It has right-boundary layer near the argument x=1. This method is employed to the two-point boundary value problem with Dirichlet's boundary conditions. The same problem is also solved numerically by Forward difference method, central difference method and upwind scheme. It is noticed that Il'in-Allen scheme converges uniformly throughout the region for any choice of the diffusion coefficient for a finer mesh. The other Finite difference methods do not converge uniformly. The advantage of this method is that, even in the boundary layer region it has uniform convergence. For mid values of the perturbation parameter the convergence in the computed solution is a little bit slower comparatively with the other perturbed parameter values. In this method we contemplated a condition on convection coefficient so that the proposed method is fast convergent to the exact solution.

CHAPTER-4

GALERKIN METHOD FOR SOLVING CERTAIN CLASS OF SINGULARLY PERTURBED TWO POINT BOUNDARY VALUE PROBLEMS WITH CUBIC B-SPLINES

INTRODUCTION

Singular perturbed two-point boundary value problems have been solved by Galerkin method with cubic B-Splines as basis functions. The basis functions have been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions is defined. A finer mesh has been taken near and around δ where the left boundary layer is located. Several examples including linear and nonlinear have been considered for testing the efficiency of the proposed method.

Differential equations occur very frequently in the mathematical modeling of physical problems in Science and Engineering. Since exact solutions for most of these problems are not available, a resort to the approximation methods for getting the solution of such problems is unavoidable. The availability of high speed digital computers has made it possible to take such a task when the approximation method involves numerical computation. The most commonly employed approximate methods, for solving such type of problems are the finite difference method and the finite element method. Even though the finite element method is somewhat difficult than the finite difference method from the point of view of computer programming, it has certain inherent advantages, namely the approximation of solutions can be obtained easily in more complicated regions etc.

The flexibility of the finite element method lies in the replacement of the domain of a problem by a mathematical model with a finite number of sub-domains which constitute the given domain. Any physical problem, mathematically modeled, can be solved by the finite element method.

In Galerkin method, the residual is made orthogonal to the basis functions. In a Galerkin method, a weak form of approximate solution for a given differential equation is exists and unique under appropriate conditions irrespective of properties of a given differential operator and weak solution is also a classical solution of the given differential equation provided sufficient attention is given to the boundary conditions [20, 55]. In this chapter we employed Galerkin method to approximate the solution of a given differential equation.

Many research workers use the Galerkin method for solving boundary value problems and initial-boundary value problems [20, 17]. In most cases the solution is a smooth function which is piecewise polynomial. To find approximate numerical solution to a given differential equation by Galerkin method, one needs a set of basis functions belonging to the space which contains all measurable admissible functions that vanish on the boundary of the domain, where the Dirichlet type of boundary conditions were given on the given differential equation.

In this chapter, we employed Galerkin method to solve a certain class of singular perturbation problems with B-splines as basis function. Infact, any differential equation whose solution changes rapidly in some parts of the interval is generally known as singular perturbation problem and also as boundary layer problem. A boundary layer by definition is a narrow region, where the solution of a differential equation changes rapidly. Further the thickness of the boundary layer tends to zero as $\varepsilon \to 0$.

Consider the following linear singular perturbed two-point boundary value problem

$$\varepsilon y''(x) + a(x)\, y'(x) + b(x) y(x) = c(x)\,;\ 0 < x < 1$$

with $y(0) = y_0$ and $y(1) = y_1$

where ε is small positive parameter ($0< \varepsilon << 1$) and y_0 , y_1 are given constants, a(x) , b(x) and c(x) are assumed to be continuously differentiable functions in [0,1]. Further, we assume that $a(x) \geq M>0$ throughout the interval [0, 1] where M is some positive constant. This assumption solely implies that the boundary layer will be in the neighborhood of x=0. Existing numerical methods produce good results only when we take step length of interval $h \leq \varepsilon$. This is very costly and time consuming process. Hence the researchers are concentrating on developing methods, which can work with reasonable step length h. For this, nowadays researchers are adopting one of the following methods.

(i) The interval is subdivided into two regions $[0,\delta]$ and $[\delta,1]$, where δ is the point near which the boundary layer is located. The region $[0,\delta]$ is called inner region and the region [δ,1] is called outer region. The problem in the inner region is treated as an initial value problem and the problem in the outer region is treated as a boundary value problem. The initial value problem in the inner region problem is solved and terminal boundary condition is obtained. Using this terminal boundary condition, the boundary value problem in the outer region problem is solved.

(ii) Using the variable mesh, one can take finer mesh around and near the point where the boundary layer is located.

Since the finite element method can be easily adaptable with variable mesh, we intend to use finite element method to solve the given singular perturbation problem.

For the case of single differential equation, it is shown in Douglas and Dupont[17] that the cubic B-splines yield 4^{th} order accurate results. Accordingly, B-splines as basis functions have been used by us in our work.

The existence of the cubic- spline interpolate S(x) to a function f(x) in closed interval [0,1] for spaced knots $0 = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-2} < x_{n-1} < x_n = 1$ is established by constructing it. The construction of S(x) is done with the help of cubic B-Splines. Introduce six additional knots x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2} and x_{n+3} such that

$x_{-3} < x_{-2} < x_{-1} < x_0$ and $x_{n+3} > x_{n+2} > x_{n+1} > x_n$.

Now the cubic B-splines Bi (x), given in [13], are defined by

$$B_i(x) = \sum_{r=i-2}^{r=i+2} \frac{(x_r - x)^3}{\prod'(x_r)}, \quad x \in [x_{i-2}, x_{i+2}]$$

$$= 0, \quad \text{otherwise}$$

where

$$(x_r - x)^3 = (x_r - x)^3, \text{ if } x_r \geq x$$
$$= 0, \text{ if } x_r \leq x$$

and $\prod(x) = (x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})(x - x_{i+2})$

It can be shown that the set $\{B_{-1}(x), B_0(x), B_1(x), B_2(x),\ldots B_n(x), B_{n+1}(x)\}$ forms a basis for the space $S_3(\pi)$ of cubic polynomial splines [52]. Schoenberg [61] has proved that the cubic B-splines are the unique non-zero splines of smallest compact support with knots at $x_{-3} < x_{-2} < x_{-1} < x_0 < x_n < x_{n+1} < x_{n+2} < x_{n+3}$.

Any cubic spline defined with a unique set of given knots [3] can be uniquely expressed as a linear combination of B-spline basis set:

$\{B_{-1}(x), B_0(x), B_1(x), B_2(x), \ldots B_n(x), B_{n+1}(x)\}$

We develop a method based on Galerkin method with B-splines as basis functions for solving a general linear singular perturbed two point boundary value problem with left boundary layer by considering

$$\varepsilon y''(x) + a(x) y'(x) + b(x)y(x) = c(x);\ 0 < x < 1$$

With $y(0) = y_0$ and $y(1) = y_1$

Where ε is small positive parameter ($0 < \varepsilon << 1$) and y_0, y_1 are given constants and $a(x) > 0$ throughout the interval [0,1].

We consider some examples of linear and nonlinear singular perturbation problems with left boundary layer. The solution for a nonlinear problem is obtained as the limit of solution of a sequence of linear problems generated by quasi-linearization technique [9]. The solution obtained, by the method developed in this chapter, for the considered examples have been compared with the exact solutions. We observed that the approximation solutions obtained by the developed method are in good agreement with the exact solutions of the problems.

LINEAR SINGULAR PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS WITH LEFT BOUNDARY LAYER

We now develop a Galerkin method with B-splines as basis functions for solving linear singular perturbed two-point boundary value problems with left-boundary layer.

$$\varepsilon y''(x) + a(x) y'(x) + b(x)y(x) = c(x);\ 0 < x < 1 \quad (4.1)$$

With $y(0) = y_0$ and $y(1) = y_1$ (4.2)

Where ε is a small positive parameter($0< \varepsilon << 1$) and y_0 , y_1 are prescribed values and $a(x) >0$ throughout the interval [0,1].

We subdivide the interval [0, 1] into subintervals by the set of n+1 distinct grid points $x_0, x_1, x_2, \ldots x_n$ such that $0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$

For the system (4.1) and (4.2) the boundary layer will be in the neighborhood of x =0. Suppose that the boundary layer is located around and near the point $x = \delta$. Take the finer mesh around and near $x= \delta$ such that the minimum of the step lengths of the subintervals is greater than ε. The procedure for finding the parameter δ is discussed in [24]. Introduce six additional knots $x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}$ and x_{n+3} such that

$x_{-3} < x_{-2} < x_{-1} < x_0$ and $x_{n+3} > x_{n+2} > x_{n+1} > x_n$.

With these grid points, the basis set of cubic B-splines $\{B_{-1}(x), B_0(x), B_1(x), .. B_{n+1}(x)\}$ has been defined. Let the approximate solution to the system (4.1) and (4.2) be given by

$$y(x) = \sum_{i=-1}^{n+1} \alpha_i B_i(x) \tag{4.3}$$

where α_i are the nodal parameters to be determined.

Since we want to solve the system (4.1) and (4.2) by the Galerkin method with cubic B-splines as basis functions, the cubic B-splines should vanish on the boundary where the Dirichlet type of boundary conditions is mentioned. But in this set of cubic B-spine basis functions: $B_{-1}(x), B_0(x), B_1(x), B_2(x), \ldots B_n(x)$ and $B_{n+1}(x)$ are not vanishing at one of the boundary points. So there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary since Dirichlet type of boundary conditions are specified on the boundary.

Using the definition of B-splines and the boundary conditions (4.2), we get the approximation y(x) , given by the equation (4.3) , at the boundary points as

$$y(0) = y(x_0) = \alpha_{-1} B_{-1}(x_0) + \alpha_0 B_0(x_0) + \alpha_1 B_1(x_0) = y_0 \tag{4.4}$$

and

$$y(1) = y(x_n) = \alpha_{n-1} B_{n-1}(x_n) + \alpha_n B_n(x_n) + \alpha_{n+1} B_{n+1}(x_n) = y_1 \tag{4.5}$$

From the above equations we get

$$\alpha_{-1} = \frac{1}{B_{-1}(x_0)}\{ y_0 - \alpha_0 B_0(x_0) - \alpha_1 B_1(x_0) \}\text{ and}$$

$$\alpha_{n+1} = \frac{1}{B_{n+1}(x_n)}\{ y_1 - \alpha_{n-1} B_{n-1}(x_n) - \alpha_n B_n(x_n) \}$$

Substituting these values of α_{-1} and α_{n+1} in (4.3) we get the approximation for y(x) as

$$y(x) = w(x) + \sum_{j=0}^{n} \alpha_j \tilde{B}_j(x) \tag{4.6}$$

Where

$$w(x) = \frac{y_0}{B_{-1}(x_0)} B_{-1}(x) + \frac{y_1}{B_{n+1}(x_n)} B_{n+1}(x) \tag{4.7}$$

$$\tilde{B}_0(x) = B_0(x) - \frac{B_0(x_0)}{B_{-1}(x_0)} B_{-1}(x) \tag{4.8(a)}$$

$$\tilde{B}_1(x) = B_1(x) - \frac{B_1(x_0)}{B_{-1}(x_0)} B_{-1}(x) \tag{4.8(b)}$$

$$\tilde{B}_i(x) = B_i(x), \quad \text{for } i = 2,3,\ldots\ldots,n-2 \tag{4.8(c)}$$

$$\tilde{B}_{n-1}(x) = B_{n-1}(x) - \frac{B_{n-1}(x_n)}{B_{n+1}(x_n)} B_{n+1}(x) \tag{4.8(d)}$$

$$\tilde{B}_n(x) = B_n(x) - \frac{B_n(x_n)}{B_{n+1}(x_n)} B_{n+1}(x) \tag{4.8(e)}$$

Here the new set of basis functions are $\tilde{B}_j$ (x), j= 0,1,2,3….., n and they vanish on the boundary. w(x) defined in (4.7) takes care of the boundary conditions (4.2)
Applying the Galerkin method with the redefined set of basis functions $\tilde{B}_j$ (x) , j =0 ,1 ,2 ,…….n to the system (4.1) and (4.2) we get

$$\int_{x_0}^{x_n} \{ \varepsilon y''(x)\tilde{B}_i(x)dx + a(x)y'(x)\tilde{B}_i(x) + b(x)y(x)\tilde{B}_i(x) \} dx = \int_{x_0}^{x_n} c(x)\tilde{B}_i(x). \tag{4.9}$$

for $i = 0,1,2.......n$

Integrating by parts the first term on the left hand side of the above equation

$$\int_{x_0}^{x_n} \varepsilon y''(x)\tilde{B}_i(x)\,dx = \varepsilon y'(x)\tilde{B}_i(x)\Big|_{x_0}^{x_n} - \int_{x_0}^{x_n} \varepsilon y'(x)\frac{d\tilde{B}_i}{dx}dx = -\int_{x_0}^{x_n} \varepsilon y'(x)\frac{d\tilde{B}_i}{dx}dx \quad (4.10)$$

for $i = 0,1,2,3,.....n$

Substituting (4.10) in (4.9) we get

$$\int_{x_0}^{x_n} \{-\varepsilon\frac{d\tilde{B}_i}{dx}\,[\frac{dw(x)}{dx}+\sum_{j=0}^{n}\alpha_j\frac{d\tilde{B}_j}{dx}]+a(x)\tilde{B}_i(x)[\frac{dw(x)}{dx}+\sum_{j=0}^{n}\alpha_j\frac{d\tilde{B}_j}{dx}]+b(x)\tilde{B}_i(x)[w(x)+\sum_{j=0}^{n}\alpha_j\tilde{B}_j(x)]\}dx$$

$$= \int_{x_0}^{x_n} c(x)\tilde{B}_i(x)\,dx \qquad \text{for } i = 0,1,2,3.............n \quad (4.11)$$

The above equation can be written as

$$\sum_{j=0}^{n}\alpha_j\{\int_{x_0}^{x_n}[-\varepsilon\frac{d\tilde{B}_i}{dx}\frac{d\tilde{B}_j}{dx}+a(x)\tilde{B}_i(x)\frac{d\tilde{B}_j}{dx}+b(x)\tilde{B}_i(x)\tilde{B}_j(x)]dx\} = \int_{x_0}^{x_n}\{c(x)\tilde{B}_i(x)+\varepsilon\frac{dw(x)}{dx}\frac{d\tilde{B}_i}{dx}$$

$$-a(x)\frac{dw(x)}{dx}\tilde{B}_i(x)-b(x)w(x)\tilde{B}_i(x)\}dx \quad \text{for } i = 0,1,2,3.........n \quad (4.12)$$

The system of equations (4.12) can be written in the matrix form as

$$K\overline{\alpha} = f \quad (4.13)$$

where $K = [k_{ij}]$, $$k_{ij} = \int_{x_0}^{x_n}[-\varepsilon\frac{d\tilde{B}_i}{dx}\frac{d\tilde{B}_j}{dx}+a(x)\tilde{B}_i(x)\frac{d\tilde{B}_j}{dx}+b(x)\tilde{B}_i(x)\tilde{B}_j(x)]dx \quad (4.14)$$

for $i = 0,1,2.........n$, $j = 0,1,2........n$, $f = [f_i]$

$$f_i = \int_{x_0}^{x_n}\left\{c(x)\tilde{B}_i(x)+\varepsilon\frac{dw(x)}{dx}\frac{d\tilde{B}_i(x)}{dx}-a(x)\frac{dw(x)}{dx}\tilde{B}_i(x)-b(x)w(x)\tilde{B}_i(x)\right\}dx \quad (4.15)$$

for $i = 0,1,2,3.........n$

$$\overline{\alpha} = [\alpha_0,\alpha_1,\alpha_2,.........\alpha_n]^T \quad (4.16)$$

A typical integral element in the matrix K, given in equation (4.14) is

$$\sum_{m=0}^{n-1} I_m \quad \text{where } I_m = \int_{x_m}^{x_{m+1}} s_i(x) s_j(x)\, z(x) dx \tag{4.17}$$

and $s_i(x)$, $s_j(x)$ are the basis functions $\tilde{B}_i(x)$ or their derivatives

It may be noted that $I_m = 0$, if $(x_{i-2}, x_{i+2}) \cap (x_{j-2}, x_{j+2}) \cap (x_m, x_{m+1}) = \phi$

Thus the stiff matrix K is a seven diagonal band matrix. The integral element (4.17) is evaluated by using the four Point Gauss-Legendre quadrature formula. The nodal parameter vector $\bar{\alpha}$ has been obtained from the system (4.13) by using the band matrix solution package.

To test the efficiency of the proposed method described in this chapter for solving the singular perturbation two point boundary value problems with left boundary layer, we considered some linear and non-linear problems. In all the selected examples, we have taken

h = min (lengths of subintervals of the given domain).

Example 4.1

Consider the following homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \quad 0 \le x \le 1 \tag{4.18}$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 1 \tag{4.19}$$

The exact solution for the above system is given by

$$y(x) = \frac{(e^{m_2} - 1)\, e^{m_1 x} + (1 - e^{m_1})\, e^{m_2 x}}{(e^{m_2} - e^{m_1})} \tag{4.20}$$

where $m_1 = \dfrac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}$, $m_2 = \dfrac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}$

We have solved the problem (4.18) and (4.19) with $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively. The approximate solutions obtained by the proposed method are compared with the exact solution in tables 4.1 (a) and 4.1 (b) for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively. From the results we can conclude that the approximation is in good agreement with the exact solution.

$\varepsilon = 10^{-3}$ h= 0.0015 δ = 0.01

x	Approximate solution	Exact solution
0.00000	1.0000000	1.0000000
0.00200	0.4568041	0.4543111
0.00400	0.3800441	0.3812461
0.00600	0.3728526	0.3720173
0.00750	0.3709753	0.3713630
0.00900	0.3718539	0.3716499
0.01050	0.3720383	0.3721471
0.01200	0.3727582	0.3726917
0.01350	0.3732188	0.3732477
0.01500	0.3738213	0.3738067
0.01750	0.3747340	0.3747413
0.02000	0.3756694	0.3756784
0.04000	0.3832526	0.3832599
0.06000	0.3910117	0.3909945
0.08000	0.3988778	0.3988851
0.10000	0.4069450	0.4069350
0.15000	0.4277724	0.4277777
0.20000	0.4497036	0.4496879
0.25000	0.4727157	0.4727203
0.30000	0.4969481	0.4969324
0.35000	0.5223812	0.5223845
0.40000	0.5491560	0.5491404
0.45000	0.5772646	0.5772666
0.50000	0.6068492	0.6068334
0.55000	0.6379137	0.6379146
0.60000	0.6706011	0.6705877
0.70000	0.7410415	0.7410401
0.80000	0.8189113	0.8188942
0.90000	0.9049323	0.9049277
1.00000	1.0000000	1.0000000

Table- 4.1(a)

$\mathcal{E}$= 10^{-4} h= 0.00015 δ = 0.009:

X	Approximate solution	Exact solution
0.00000	1.00000	1.00000
0.00020	0.4560560	0.4535159
0.00040	0.3783315	0.3796358
0.00055	0.3713096	0.3707004
0.00070	0.3684529	0.3687498
0.00085	0.3685201	0.3683575
0.00100	0.3682148	0.3683130
0.00115	0.3683900	0.3683459
0.00130	0.3683679	0.3683962
0.00150	0.3684776	0.3684686
0.00175	0.3685433	0.3685606
0.00200	0.3686297	0.3686527
0.00500	0.3697476	0.3697602
0.01000	0.3716192	0.3716135
0.01500	0.3734570	0.3734760
0.02000	0.3753495	0.3753479
0.03000	0.3791019	0.3791198
0.04000	0.3829309	0.3829296
0.06000	0.3906467	0.3906645
0.08000	0.3985607	0.3985557
0.10000	0.4065937	0.4066062
0.15000	0.4274547	0.4274513
0.20000	0.4493461	0.4493649
0.25000	0.4724050	0.4724020
0.30000	0.4966005	0.4966201
0.35000	0.5220817	0.5220797
0.40000	0.5488243	0.5488446
0.45000	0.5769824	0.5769815
0.50000	0.6065399	0.6065609
0.550000	0.6376565	0.6376569
0.60000	0.6703249	0.6703469
0.65000	0.7047110	0.7047127
0.70000	0.7408174	0.7408404
0.75000	0.7788171	0.7788202
0.80000	0.8187230	0.8187471
0.85000	0.8607160	0.8607209
0.90000	0.9048215	0.9048464
0.95000	0.9512262	0.9512342
1.00000	1.00000	1.00000

Table-4.1 (b)

Example 4.2

Consider the following non- homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \quad 0 \le x \le 1 \tag{4.21}$$

with $y(0) = 0$ and $y(1) = 1$

$$y(x) = x(x+1-2\varepsilon) + \frac{(2\varepsilon-1)(1-e^{\frac{-x}{\varepsilon}})}{(1-e^{\frac{-x}{\varepsilon}})} \tag{4.22}$$

We have solved the problem (4.21) with $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively. The approximate solutions obtained by the proposed method compared with the exact solution (4.22) in tables 4.2(a) and 4.2(b) for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively. From the results we can conclude that the approximation is in good agreement with the exact solution.

Case1 : $\varepsilon = 10^{-3}$,h= 0.0015 and δ = 0.009

X	Approximate solution	Exact solution
0.00000	-0.00000002	0.00000000
0.00200	-0.85699360	-0.86093540
0.004000	-0.97758870	-0.97571300
0.00600	-0.98817260	-0.98950220
0.00750	-0.99050180	-0.98990680
0.00900	-0.98847780	-0.98881390
0.01050	-0.98753940	-0.98738320
0.01200	-0.98575430	-0.98587390
0.01350	-0.98437370	-0.98434340
0.01500	-0.98276670	-0.98280470
0.01750	-0.98022520	-0.98022870
0.02000	-0.97763900	-0.97764000
0.04000	-0.95647660	-0.95648000
0.06000	-0.93447790	-0.93452000
0.08000	-0.91175710	-0.91176000
0.10000	-0.88816940	-0.88820000
0.15000	-0.82579450	-0.82580000
0.20000	-0.75836100	-0.75840000
0.25000	-0.68599440	-0.68600000
0.30000	-0.60856180	-0.60860000
0.35000	-0.52619370	-0.52620000
0.40000	-0.43876260	-0.43880000
0.45000	-0.34639280	-0.34640000
0.50000	-0.24896340	-0.24900000
0.55000	-0.14659230	-0.14660000
0.60000	-0.03916838	-0.03919995
0.70000	0.19060910	0.19060000
0.80000	0.44043510	0.44040000
0.90000	0.71021060	0.71020000
1.00000	1.00000000	1.00000000

Table-4.2(a)

Case ii: $\varepsilon = 10^{-4}$, h= 0.0015 and δ = 0.009

x	Approximate solution	Exact solution
0.00000	0.00000000	0.00000000
0.00020	-0.86029370	-0.86429180
0.00040	-0.98317150	-0.98108800
0.00055	-0.99422230	-0.99516390
0.00070	-0.99867910	-0.9981880
0.00085	-0.99851080	-0.99874600
0.00100	-0.99893080	-0.99875380
0.00115	-0.99859070	-0.99863870
0.00130	-0.99856280	-0.99849630
0.00150	-0.99830530	-0.99829780
0.00175	-0.99809630	-0.99804730
0.00200	-0.99785460	-0.99779640
0.00500	-0.99481750	-0.99477600
0.01000	-0.98971440	-0.98970200
0.01500	-0.98462950	-0.98457800
0.02000	-0.97942300	-0.997940400
0.03000	-0.96895580	-0.96890600
0.04000	-0.95822740	-0.95820800
0.06000	-0.93626150	-0.93621200
0.08000	-0.91342900	-0.91341600
0.10000	-0.88986090	-0.88982000
0.15000	-0.82734430	-0.82733000
0.20000	-0.75988980	-0.75984000
0.25000	-0.68736340	-0.68735000
0.30000	-0.60990980	-0.60986000
0.35000	-0.52738350	-0.52737000
0.40000	-0.43992980	-0.43988000
0.45000	-0.34740370	-0.34739000
0.50000	-0.24994970	-0.24990000
0.550000	-0.14742380	-0.14741000
0.60000	-0.03996952	-0.03991995
0.65000	0.07255605	0.07256994
0.70000	0.19001050	0.19006000
0.75000	0.31253600	0.31255000
0.80000	0.43999070	0.44004000
0.85000	0.57251580	0.57253010
0.90000	0.70997110	0.71001990
0.95000	0.85249320	0.85251000
1.00000	1.00000000	1.00000000

Table-4.2(b)

Example 4.3

Consider the following homogeneous singular perturbation problem

$$\varepsilon y''(x) + (1 - \frac{x}{2}) y'(x) - \frac{1}{2} y(x) = 0, \qquad 0 \le x \le 1 \tag{4.23}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1 \tag{4.24}$$

The exact solution for the above system is given by

$$y(x) = \frac{1}{2\text{-}x} - \frac{1}{2} \; e^{\frac{-(x - \frac{x^2}{4})}{\varepsilon}} \tag{4.25}$$

We have solved the problem (4.23) and (4.24) with $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively. The approximate solutions obtained by the proposed method compared with the exact solution (4.25) in tables4.3 (a) and 4.3(b) for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively. From the results we can conclude that the approximated solution is in good agreement with the exact solution.

When $\varepsilon = 10^{-3}$ **, h= 0.0015 and δ = 0.01:**

x	Approximate solution	Exact solution
0.00000	0.00000000	0.00000000
0.00200	0.43143500	0.43276520
0.00400	0.49348850	0.49180750
0.00600	0.50033520	0.50025390
0.00750	0.50265150	0.50160160
0.000900	0.50278040	0.50219720
0.01050	0.50345490	0.50262470
0.01200	0.50370730	0.50301490
0.01350	0.50416470	0.50339720
0.01500	0.50451180	0.50377820
0.01750	0.50516460	0.50441360
0.02000	0.50580300	0.50505050
0.04000	0.51095780	0.51020410
0.06000	0.51620030	0.51546390
0.08000	0.52159140	0.52083330
0.10000	0.52706180	0.52631580
0.15000	0.54130300	0.54054050
0.20000	0.55630370	0.55555560
0.25000	0.57219510	0.57142860
0.30000	0.58898430	0.58823530
0.35000	0.60682340	0.60606060
0.40000	0.62574110	0.62500000
0.45000	0.64590950	0.64516130
0.50000	0.66738650	0.66666670
0.55000	0.69037260	0.68965520
0.60000	0.71496650	0.71428570
0.70000	0.76985300	0.76923080
0.80000	0.83382150	0.83333330
0.90000	0.90940120	0.90909090
1.00000	1.00000000	1.00000000

Table-4.3(a)

When $\varepsilon = 10^{-4}$, h= 0.00015 and $\delta = 0.009$

x	Approximate solution	Exact solution
0.00000	0.00000000	0.00000000
0.00020	0.43043570	0.43237560
0.00040	0.49204900	0.49093850
0.00055	0.49769060	0.49809260
0.00070	0.50003340	0.49971850
0.00085	0.50006230	0.50011070
0.00100	0.50038530	0.50022740
0.00115	0.50032800	0.50028260
0.00130	0.50042680	0.50032410
0.00150	0.50044830	0.50037520
0.00175	0.50053190	0.50043790
0.00200	0.50059910	0.50050050
0.00500	0.50134340	0.50125310
0.01000	0.50258850	0.50251260
0.01500	0.50387380	0.50377830
0.02000	0.50512970	0.50505050
0.03000	0.50770910	0.50761420
0.04000	0.51028380	0.51020410
0.06000	0.51555930	0.51546390
0.08000	0.52091040	0.52083330
0.10000	0.52640750	0.52631580
0.15000	0.54061920	0.54054050
0.20000	0.55565320	0.55555560
0.25000	0.57150760	0.57142860
0.30000	0.58833400	0.58823530
0.35000	0.60614000	0.60606060
0.40000	0.62509890	0.62500000
0.45000	0.64523990	0.64516130
0.50000	0.66676450	0.66666670
0.55000	0.68973150	0.68965520
0.60000	0.71438070	0.71428570
0.65000	0.74081240	0.74074070
0.70000	0.76932000	0.76923080
0.75000	0.80006340	0.80000000
0.80000	0.83341220	0.83333330
0.85000	0.86961480	0.86956520
0.90000	0.90915200	0.90909090
0.95000	0.95240890	0.95238100
1.00000	1.000000000	1.00000000

Table-4.3(b)

Example 4.4

Consider the non-linear singular perturbation problem

$$\varepsilon y''(x) + 2y'(x) + e^{y} = 0, \qquad 0 \le x \le 1 \qquad (4.26)$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 0 \qquad (4.27)$$

Applying Quasi-linearization technique [9] to the equation (4.26) with the boundary conditions (4.27) , we get a sequence of linear problems as

$$\varepsilon\, y''_{r+1}(x) + 2y'_{r+1} + e^{y_r} y_{r+1} = (y_r - 1)\, e^{y_r}, \qquad r = 0,1,2,3.... \qquad (4.28)$$

We solved the system of equations (4.28) along with the boundary conditions (4.27) by the Galerkin method by taking $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$.

For the above problem (4.26) with (4.27) , we have chosen Bender and Orszag's uniformly valid approximation for comparisons[10]

$$y(x) = \log\frac{2}{1+x} - e^{\frac{-2x}{\varepsilon}} \log 2.$$

The approximate solution obtained by the proposed method with $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ are compared with Bender and Orszag's uniformly valid approximation in tables 4.4(a) and 4.4 (b) respectively.

ε =10^{-3} , h= 0.0015 and δ = 0.01

X	Approximate solution	Bender and Orszag's Solution
0.00000	0.00000002	0.00000000
0.00200	0.65748040	0.67845370
0.00400	0.70612440	0.68892260
0.00600	0.66894180	0.68716080
0.00750	0.69315830	0.68567500
0.000900	0.67522500	0.68418740
0.01050	0.68431480	0.68270190
0.01200	0.67576530	0.68121860
0.01350	0.67898120	0.67973750
0.01500	0.67481950	0.67825860
0.01750	0.67409990	0.67579850
0.02000	0.67159250	0.67334460
0.04000	0.65224490	0.65392650
0.06000	0.63139740	0.63487830
0.08000	0.61466250	0.61618610
0.10000	0.59493630	0.59783700
0.15000	0.55188670	0.55338530
0.20000	0.50765000	0.51082560
0.25000	0.46869290	0.47000360
0.30000	0.42775760	0.43078290
0.35000	0.39185340	0.39304260
0.40000	0.35379040	0.35667490
0.45000	0.32049830	0.32158360
0.50000	0.28491780	0.28768210
0.55000	0.25391780	0.25489220
0.60000	0.22074560	0.22314350
0.70000	0.16161750	0.16251890
0.80000	0.10290410	0.10536050
0.90000	0.05045704	0.05129331
1.00000	0.00000000	0.00000000

Table-4.4(a)

When $\varepsilon = 10^{-4}$, h = 0.00015 and δ = 0.009

X	Approximate solution	Bender and Orszag's solution
0.00000	0.00000001	0.00000000
0.00020	0.66487960	0.68025180
0.00040	0.71735530	0.69251470
0.00055	0.68253430	0.69258580
0.00070	0.70516570	0.69244680
0.00085	0.69029840	0.69229750
0.00100	0.70000280	0.69214770
0.00115	0.69327410	0.69199780
0.00130	0.69723960	0.69184800
0.00150	0.69440290	0.69164830
0.00175	0.69625960	0.69139870
0.00200	0.69628190	0.69114920
0.00500	0.69256170	0.68815960
0.01000	0.68608380	0.68319680
0.01500	0.68313440	0.67825860
0.02000	0.67649420	0.67334460
0.03000	0.66833750	0.66358840
0.04000	0.65701630	0.65392650
0.06000	0.63951640	0.63487830
0.08000	0.61878140	0.61618610
0.10000	0.60188630	0.59783700
0.15000	0.55582390	0.55338530
0.20000	0.51506570	0.51082560
0.25000	0.47210980	0.47000360
0.30000	0.43478810	0.43078290
0.35000	0.39490450	0.39304260
0.40000	0.36047190	0.35667490
0.45000	0.32323650	0.32158360
0.50000	0.29129870	0.28768210
0.55000	0.25636300	0.25489220
0.60000	0.22660280	0.22314350
0.65000	0.19368250	0.19237190
0.70000	0.16583970	0.16251890
0.75000	0.13470030	0.13353140
0.80000	0.10855740	0.10536050
0.85000	0.07900877	0.07796153
0.90000	0.05435734	0.05129331
0.95000	0.02640168	0.02531781
1.00000	0.00000000	0.00000000

Table-4.4 (b)

CONCLUSIONS

Convection-diffusion problems form a class of singular perturbation problems. Singular perturbed two-point boundary value problems have been solved by Galerkin method with cubic B-Splines as basis functions. The basis functions have been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions is applied. A finer mesh has been taken near and around δ where the left boundary layer is located. Many examples including linear and nonlinear problems have been considered for testing the efficiency of the proposed method. The proposed Galerkin method has given the computational results which are very much close to the analytical solutions which are available in the literature for a fine mesh size h. Though the diffusion coefficient value allowed to take very lower, the convergence existed to the reasonable accuracy through out the region. Byandlarge this method is very efficient method and easily implemented on a digital computer by writing the suitable numerical code. From the results we observed that the approximation solutions obtained by the developed method are in good agreement with the exact solutions of the selected problems.

CHAPTER-5

NUMERICAL INTEGRATION METHOD FOR STEADY –STATE CONVECTION-DIFFUSION PROBLEM

INTRODUCTION

In this chapter, a numerical integration method is presented for solving a general steady-state convection problem or singularly perturbed two-point boundary value problem. The governing second-order differential equation is replaced by an approximate first-order differential equation with a small deviating argument. Then the Simpson one-third formula is used to obtain the three term recurrence relationship. The proposed method is iterative on the deviating argument. To test and validity of this method we have solved several model linear problems with left-end boundary layer or right-end boundary layer or an internal layer and offered the computational results.

Convection-diffusion problems occur very frequently in the fields of science and engineering such as fluid dynamics, specifically the fluid flow problems involving large Reynolds number and other problems in the great world of fluid motion. The numerical treatment of singular perturbation problems is far from trivial because of the boundary layer behavior of the solution. However, the area of convection-diffusion problems is a field of increasing interest to applied mathematicians.

The survey paper by Kadalbajoo and Reddy [30], gives an intellectual outline of the singular perturbation problems and their treatment starting from Prandtl's paper [51] on fluid dynamical boundary layers. This survey paper will remain as one of the most readable source on convection-diffusion or singular perturbation problems.

In this chapter, a numerical integration method is presented for solving general singularly perturbed two-point boundary value problems. The main advantage of this method is that it does not require very fine mesh size. The original second-order differential equation is replaced by an approximate first-order differential equation with a small deviating argument. Then, the Simpson one-third formula is used to obtain the three term recurrence relationship. Thomas Algorithm is applied to solve the resulting tri diagonal algebraic system of equations. The proposed method is iterative on the deviating argument. The method is to be repeated for different choices of the deviating argument until the solution profile stabilizes. To examine the applicability of the proposed method, we have solved several model linear problems with left-end boundary layer or right –end boundary layer or an internal layer and presented the numerical results. It is observed that the numerical integration method approximates the exact solution extremely well.

NUMERICAL INTEGRATION METHOD

For the sake of convenience we call our method the 'Numerical Integration Method'. To set the stage for the numerical integration method, we consider the following Governing linear Convection-diffusion two-point boundary value problem:

$$\varepsilon y''(x) + a(x) y'(x) + b(x)y(x) = f(x); \; 0 \le x \le 1 \tag{5.1}$$

With $y(0) = \alpha$ and $y(1) = \beta$ (5.2)

Where ε is a small positive parameter called diffusion parameter which lies in the interval $0 < \varepsilon \le 1$; α and β are given constants; a(x) , b(x) and f(x) considered to be sufficiently continuously differentiable functions in [0,1]. Furthermore, we assume that $a(x) \ge M > 0$ throughout the interval [0,1], where M is some positive constant. This assumption purely implies that the boundary layer will be in the neighborhood of x=0.

Let δ be a small positive deviating argument ($0 < \delta \le 1$). By applying Taylor series expansions in the neighborhood of the point x, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{5.3}$$

and consequently, Eq. (5.1) is replaced by the following first-order differential equation with a small deviating argument.

$$\frac{\delta^2}{2} y''(x) = y(x-\delta) - y(x) + \delta y'(x) \Rightarrow y''(x) = \frac{2}{\delta^2}[y(x-\delta) - y(x) + \delta y'(x)] \text{ So that}$$

$$(5.1) \qquad \Rightarrow \frac{2\varepsilon}{\delta^2}[y(x-\delta) - y(x) + \delta y'(x)] + a(x) y'(x) + b(x) y(x) = f(x); 0 \le x \le 1$$

$$\Rightarrow 2\varepsilon y(x-\delta) - 2\varepsilon y(x) + 2\varepsilon\delta y'(x) + a(x) y'(x)\delta^2 + b(x) y(x)\delta^2 = \delta^2 f(x)$$

$$\Rightarrow [2\varepsilon 2 + a(x)\,\delta^2]\,(y'(x)) + [b(x)\,\delta^2 - 2\varepsilon\varepsilon\, y(x) = \delta^2\, f(x) - 2\,\varepsilon\, y(x-\delta)$$

$$\Rightarrow y'(x) = \frac{\delta^2\, f(x) - 2\,\varepsilon\, y(x-\delta)}{2\,\varepsilon\,\delta + a(x)\;\delta^2}\, y(x-\delta) + \frac{(2\,\varepsilon - b(x)\;\delta^2)}{2\,\varepsilon\,\delta + a(x)\;\delta^2}\; y(x)$$

$$\Rightarrow y'(x) = \frac{-2\,\varepsilon}{2\,\varepsilon\,\delta + a(x)\,\delta^2}\, y(x-\delta) + \frac{2\varepsilon - b(x)\,\delta^2}{2\,\varepsilon\,\delta + a(x)\,\delta^2}\, y(x) + \frac{\delta^2\, f(x)}{2\;\varepsilon\,\delta + a(x)\,\delta^2} \tag{5.4}$$

(5.4) can be re-written as

$$y'(x) = p(x)\, y(x-\delta) + q(x)\, y(x) + r(x) \text{ for } \delta \le x \le 1 \tag{5.5}$$

Where

$$p(x) = \frac{-2\varepsilon}{2\,\varepsilon\,\delta + \delta^2\; a(x)} \tag{5.6}$$

$$q(x) = \frac{2\varepsilon - \delta^2\; b(x)}{2\,\varepsilon\,\delta + \delta^2\; a(x)} \tag{5.7}$$

$$r(x) = \frac{\delta^2\; f(x)}{2\,\varepsilon\,\delta + \delta^2\; a(x)} \tag{5.8}$$

We now divide the interval [0,1] into N equal parts with mesh size h, i.e., h=1/N and x_i = ih for i= 1,2,3…….N. Integrating equation (5.5) in $[x_{i-1},x_{i+1}]$ we get

$$y(x_{i+1}) - y(x_{i-1}) = \int_{x_{i-1}}^{x_{i+1}} [p(x)\, y(x-\delta) + q(x)\, y(x) + r(x)]\, dx \tag{5.9}$$

By making use of the Newton-Cotes formula when n=2 i.e. by applying Simpson's one-third rule

$$y(x_{i+1}) - y(x_{i-1}) = \frac{h}{3}[p(x_{i+1})y(x_{i+1} - \delta) + 4\,p(x_i)\; y(x_i - \delta) + p(x_{i-1} - \delta)$$
$$+ (p_{i+1} + p_{i-1})\;\; [y(x_{i+1} - \delta) + y(x_{i-1} - \delta)] + q(x_{i+1})\; y(x_{i+1}) + q(x_{i-1})\; y(x_{i-1}) + q(x_{i+1})y(x_{i+1})$$
$$+ 4q(x_i)y(x_i) + q(x_{i-1})y(x_{i-1}) + r(x_{i+1}) + 4r(x_i) + r(x_{i-1}) + r(x_{i+1}) + r(x_{i-1})\,] \tag{5.10}$$

Again by means of Taylor's series expansion, we have

$$y(x-\delta) \cong y(x) - \delta\, y'(x)$$

and then by approximating $y'(x)$ by linear interpolation method we get

$$y(x_i-\delta)\cong y(x_i)-\frac{\delta[y(x_{i+1})-y(x_{i-1})]}{2h}$$

$$=y(x_i)+\frac{\delta}{2h}y(x_{i-1})-\frac{\delta}{2h}y(x_{i+1}) \qquad (5.11)$$

Similarly

$$y(x_{i-1}-\delta)\cong(1+\frac{\delta}{h})\,y(x_{i-1})-\frac{\delta}{h}y(x_i) \qquad (5.12)$$

$$y(x_{i+1}-\delta)=(1-\frac{\delta}{h})\,y(x_{i+1})+\frac{\delta}{h}y(x_i) \qquad (5.13)$$

Hence making use of (5.11),(5.12),(5.13) in (5.10), it can be written as

$$y_{i+1}-y_{i-1}=\frac{h}{3}[\,p_{i+1}[(1-\frac{\delta}{h})\,y_{i+1}+\frac{\delta}{h}\,y_i]+4p_i[y_i-\frac{\delta}{2h}\,y_{i+1}+\frac{\delta}{2h}\,y_{i-1}]+p_{i-1}[(1+\frac{\delta}{h})\,y_{i-1}-\frac{\delta}{h}y_i]$$

$$+(p_{i+1}+p_{i-1})[(1-\frac{\delta}{h})y_{i+1}+\frac{\delta}{h}y_i+(1+\frac{\delta}{h})\,y_{i-1}-\frac{\delta}{h}y_i+2q_{i+1}y_{i+1}+2q_{i-1}y_{i-1}+4q_iy_i+2r_{i+1}+4r_i+2r_{i-1}]$$

$$[-1-\frac{2p_i\,\delta}{3}-\frac{h}{3}p_{i-1}(1+\frac{\delta}{2h})-\frac{h}{3}(p_{i+1}+p_{i-1})(1+\frac{\delta}{h})-\frac{2h}{3}q_{i-1}]\,y_{i-1}+[\frac{\delta\,p_{i-1}}{3}-\frac{\delta}{3}p_{i+1}-\frac{4hp_i}{3}$$

$$-\frac{4hq_i}{3}]\,y_i+[1-\frac{h}{3}p_{i+1}(1-\frac{\delta}{h})+\frac{2p_i\delta}{3}-\frac{h}{3}(p_{i+1}+p_{i-1})(1-\frac{\delta}{h})-\frac{2h}{3}q_{i+1}]\,y_{i+1}$$

$$=\frac{2h}{3}[r_{i+1}+2r_i+r_{i-1}] \qquad (5.14)$$

(5.14) can be written in the standard form as

$$A_i\;y_{i-1}+B_i\;y_i+C_i\;y_{i+1}=D_i \qquad (5.15)$$

where $$A_i=-1-\frac{2p_i\,\delta}{3}-\frac{h}{3}p_{i-1}(1+\frac{\delta}{2h})-\frac{h}{3}(p_{i+1}+p_{i-1})(1+\frac{\delta}{h})-\frac{2h}{3}q_{i-1} \qquad (5.16)$$

$$B_i=\frac{\delta\,p_{i-1}}{3}-\frac{\delta}{3}p_{i+1}-\frac{4hp_i}{3}-\frac{4hq_i}{3} \qquad (5.17)$$

$$C_i=1-\frac{h}{3}p_{i+1}(1-\frac{\delta}{h})+\frac{2\,p_i\delta}{3}-\frac{h}{3}(p_{i+1}+p_{i-1})(1-\frac{\delta}{h})-\frac{2\,h}{3}q_{i+1} \qquad (5.18)$$

$$D_i=\frac{2h}{3}[r_{i+1}+2r_i+r_{i-1}] \qquad (5.19)$$

Here $y_i=y(x_i)$, $p_i=p(x_i)$, $q_i=q(x_i)$ and $r_i=r(x_i)$. Equation (5.16) gives a system of (N-1) equations with (N+1) unknown's y_0 to y_N. The two given boundary conditions (5.2) together with these (N-1) equations are then sufficient to solve for the unknowns y_0 to y_N. The solution of the Tri-diagonal system (5.15) can be obtained by using an efficient algorithm called 'Thomas Algorithm. In this algorithm we set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \tag{5.20}$$

Where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ are to be determined from (5.20) we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{5.21}$$

Substituting (5.21) in (5.15) we get

$$y_i = \frac{C_i}{B_i - A_i W_{i-1}} y_{i+1} + \frac{A_i T_{i-1} - D_i}{B_i - A_i W_{i-1}} \tag{5.22}$$

By comparing (5.20) and (5.22) , we can get

$$W_i = \frac{C_i}{B_i - A_i W_{i-1}} \tag{5.23}$$

$$T_i = \frac{A_i T_{i-1} - D_i}{B_i - A_i W_{i-1}} \tag{5.24}$$

To solve these recurrence relations for i=1,2, 3,…….N-1; we need to know the initial conditions for W_0 and T_0. This can be done by considering (5.2)

$$y_0 = \alpha = W_0 y_1 + T_0 \tag{5.25}$$

If we choose $W_0=0$, then $T_0 = \alpha$. With these initial values , we compute sequentially W_i and T_i for i=1,2,3,….N-1;from (5.24) and (5.25) in the forward process and then obtain y_i in the backward process from (5.20) using (5.2).

Repeat the numerical scheme for different choices of δ (deviating argument, satisfying the conditions $(0 < \delta \le 1)$, until the solution profiles do not differ significantly from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$\left| y(x)^{m+1} - y(x)^m \right| \le \rho, 0 \le x \le 1 \tag{5.26}$$

Where $y(x)^m$ the solution for the mth is iterate of δ, and ρ is the prescribed tolerance bound.

LINEAR PROBLEMS

We considered the applicability of the numerical integration method; we have applied it to linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in the literature and approximate solutions are available for comparison.

Example 5.1:

Consider the following homogeneous Singular value perturbation problem from Kevorkian and Cole [36], p.33, Eqs. (2.3.26) and (2.3.27) with α =0:

$\varepsilon\, y''(x) + y'(x) = 0,\ 0 \le x \le 1$ with $y(0) = 0$ and $y(1) = 1$

The exact solution is given by

$$y(x) = \frac{(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))}$$

The computational results are presented in Table 5.1(a) and 5.1(b) for $\varepsilon = 10^{-3}$, 10^{-4} respectively.

Computational results for Example5.1

(a) $\varepsilon=10^{-3}$, h=0.01.

X	y(x)			Exact solution
	δ=0.008	δ=0.009	δ=0.01	
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	0.9876486	0.9899944	0.9917358	1.0000000
0.04	0.9998419	0.9998944	0.9999319	1.0000000
0.06	0.9999925	0.9999934	0.9999995	1.0000000
0.08	0.9999945	0.9999945	1.0000000	1.0000000
0.10	0.9999946	0.9999948	1.0000000	1.0000000
0.20	0.9999954	0.9999952	1.0000000	1.0000000
0.40	0.9999964	0.9999964	1.0000000	1.0000000
0.60	0.9999976	0.9999976	1.0000000	1.0000000
0.80	0.9999988	0.9999988	1.0000000	1.0000000
1.00	1.00000000	1.00000000	1.0000000	1.0000000

Table. 5.1(a)

(b) $\varepsilon= 10^{-4}$ and h= 0.01:

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x	δ=0.007	δ=0.008	δ=0.009	Exact soln
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	0.9998016	0.9998477	0.9998792	1.0000000
0.04	0.9999999	1.0000000	1.0000000	1.0000000
0.06	1.0000000	1.0000000	1.0000000	1.0000000
0.08	1.0000000	1.0000000	1.0000000	1.0000000
0.10	1.0000000	1.0000000	1.0000000	1.0000000
0.20	1.0000000	1.0000000	1.0000000	1.0000000
0.40	1.0000000	1.0000000	1.0000000	1.0000000
0.60	1.0000000	1.0000000	1.0000000	1.0000000
0.80	1.0000000	1.0000000	1.0000000	1.0000000
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table.5.1 (b)

Example 5.2

Consider the following homogeneous Singular perturbation problem from Bender and Orsag[10] ,p.480. Problem 9.17 with α =0:

$$\varepsilon y''(x) + y'(x) - y(x) = 0,\ 0 \le x \le 1 \text{ with } y(0) = 0 \text{ and } y(1) = 1$$

The exact solution is given by

$$y(x) = \frac{(e^{m_2} - 1)\, e^{m_1 x} + (1 - e^{m_1})\, e^{m_2 x}}{(e^{m_2} - e^{m_1})} \qquad \text{where}$$

$$m_1 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon} \;;\; m_2 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}$$

Computational results for Example 5.2 are furnished in table **5.2(a)** and **5.2(b)**.

Case-1: ε=0.001, h=0.01

X	y(x)			Exact solution
→	δ=0.008	δ=0.009	δ=0.01	
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3834784	0.3819605	0.3808348	0.3756784
0.04	0.3834410	0.3833556	0.3832939	0.3832599
0.06	0.3910826	0.3910290	0.3909866	0.3909945
0.08	0.3989720	0.3989188	0.3988770	0.3988851
0.10	0.4070216	0.4069688	0.4069269	0.4069350
0.20	0.4497731	0.4497210	0.4496799	0.4496879
0.40	0.5492185	0.5491707	0.5491330	0.5491404
0.60	0.6706514	0.6706123	0.6705816	0.6705877
0.80	0.8189330	0.8189092	0.8188905	0.8188942
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table 5.2 (a)

Case-2 : $\varepsilon = 10^{-4}$ and h=0.01 :

X	y(x)			Exact solution
→	δ=0.009	δ=0.008	δ=0.007	
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3754246	0.3754509	0.3754841	0.3753479
0.04	0.3829308	0.3829373	0.3829417	0.3829296
0.06	0.3906657	0.3906722	0.3906766	0.3906645
0.08	0.3985569	0.3985633	0.3985675	0.3985557
0.10	0.4066074	0.4066138	0.4066185	0.4066062
0.20	0.4493662	0.4493724	0.4493767	0.4493649
0.40	0.5488456	0.5488514	0.5488553	0.5488445
0.60	0.6703477	0.6703524	0.6703555	0.6703469
0.80	0.8187476	0.8187505	0.8187524	0.8187471
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table.5.2 (b)

Example 5.3.

Consider the following non-homogeneous Singular perturbation problem

$$\varepsilon\, y''(x) + y'(x) = 1 + 2x, \quad 0 \le x \le 1$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\,\frac{(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))}$$

The computational results are presented in Table 5.3(a) and 5.3(b) for $\varepsilon=10^{-3}$, 10^{-4} respectively.

Computational results for Example 5.3

(a) $\varepsilon=10^{-3}$, h=0.01.

X	Y(x)			Exact solution
→	δ=0.008	δ=0.009	δ=0.01	
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	-0.9648339	-0.9674433	-0.9693918	-0.9776401
0.04	-0.9558469	-0.9561658	-0.9564114	-0.9564800
0.06	-0.9340471	-0.9343091	-0.9345188	-0.9345200
0.08	-0.9112990	-0.9115545	-0.9117596	-0.9117600
0.10	-0.8877492	-0.8879992	-0.8881995	-0.8882000
0.20	-0.7579996	-0.7582219	-0.7583995	-0.7584000
0.40	-0.4385004	-0.4386670	-0.4387995	-0.4388000
0.60	-0.0390007	-0.0391119	-0.0391996	-0.0391999
0.80	0.4404994	0.4404438	0.4404002	0.4404000
1.00	1.0000000	1.00000000	1.00000000	1.00000000

Table 5.3(a)

(b) $\varepsilon= 10^{-4}$ and h= 0.01

X	δ=0.007	δ=0.008	δ-0.009	Exact Solution
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	-0.9791212	-0.9792020	-0.9792610	-0.9794040
0.04	-0.6581250	-0.9581596	-0.9581869	-0.9582080
0.06	-0.9361311	-0.9361844	-0.9361909	-0.9362120
0.08	-0.9133368	-0.9133694	-0.9133958	-0.9134160
0.10	-0.8897421	-0.8897744	-0.8897998	-0.8898200
0.20	-0.7597710	-0.7597994	-0.7598217	-0.7598400
0.40	-0.4398281	-0.4398495	-0.4398661	-0.4398800
0.60	-0.0398852	-0.0398996	-0.0399109	-0.0399199
0.80	0.4400573	0.4400503	0.4400447	0.4400400
1.00	1.00000000	1.00000000	1.00000000	1.00000000

Table.5.3 (b)

RIGHT END BOUNDARY LAYER PROBLEMS

We now describe the numerical integration method for solving problems with the boundary layer at the right-end of the underlying interval. To be specific we consider the following singular perturbation problem.

$$\varepsilon\, y''(x)+a(x)\,y'(x)+b(x)\,y(x)=f(x)\ ;\ 0\le x\le 1 \qquad (5.27)$$

$$\text{with } y(0)=\alpha \text{ and } y(1)=\beta \qquad (5.28)$$

Where ε is a small positive parameter ($0<\varepsilon<<1$); α, β are given constants; a(x), b(x) and f(x) are assumed to be sufficiently continuously differentiable functions in [0, 1].

Here we are assumed that $a(x) \le M < 0$ throughout the interval [0,1] where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of x=1.

The evaluation of the right-end boundary layer for (5.27) and (5.28) is similar to that of the left-end boundary layer but there are some differences worth noting. By using Taylor series expansion in the neighborhood of the point x, we have consequently, Eq. (5.27) is replaced by the following first-order differential equation with a small deviating argument.

$$2\varepsilon\, y(x+\delta)-2\varepsilon\, y(x)-2\varepsilon\,\delta\, y'(x)+\delta^2 a(x)y'(x)+\delta^2 b(x)y(x)=\delta^2 f(x) \qquad (5.29)$$

Transition from Equation (5.27) to (5.29) is exists, because of the condition that δ is small Viz., ($0<\delta<<1$).

We rewrite equation (5.29) in the following convenient form:

$$y'(x)=p(x)\,y(x+\delta)+q(x)y(x)+r(x) \text{ for } 0\le x\le 1-\delta \qquad (5.30)$$

Where

$$p(x) = \frac{-2\varepsilon}{\delta^2 a(x) - 2\varepsilon\delta} \tag{5.31}$$

$$q(x) = \frac{2\varepsilon - \delta^2 b(x)}{\delta^2 a(x) - 2\varepsilon\delta} \tag{5.32}$$

$$r(x) = \frac{\delta^2 f(x)}{\delta^2 a(x) - 2\varepsilon\delta} \tag{5.33}$$

We now divide the interval [0, 1] into N equal parts with mesh size h, i.e., h=1/N and $x_i = ih$ for i= 1, 2, 3…….N. Integrating equation (5.30) in $[x_{i-1}\ x_{i+1}]$ we get

$$y(x_{i+1}) - y(x_{i-1}) = \int_{x_{i-1}}^{x_{i+1}} [p(x)\, y(x-\delta) + q(x)y(x) + r(x)]\, dx \tag{5.34}$$

By making use of the Newton-Cotes formula when n=2 i.e. applying Simpson's one-third rule approximately, we obtain

$$y(x_{i+1}) - y(x_{i-1}) = \frac{h}{3}[p(x_{i+1})y(x_{i+1}+\delta) + 4\,p(x_i)\,y(x_i+\delta) + p(x_{i-1})y(x_{i-1}-\delta) + q(x_{i+1})\,y(x_{i+1}) + q(x_{i-1})\,y(x_{i-1}) + 4q(x_i)y(x_i) + r(x_{i+1}) + 4r(x_i) + r(x_{i-1})) \tag{5.35}$$

By means of Taylor's series expansion we have

$$y(x_i + \delta) \cong y(x_i) + \delta\, y'(x_i)$$

and then by approximating $y'(x)$ by interpolation formula, we get

$$y(x_i + \delta) = y(x_i) + \delta\left[\frac{y(x_{i+1}) - y(x_i)}{h}\right] = (1 - \frac{\delta}{h})\, y(x_i) + \frac{\delta}{h}\, y(x_{i+1}) \tag{5.36}$$

and similarly we have

$$y(x_{i-1}+\delta)\cong(1-\frac{\delta}{h})\,y(x_{i-1})+\frac{\delta}{h}\,y(x_i) \tag{5.37}$$

$$y(x_{i+1}+\delta)\cong(1+\frac{\delta}{h})\,y(x_{i+1})-\frac{\delta}{h}\,y(x_i) \tag{5.38}$$

Apply (5.36), (5.37) and (5.38) in. (5.35) we have

$$y(x_{i+1})-y(x_{i-1})=\frac{h}{3}[\,p_{i+1}[(1+\frac{\delta}{h})\,y_{i+1}-\frac{\delta}{h}\,y_i]+4p_i[(1-\frac{\delta}{h})y_i+\frac{\delta}{h}\,y_{i+1}]$$
$$+p_{i-1}[(1-\frac{\delta}{h})\,y_{i-1}+\frac{\delta}{h}\,y_i]+q_{i+1}y_{i+1}+q_{i-1}y_{i-1}$$
$$+4q_i y_i+2r_{i+1}+4r_i+r_{i-1}\,] \tag{5.39}$$

Now rearranging equation (5.39) in the three point form. i.e. three term recurrence relation

$$A_i\,y_{i-1}+B_i\,y_i+C_i\,y_{i+1}=D_i \tag{5.40}$$

$$A_i=-[\frac{h}{3}p_{i-1}(1-\frac{\delta}{h})+\frac{h}{3}q_{i-1}+1] \tag{5.41}$$

$$B_i=-[\frac{\delta}{3}p_{i-1}+\frac{4hp_i}{3}(1-\frac{\delta}{h})-\frac{\delta\,p_{i+1}}{3}+\frac{4hq_i}{3}] \tag{5.42}$$

$$C_i=1-\frac{h}{3}\,p_{i+1}(1+\frac{\delta}{h})-\frac{4\,p_i\delta}{3}-\frac{h}{3}q_{i+1} \tag{5.43}$$

$$D_i=\frac{h}{3}[r_{i+1}+4r_i+r_{i-1}] \tag{5.44}$$

And $y_i=y(x_i)$, $p_i=p(x_i)$, $q_i=q(x_i)$ and $r_i=r(x_i)$

Now we can solve (5.40) the system of equations of order (N-1) in terms of (N-1) unknowns y_1 , y_2 , y_3 ,……….y_{N-1} by means of famous efficient Thomas algorithm. Repeat the numerical scheme for different choices of δ the deviating argument, satisfying the condition $0<\delta<<1$, until the solution profiles do not differ materially from iteration to iteration.

Example 5.4:

To demonstrate the applicability of the numerical integration method, we will discuss one singular perturbation problem with right-end boundary layer.

$$\varepsilon y''(x) - y'(x) = 0; \qquad 0 \le x \le 1$$
$$\text{with } y(0) = 1 \text{ and } y(1) = 0$$

In this example we have a(x) =-1, b(x) = 0 and f(x) =0. Further we have a boundary layer of width O(ε) at x =1

The exact solution is given by $y(x) = \dfrac{1-\exp(\frac{x-1}{\varepsilon})}{1-\exp(\frac{-1}{\varepsilon})}$

The computational results are presented in Table 5.4(a) and 5.4(b), for ε = 10^{-3}, 10^{-4} respectively.

Computational results for Example5.4:

ε **= = 10^{-3} and h=0.01**

x	δ =0.008	δ =0.009	δ =0.01	Exact solution
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.9999989	0.9999997	1.0000000	1.0000000
0.40	0.9999975	0.9999997	1.0000000	1.0000000
0.60	0.9999962	0.9999997	1.0000000	1.0000000
0.80	0.9999948	0.9999997	1.0000000	1.0000000
0.90	0.9999942	0.9999997	1.0000000	1.0000000
0.92	0.9999940	0.9999997	1.0000000	1.0000000
0.94	0.9999920	0.9999987	0.9999995	1.0000000
0.96	0.9998413	0.9998997	0.9999316	1.0000000
0.98	0.9876480	0.9899997	0.9917356	1.0000000
1.00	0.0000000	0.0000000	0.0000000	0.0000000

Table.5. 4(a)

***ε* = 10^{-4} and h=0.01**

x	δ =0.007	δ =0.008	δ =0.009	Exact solution
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	1.0000000	1.0000000	1.0000000	1.0000000
0.40	1.0000000	1.0000000	1.0000000	1.0000000
0.60	1.0000000	1.0000000	1.0000000	1.0000000
0.80	1.0000000	1.0000000	1.0000000	1.0000000
0.90	1.0000000	1.0000000	1.0000000	1.0000000
0.92	1.0000000	1.0000000	1.0000000	1.0000000
0.94	1.0000000	1.0000000	1.0000000	1.0000000
0.96	1.0000000	1.0000000	1.0000000	1.0000000
0.98	0.9998017	0.9998475	0.9998476	1.0000000
1.00	0.0000000	0.0000000	0.0000000	0.0000000

Table. 5.4(b)

CONCLUSIONS

As mentioned the numerical integration method is iterative on the deviating argument δ. The process is to be repeated for different choices of δ (deviating argument), until the solution profiles do not differ materially from iteration to iteration. The choice of δ is not unique but can assume any number of values satisfying the condition, $0 < \delta << 1$. To reduce the amount of computational time, we fix the mesh size h and vary the deviating argument δ. Finally, we pick up the smallest value of δ which produces the required accuracy. We have implemented this method on various problems with a left-end boundary layer and right-end boundary layer by taking different values for ε. The computational results are presented in Tables 5.1(a) - 5.4(b). We have given here only a few values although the solutions are computed at all the points with mesh size h. It can be observed from the tables that the present method approximates the exact solution very well. This shows the efficiency and accuracy of the present method.

We have observed that the numerical integration method is capable of solving general convection-diffusion type of singularly perturbed two-point boundary value problems. This method provides an alternative and supplementary technique to the conventional ways of solving singular perturbation problems. It is a practical method, easily adaptable on a computer to solve singular perturbation problems with a modest amount of problem preparation.

PART-III

CHAPTER-6

ARTIFICIAL DIFFUSION – CONVECTION PROBLEM IN ONE DIMENSION

INTRODUCTION

This chapter deals with a convection-diffusion problem in one-dimension with variable coefficient wherein an artificial – diffusion term is present. As a closed form solution, in general, is not possible the classical Frobenious method of series solution was used to solve the governing differential equation. Further the problem is also solved by making use of a central difference scheme. The Frobenious series solution is numerically computed and the results are compared with those obtained by central difference scheme. The results are depicted through graphs and the results obtained by both the methods seem to be in good agreement. It is observed that the artificial diffusion term plays a significant role in the behaviour of the solution.

Martin Stynes in his exemplary contribution [66] has presented an excellent survey of steady-state convection-diffusion problems. Quoting Morton [40], Stynes observes that while the most common source of convection-diffusion problem is through linearization of Navier-Stokes equations with large Reynolds number, there are at least ten diverse situations where such equation occurs.

In the present chapter we considered a convection-diffusion equation with a slight modification made in the diffusion coefficient, such diffusion coefficient is apparently increased with small quantity to analyze the nature of solution in the boundary layer region. The reason behind this, in chapter-2 the steady-state convection-diffusion problem in one dimension has a numerical solution which has oscillatory nature in the boundary layer region. In the present chapter we proposed to study a convection-diffusion problem with variable coefficients wherein the diffusion coefficient in chapter-2 is apparently increased by adding an artificial diffusion term to the diffusion coefficient which is merely a numerical quantity.

The revised differential equation is solved first by the classical series solution method of Frobenious. Subsequently the differential equation is also solved numerically making use of a central difference scheme. The solution is obtained by Frobenious method is numerically computed for a given diffusion parameter and is compared with the numerical solution. The results are seem to be in good agreement. The artificial diffusion term introduced seems to have influenced the boundary layer thickness and in the present case the boundary layer thickness is reduced in comparison with that obtained in chapter-2.

ANALYTICAL SOLUTION

In the case of Convection – Diffusion problem

$$-\varepsilon\frac{d^2u}{dx^2}+\frac{du}{dx}=1 \qquad \text{With the boundary conditions } u(0)=u(1)=0 \tag{6.1}$$

Analytical solution of (6.1) is

$$u(x)=x-\frac{e^{-(1-x)/\varepsilon}-e^{-1/\varepsilon}}{1-e^{-1/\varepsilon}} \text{ for } 0\le x\le 1 \tag{6.2}$$

the associated graphs of the solution (6.2) and the computed solution of (6.1) by using central difference scheme are shown here.

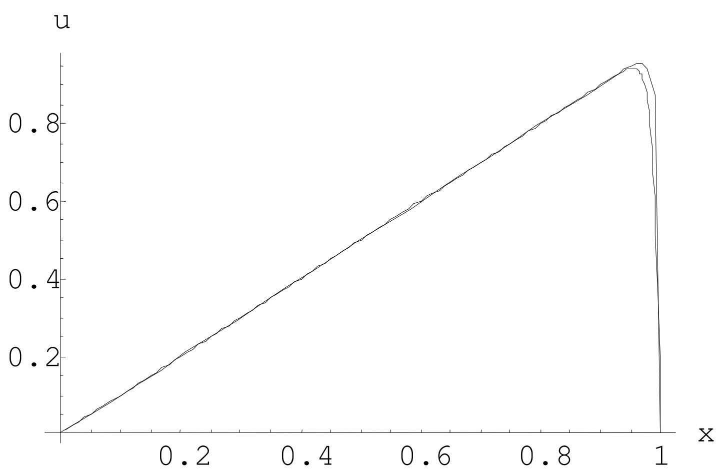


Figure.6.1

Now we shall consider the two–point boundary value artificial diffusion – convection problem in one-dimension given by

$$-(\varepsilon+\frac{hx}{2})\frac{d^2u}{dx^2}+x\frac{du}{dx}+u=1 \qquad \text{with } u(0)=0,\ u(1)=0 \tag{6.3}$$

Let $p(x)=-\dfrac{x}{(\varepsilon+\frac{h\,x}{2})}$, $q(x)=-\dfrac{1}{(\varepsilon+\frac{h\,x}{2})}$, $r(x)=-\dfrac{1}{(\varepsilon+\frac{h\,x}{2})}$ and (6.3) be brought to the standard form:

$$\frac{d^2u}{dx^2}+p(x)\frac{du}{dx}+q(x)\,u=r(x) \qquad \text{with } u(0)=0,\ u(1)=0 \tag{6.4}$$

The differential equation (6.4) is linear with variable coefficients. Closed form solution for this equation seems to be out of reach. Hence we propose to solve it by applying series solution method due to Frobenius.

x =0 is an ordinary point of (6.4) , its every solution can be expressed as a series of the form

$$u=\sum_{k=0}^{\infty} a_k\, x^k \tag{6.5}$$

Writing (6.5) and the expressions of

$$\frac{du}{dx}=\sum_{k=0}^{\infty} a_k k\, x^{k-1} \quad , \quad \frac{d^2u}{dx^2}=\sum_{k=0}^{\infty} a_k\, k\,(k-1)\, x^{k-2} \tag{6.6}$$

in (6.3) we have

$$-(\varepsilon+\frac{hx}{2})\sum_{k=2}^{\infty} a_k\, k\,(k-1)\, x^{k-2} + x\sum_{k=1}^{\infty} a_k\, k\, x^{k-1} + \sum_{k=0}^{\infty} a_k x^k = 1$$

The expressions for $a_2, a_3, a_4, a_5 \ldots\ldots.$ in terms of a_0, a_1 are given by

$$a_2=\frac{a_0-1}{2\varepsilon},\ a_3=\frac{h-h\,a_0+4\varepsilon\, a_1}{12\varepsilon^2},\ a_4=\frac{6\varepsilon(a_0-1)-(h^2-h^2 a_0-4a_1 h\,\varepsilon)}{48\,\varepsilon^3}$$

$$a_5=\frac{8\,\varepsilon\,(h-h\,a_0+4\,a_1\,\varepsilon)-3\,h\,(6\,\varepsilon\,a_0-6\,\varepsilon-h^2+h^2\,a_0-4\,a_1\,h\,\varepsilon)}{480\,\varepsilon^4} \quad \text{Etc.,}$$

On comparison of coefficients of lowest degree terms of x to zero, to determine the coefficients in terms of a_0, a_1 numerically, the recurrence relation may be obtained as

$$a_{n+2} = \frac{1}{\varepsilon(n+2)}\left[a_n - \frac{nh}{2} a_{n+1}\right],\ n = 2, 3, 4.... \qquad (6.7)$$

These coefficients are related in terms of a_0 and a_1

0n substitution of all the values in equation (6.4) and the boundary conditions u(0) = 0, u (1) = 0 the series solution may be obtained for h=0.01 , ε = 0.05 as

$$u = 1.626954733x - 10x^2 + 11.17969822x^3 - 50.55848491x^4 + 47.75233197x^5 + ... \qquad (6.8)$$

The approximated graph of (6.8) which is the solution of (6.3) is given below

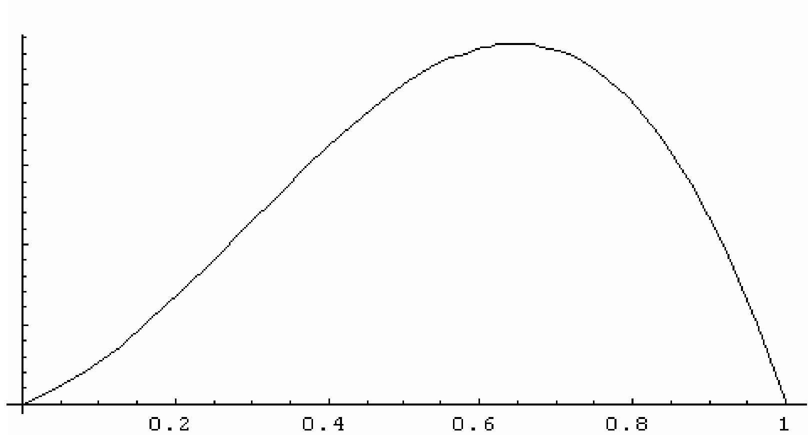


Figure 6.*2*

which satisfies the condition of convergence in the interval $0 < x < 1$ by virtue of D'alembert's ratio test. The condition of convergence can be established by introducing partial sums.

FINITE DIFFERENCE METHOD

Consider the artificial diffusion – convection equation

$$-\left(\varepsilon + \frac{hx}{2}\right)\frac{d^2u}{dx^2} + x\frac{du}{dx} + u = 1 \quad \text{with } u(0) = 0, u(1) = 0 \qquad (6.9)$$

Apply central difference scheme to the above differential equation where

$$u'(x) = \frac{u_{i+1} - u_{i-1}}{2h} \quad \text{and } u''(x) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \tag{6.10}$$

where $u_i = u(x_i)$. , x= ih on substitution of (6.10) in (6.9) we get

$$-(\varepsilon + \frac{ih^2}{2}) \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + ih \frac{u_{i+1} - u_{i-1}}{2h} + u_i = 1 \tag{6.11}$$

The final transformed difference scheme is

$$a_i u_{i+1} + b_i u_i + c_i u_{i-1} = d_i \tag{6.12}$$

Where $a_i = -\varepsilon, b_i = 2\varepsilon + h^2(1+i), c_i = -(\varepsilon + ih^2), d_i = h^2$

The boundary conditions u(0) = u(1) = 0 are represented by $u_0 = 0$, $u_N = 0$

Equation (6.12) represents a Tri-diagonal Matrix of the form

$$A\vec{u} = \vec{D} \tag{6.13}$$

where the coefficient matrix A is of order n-1 The Non-homogeneous linear system (6.13) is solved by applying Thomas algorithm. Here the Coefficient matrix is a Monotonic matrix. This concept incorporated reduces the variations in the computed solution. The computed result with corresponding graph is shown below.

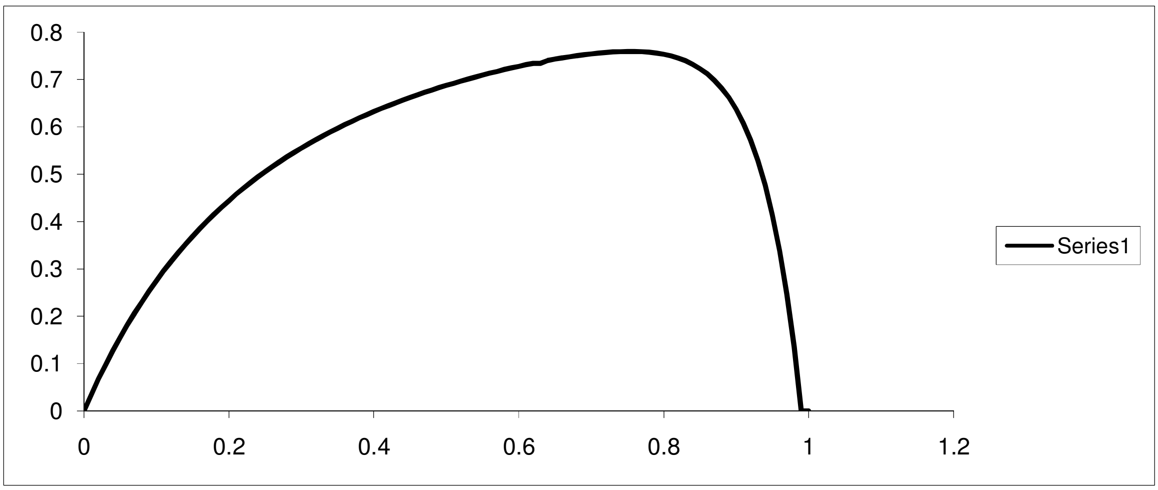


Figure.6.3

Numerical solution of artificial diffusion – convection equation

X	u	x	u	X	u	x	u
0	0	0.26	0.5157	0.51	0.6919	0.76	0.7591
0.01	0.0345	0.27	0.5261	0.52	0.6964	0.77	0.7585
0.02	0.0672	0.28	0.5361	0.53	0.7008	0.78	0.7574
0.03	0.0982	0.29	0.5457	0.54	0.705	0.79	0.7556
0.04	0.1275	0.3	0.555	0.55	0.7091	0.8	0.753
0.05	0.1553	0.31	0.564	0.56	0.7131	0.81	0.7495
0.06	0.1817	0.32	0.5726	0.57	0.7169	0.82	0.745
0.07	0.2068	0.33	0.5809	0.58	0.7206	0.83	0.7392
0.08	0.2307	0.34	0.589	0.59	0.7242	0.84	0.7319
0.09	0.2534	0.35	0.5968	0.6	0.7276	0.85	0.7229
0.1	0.275	0.36	0.6043	0.61	0.731	0.86	0.7118
0.11	0.2956	0.37	0.6115	0.62	0.7341	0.87	0.6982
0.12	0.3152	0.38	0.6185	0.63	0.7342	0.88	0.6816
0.13	0.3339	0.39	0.6253	0.64	0.7401	0.89	0.6615
0.14	0.3518	0.4	0.6319	0.65	0.7428	0.9	0.6371
0.15	0.3689	0.41	0.6383	0.66	0.7454	0.91	0.6076
0.16	0.3853	0.42	0.6444	0.67	0.7479	0.92	0.5719
0.17	0.4009	0.43	0.6504	0.68	0.7501	0.93	0.5288
0.18	0.4159	0.44	0.6562	0.69	0.7522	0.94	0.4768
0.19	0.4302	0.45	0.6618	0.7	0.754	0.95	0.4139
0.2	0.444	0.46	0.6672	0.71	0.7556	0.96	0.338
0.21	0.4572	0.47	0.6724	0.72	0.757	0.97	0.2461
0.22	0.4698	0.48	0.6775	0.73	0.758	0.98	0.1348
0.23	0.482	0.49	0.6825	0.74	0.7588	0.99	0.0001
0.24	0.4937	0.5	0.6873	0.75	0.7591	1	0
0.25	0.5049						

Table 6.1

CONCLUSIONS

It has been observed that the graphs shown in Fig(6.1) , Fig(6.2) , Fig(6.3) maintain character preserving phenomena over (0,1). Especially in the interval of smooth region steep down fall of the graph coinciding with the actual solution is significant thing of considerable order. For small ε the equation is dominated by the convection term. The boundary or interior layers may appear along downstream of the convection direction i.e., after the smooth region the diffusion effect is visible in the interval (δ, 1). The amount $h\,a(x)/2$ by which the diffusion coefficient was apparently increased is called the artificial diffusion. Stable solution is observed under the influence of the artificial-diffusion. The exact solution is non-zero almost everywhere except at the boundary points and approaches to zero in a narrow boundary layer sub-interval very close to the point x=1. The numerically computed values of u also support this statement vide Table-6.1. The computed solution and the series solution exhibit good agreement on the convection-diffusion phenomena almost throughout the region. When diffusion is more (Artificial diffusion), then the computed layers are smeared.

CHAPTER-7

NUMERICAL STUDY OF CONVECTION –DIFFUSION PROBLEM IN TWO-DIMENSIONAL SPACE

INTRODUCTION

The convection-diffusion problem in two-dimensional space is solved on a unit square mesh with the prescribed boundary conditions by finite difference method where in central difference scheme is employed. In the process finite difference scheme of Standard five point formula was employed. Initial approximations to temperature distribution function were given on the basis suitable to physical nature of the problem by intuition. The results thus obtained are plotted through graphs and the physical nature of the problem is discussed. It is observed that there is a boundary layer at the specific values of arguments.

Consider the elliptic operator whose second order derivatives are multiplied by a parameter ε that is close to zero. These derivatives model diffusion while first-order derivatives are associated with the convective or transport process. In classical problems ε is not close to zero. Here the two-dimensional convection-diffusion problem is studied. Diffusion term play an important role at the boundary layer near the arguments x=1, y=1 which makes rapid changes in the solution at the boundary layer. In the two-dimensional convection- diffusion problem the differential equation got converted to difference equation. The corresponding finite difference scheme is solved by using standard five point formula with the initial guess values. Here we have selected the relation between mesh size (h) and the perturbation parameter (ε) in such a way that the numerical solution gives a stable solution. To summarize, when a standard numerical method is applied to a convection-diffusion problem, when there is too little diffusion then the computed solution is often oscillatory, while if there is superfluous diffusion term, the computed layers are smeared.

We can see that the solution of this problem has a convective nature on most of the domain of the problem, and the diffusive part of the differential operator is influential only in the certain narrow sub-domain. In this region the gradient of the solution is large. This nature is evidenced by steep down fall of solution near the boundary.

In the linear convection-diffusion problem with variable coefficient, transport mechanism dominates where as diffusion effects are confined to a reasonably small part of the domain. The coefficient of diffusion causes the oscillations at the boundary layer. The solution pattern shows that at the boundary layer diffusion term play significant role. For low Peclet number we may get the stable solution.

ANALYTICAL SOLUTION

In this chapter, the diffusion coefficient ε is a small positive parameter and coefficient of convection a(x, y) is continuously differentiable function that is Holder continuous on $\bar{\Omega}$ the closure of Ω.

In two dimensions, the governing convection-diffusion equation is

$$-\varepsilon\,\Delta u(x, y) + a(x, y)\,\nabla u\,(x, y) + b(x, y)\,u(x, y) = f(x, y)$$

$$\text{on} \quad \Omega \subset R^2 \text{ with } u(x,y) = g(x,y) \text{ on } \partial\Omega \qquad (7.1)$$

where $0 < \varepsilon << 1$, and the functions a, b and f are assumed to be Holder continuous on $\vec{\Omega}$, the closure of Ω. Here we also assume that $b \geq 0$ on $\vec{\Omega}$. Here Ω is any bounded domain in R^2 with a piecewise Lipchitz–continuous boundary $\partial\Omega$. Let us suppose that g is continuous except perhaps for a jump discontinuity at a single point.

The differential operator L is elliptic so (7.1) possess a solution in the region defined. Here L also satisfies the Maximum principle which is discussed in [59]. Assume that the absolute value of ‘a’ is close to 1 so that convection dominates diffusion. In the problem that we consider, the solution u(x,y) of (7.1) has an asymptotic structure similar to that of one-dimensional problem which was discussed in chapter-2. We can write u as the sum of the solutions to a first-order partial differential equation, u at layer(s) with order O(ε) term.

To make this more precise, we divide the boundary $\partial\Omega$ into 3 parts

Inflow boundary $\quad \partial^-\Omega = \{x \in \partial\Omega : a.n < 0\}$,

Outflow boundary $\quad \partial^+\Omega = \{x \in \partial\Omega : a.n > 0\}$

Tangential flow boundary $\quad \partial^0\Omega = \{x \in \partial\Omega : a.n = 0\}$, ……. (7.2)

Where **n** is the outward-pointing unit normal to $\partial\Omega$.

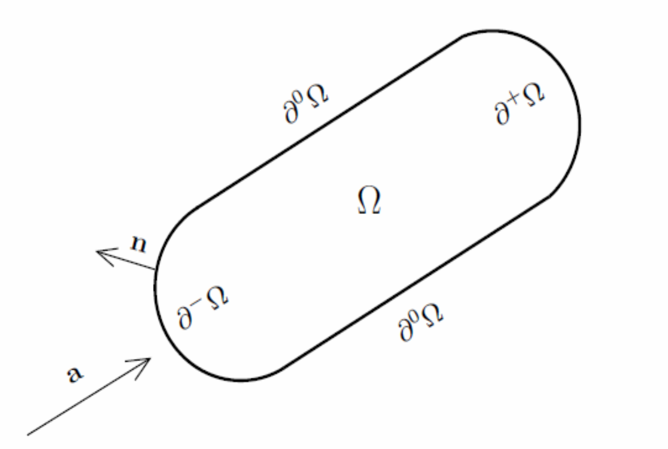


Figure.7.1 (Partition of $\partial\Omega$)

A typical solution u will have boundary layers–narrow regions close to $\partial\,\Omega$ where $|\nabla u|$ is large along $\partial^{+}\Omega$ and $\partial^{0}\Omega$. As in one-dimensional problems exceptional Dirichlet boundary conditions on g can eliminate these layers. On most of Ω , u is approximately equal to $u_0(x,y)$. Then the solution of the reduced problem

$$a(x,y)\,\nabla u_0\,(x,y) + b(x,y)\,u(x,y) = f(x,y) \ \text{ on } \Omega,\ u_0 = g \text{ on } \partial^{-}\Omega \qquad (7.3)$$

This first-order partial differential equation (7.3) has characteristic curves which are the parameterized curves (x (t), y(t)) in Ω defined by

$$x^{1}(t) = a_1(x,y),\ y^{1}(t) = a_2(x,y) \qquad (7.4)$$

with initial data (x (0), y (0)) = (x', y') where (x', y') is any point in $\partial^{-}\Omega$. Thus one such curve radiates into Ω from each point in $\partial^{-}\Omega$.

Exponential Boundary Layers

Consider the boundary value Problem (7.1)

$-\varepsilon\,\Delta u + b(x, y)\,\nabla u + c(x, y)\,u = f(x, y)$ in $\Omega = (0, 1) \times (0, 1)$,

$u = 0$ on the boundary Γ

Assume that the data are smooth and that $c \geq 0$ with $b = (b_1, b_2)$ where $b_1 > 0$ and $b_2 > 0$. Then the sub characteristics behave as in Figure.7. 2 and the reduced problem is defined as

$$b.\nabla u_0 + c u_0 = f, u_0\big|_{x=0} = u_0\big|_{y=0} = 0 \tag{7.5}$$

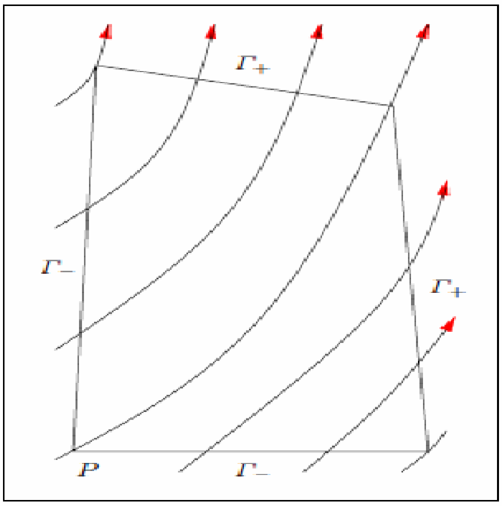


Figure.7.2 Sub Characteristics through a Corner

We expect exponential boundary layers at x = 1 and at y = 1. The asymptotic approximation with transformation $\xi = (1-x)/\varepsilon$ and $\eta = (1-y)/\varepsilon$ takes the form:

$$u_{Asy}^{*}(x, y) = u_0(x, y) - u_0(1, y)\exp\left[-b_1(1, y)\frac{1-x}{\varepsilon}\right] - u_0(x,1)\exp\left[-b_2(x,1)\frac{1-y}{\varepsilon}\right] \tag{7.6}$$

equation (7.6) is inaccurate near the corner point (1,1) because the boundary layer terms overlap there. Consequently we add a corner layer correction which is the solution of

$$-\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}\right) - b_1(1,1)\frac{\partial \omega}{\partial \xi} - b2(1,1)\frac{\partial \omega}{\partial \eta} = 0 \quad \text{On } (0,\infty) \text{ X } (0,\infty) \tag{7.7}$$

With the use of the transformations $\xi = (1-x)/\varepsilon$ and $\eta = (1-y)/\varepsilon$

We obtain

$u_{Asy}(x,y) = u_{Asy}^{*}(x,y) + u_0(1,1)\exp\left[-b_1(1,1)\frac{(1-x)}{\varepsilon}\right]\exp\left[-b_2(1,1)\frac{1-y}{\varepsilon}\right]$

if $U_0 \in C^2(\Omega)\cap C(\bar{\Omega})$, the classical comparison principle gives

$$\|u - u_{asy}\| \leq C\varepsilon \qquad (7.8)$$

Here C is generic constant which is independent of ε. Layers along $\partial^+\Omega$ are called regular or exponential boundary layers. Writing $\vec{n} = (n_1, n_2)$ for the unit outward-pointing normal to the $\partial\Omega$, then near $\partial^+\Omega$, exponential layers are essentially multiples of the function Exp[-(**a.n**) d(x,y), $\partial^+\Omega/\varepsilon$] , where d(x,y), denote the distance from the point (x,y) to the out-flow boundary $\partial^+\Omega$. Thus in cross-section perpendicular to $\partial^+\Omega$ these layers are very much similar to the boundary layers that in one -dimension. Their first order derivatives in the direction perpendicular to the boundary have magnitude $O(\frac{1}{\varepsilon})$, and the width of the layer is O (ε ln(1/ε)).

FINITE DIFFERENCE METHOD:

Consider the two-dimensional convection-diffusion problem

$$-\varepsilon\,\Delta u(x,y) + \frac{\partial u}{\partial x} = 1 \qquad \text{Equivalently}$$

$$-\varepsilon\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \frac{\partial u}{\partial x} = 1 \quad \text{defined in the region } \Omega = (0,1)\times(0,1)$$

$u(x,y) = 0$ on the boundary $\partial\Omega$ (7.9)

i.e., u(0,0) = 0 , u(i,0) = 0 , u(0,j) = 0 , u(1,1) = 0   i,j =1,2,3……….n

As a closed form solution, in general, is not possible so we solve the problem by using Discretization method.

Discretize the above differential equation (7.9) by using central difference approximations

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}], \frac{\partial^2 u}{\partial y^2} = \frac{1}{k^2}[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] \qquad (7.10)$$

and $$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \tag{7.11}$$

Apply equation (7.10), (7.11) in (7.9) to get a difference equation of the form with h = k on the square Region.

$$\frac{-\varepsilon}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] + \frac{u_{i+1,j} - u_{i-1,j}}{2h} = 1$$

The final transformed difference scheme is

$$u_{i,j} = \frac{1}{8\varepsilon}\left[2h^2 - (-2\varepsilon + h)\, u_{i+1,j} + (2\varepsilon + h)\, u_{i-1,j} + 2\varepsilon u_{i,j+1} + 2\varepsilon\, u_{i,j-1}\right] \tag{7.12}$$

Select $\varepsilon = 0.05$, h = 0.01 so that we can expect a stable solution. Apply the standard five point formula on (7.12) by selecting the initial approximations we can get values of u at each nodal point. The associated graph is as plotted below in figure. 7.3

The values of u have been computed for the ranges of x =0 to x =1 and y=0 to y=1with spacing h=k=0.01. There are as many as 99x99 entries in the tabulated output. Here we are furnishing values of u corresponding to: x=0 to x=0.1 with step size 0.01, then for x=0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 with step size 0.1. Finally values of u for x=0.9 to x=1 are presented in Table.7.1 with step size 0.01.

Temperature Distribution (U-values):

x \ y	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
0	0	0	0	0	0	0	0	0	0	0	0
0.01	0	0.08516	0.15355	0.20461	0.24031	0.26388	0.2612	0.2617	0.2644	0.2766	0.281
0.02	0	0.15252	0.27489	0.29882	0.4282	0.469045	0.27869	0.28759	0.29274	0.29561	0.29717
0.03	0	0.19734	0.35617	0.29882	0.55394	0.60592	0.49433	0.50929	0.51781	0.5225	0.525
0.04	0	0.22099	0.39993	0.29882	0.62246	0.68051	0.63776	0.6564	0.6669	0.67259	0.6756
0.05	0	0.23024	0.41771	0.55635	0.65112	0.71183	0.71583	0.73637	0.74784	0.75404	0.75431
0.06	0	0.23269	0.42273	0.56482	0.65963	0.72121	0.74865	0.76998	0.78186	0.78825	0.80403
0.07	0	0.23306	0.42358	0.56482	0.66121	0.72297	0.75851	0.78009	0.79209	0.79854	0.80403
0.08	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76038	0.78201	0.79403	0.80049	0.80403
0.09	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.1	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.2	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.3	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.4	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.5	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.6	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.7	0	0.23309	0.42365	0.56482	0.66135	0.72312	0.76054	0.78218	0.7942	0.80066	0.80403
0.8	0	0.23309	0.42365	0.56482	0.66134	0.72312	0.76053	0.78215	0.7942	0.80066	0.80061
0.9	0	0.23293	0.42386	0.56211	0.65395	0.71856	0.74116	0.66777	0.79405	0.80056	0.7012
0.91	0	0.23274	0.42196	0.5593	0.64698	0.71414	0.72493	0.59541	0.79402	0.79984	0.701
0.92	0	0.23234	0.42014	0.5539	0.63427	0.70592	0.69724	0.55345	0.794	0.79899	0.7
0.93	0	0.2315	0.41656	0.54389	0.61204	0.69115	0.6521	0.50432	0.79399	0.79734	0.6989
0.94	0	0.22981	0.40978	0.52605	0.57492	0.66568	0.58233	0.5	0.78802	0.77796	0.68989
0.95	0	0.22652	0.39743	0.49574	0.51627	0.62374	0.48118	0.48997	0.78238	0.75976	0.6885
0.96	0	0.23038	0.376	0.49574	0.42941	0.5583	0.44878	0.48112	0.67	0.72924	0.68232
0.97	0	0.20944	0.34084	0.43377	0.31047	0.46257	0.3052	0.46128	0.6022	0.68025	0.60723
0.98	0	0.19102	0.28684	0.27189	0.16263	0.33297	0.3	0.42134	0.49614	0.49614	0.49978
0.99	0	0.11989	0.11186	0.14338	0.16163	0.17356	0.1794	0.18236	0.1838	0.18446	0.18477
1	0	0	0	0	0	0	0	0	0	0	0

Table. 7.1

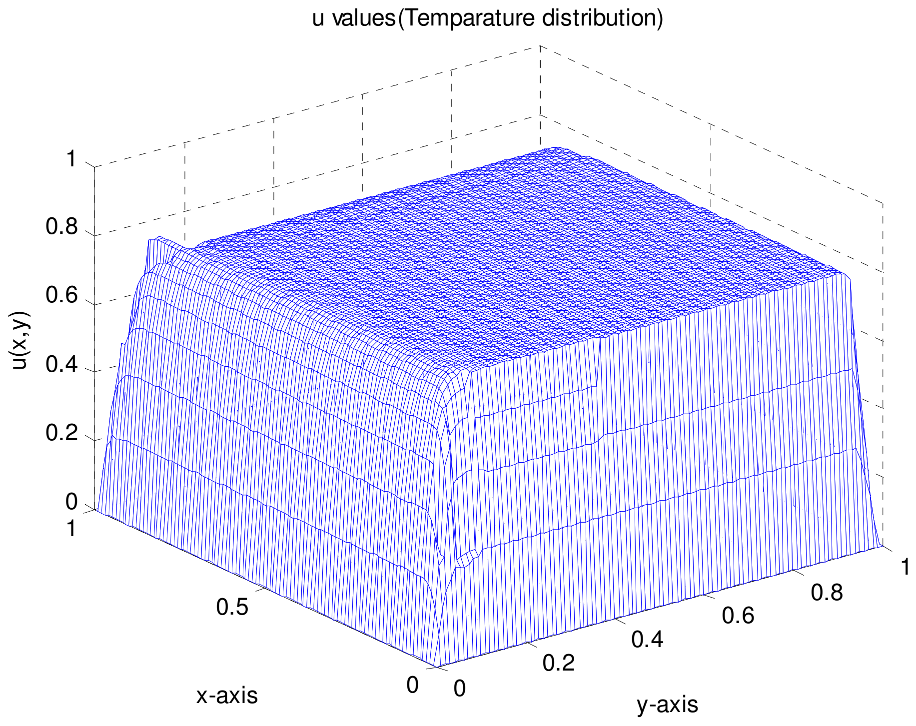


Figure.7.3

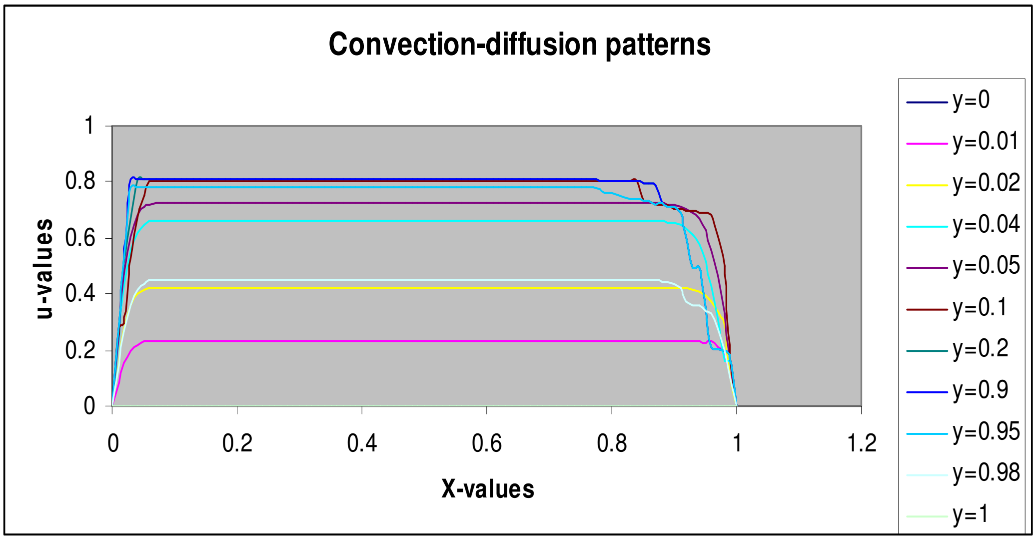


Figure. 7.4

CONCLUSIONS

The inflow boundary $\partial^{-}\Omega$ is the side x = 0 of $\vec{\Omega}$; the tangential flow boundary comprises of the sides y = 0 and y = 1; the outflow boundary is the remaining side x = 1. From (7.4) each sub characteristic is parameterized by $x'(t) = 1, y'(t) = 0$ so that we can get $x = t$ as admissible solution and the sub characteristics are the lines $y = k$ (arbitrary).

On most of Ω from Figure.7.3, Figure.7.4 it is evidenced that u(x, y) ≈ x in the region. The side x = 1 of $\vec{\Omega}$ is the outflow boundary $\partial^{+}\Omega$ and an exponential layer appears there. The tangential flow boundaries y = 0 and y = 1 have characteristic boundary layers that grow in strength as x moves from 0 to 1 because of the increasing discrepancy between u_0 and the boundary conditions. On most of the region convection process dominates where as diffusion process is visible only at the neighborhood of the corner point (1,1). For low Peclet number convection process dominates in the region identified. When values of x are in the range 0.08-0.81 values of u are found to be constant for any choice of values of y, means there is no effect of diffusion. Infact naturally u lies in the smooth region, as mentioned above prior to the boundary layer region. For high Peclet number solutions are essentially of pure convection flows. The solution possesses an interior layer starting at (0, 0.8). On the boundary x=1 and on the right part of the boundary y=0 exponential layers are developed.

PART-IV

CHAPTER-8

NUMERICAL STUDY OF WAVE PROPAGATION IN A NON-LINEAR MEDIUM DUE TO IMPACT

INTRODUCTION

Two bodies which have distinct velocities in the same direction come into contact, an impact occurs. Within the impact analysis i.e., in the displacement of the bodies after impact, the impact force is a function of time‘t’ which is acting like a compression force. The impact time is very short and the stresses generated are high. We studied non-linear material behaviour in the one-dimensional case after impact. The wave propagation is studied by means of material nature. Here we considered two bodies with same material property with some non-linearity. Nonlinearity is studied after impact. The objective of this chapter is to present a numerical study of propagating pulsed and harmonic waves in nonlinear media using a Finite difference scheme. This study focuses on longitudinal, one-dimensional wave propagation. In the finite difference scheme Non-linear system is reduced to a linear system by quasi-linearization method. The numerically obtained results reveal the material nature.

FORMULATION OF THE PROBLEM

A bar -1 of length L_1 impacts another bar- 2 of length ‘ L_2‘. Both bars have the same material properties and non-linear nature. The left bar has an initial velocity of V_0, whereas the right bar is at rest.

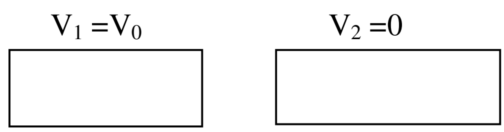


Figure. 8.1

(LONGITUDINAL IMPACT OF TWO BARS)

Here c (u) ≥ 0, $R_N \leq 0$ & R_N c (u) =0 (8.1)

Here Reaction force (R_N), Normal gap c(u) are perpendicular to one another. Furthermore one has to fulfill the initial and boundary conditions of the problem stated in the above figure and the standard contact conditions (8.1) which describe that no penetration can occur at the contact point and also that the contact force is a compression force. In this problem our interest is to study when two objects are selected in the figure (8.1) which are not linear in nature.

For materials under plastic deformation, Materials with distributed damage, linear elastic Hooke's law is usually inadequate to describe their nonlinear, inelastic behavior. Various constitutive laws have been proposed. Here we study the class of materials whose behavior can be described by the following stress-strain relationship.

$$\frac{\partial\sigma(\varepsilon,\varepsilon^1)}{\partial\varepsilon}=g(\varepsilon)-\alpha s[\sigma(\varepsilon_0)-f(\varepsilon_0)]e^{\alpha s[\varepsilon_0-\varepsilon]}-\alpha s\int_{\varepsilon_0}^{\varepsilon}[g(\tau)-\frac{df(\tau)}{d\tau}]e^{\alpha s(\tau-\varepsilon)}\,d\tau \tag{8.2}$$

Where ε_0 is the initial strain s=sign (ε^1), α is a constant, and f (ε) and g (ε) are functions to be determined experimentally for a given material.

A special case of (8.2) namely with no initial stress and strain is considered as

$$\frac{\partial\sigma(\varepsilon,\varepsilon^1)}{\partial\varepsilon}=g(\varepsilon)+\alpha s f(0)e^{-\alpha s\varepsilon}-\alpha s\int_0^{\varepsilon}\left[g\{\tau\}\frac{df(\tau)}{d\tau}\right]e^{\alpha s[\tau-\varepsilon]}\,d\tau \tag{8. 2(a)}$$

set $\alpha = 0$,

$$g(\varepsilon)=E(1-\gamma\varepsilon-\delta\varepsilon^2) \tag{8.3}$$

One can reduce the stress-strain relationship of (8.2(a)) to the well-known nonlinear elastic constitutive law,

$$\frac{\partial\sigma(\varepsilon,\varepsilon')}{\partial\varepsilon}=E(1-\gamma\varepsilon-\delta\varepsilon^2-.....) \tag{8.4}$$

Where E is the second order Elastic (Young's) modulus. Eγ is called the third order elastic constant, Equation (8.4) was derived by Landau and Lifshitz (1959) by expanding the strain energy density function for hyper-elastic materials.

Equations (8.4) do not show any hysteresis in the stress-strain relationship. The hysteretic behavior is accounted for by using a nonzero α. Means, call α the hysteresis parameter.

Using the equation of motion by Achenbach [2]

$$\frac{1}{\rho}\frac{\partial \sigma}{\partial x} = \frac{\partial^2 u}{\partial t^2} \tag{8.5}$$

Where u(x, t) is the displacement in the x-direction, ρ is the mass density, and $\sigma(x,t)$ is the normal stress in the x-direction. For the small strain deformation considered here, the normal strain in the x-direction is

$$\varepsilon = \frac{\partial u}{\partial x} \tag{8. 5(a)}$$

By using formula (8.5(a)) with c= $\sqrt{\frac{E}{\rho}}$ and $\sigma = \sigma(\varepsilon, \varepsilon')$ in (8.5) we have

$$\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = [\frac{1}{E}\frac{\partial \sigma}{\partial \varepsilon} - 1]\frac{\partial^2 u}{\partial x^2} \tag{8.6}$$

Where E is the elastic Young's modulus and 'c' can be considered as the phase velocity. This nonlinear equation is solved by applying finite difference method. In the middle of the process iteration across the time –step concept is introduced to overcome the Non-linearity of equations.

From equation (8.4) in the case of a nonlinear material considered by the first two terms only so that we get

$$\frac{\partial \sigma(\varepsilon, \varepsilon')}{\partial \varepsilon} = E(1 - \gamma\varepsilon) \tag{8.7}$$

$$\sigma = E\ (\varepsilon - \frac{1}{2}\gamma\varepsilon^2) \tag{8.8}$$

Clearly, when $\gamma = 0$, the material is linear elastic. The parameter γ indicates the amount of material nonlinearity. The parameter γ defined here is identical to the acoustic nonlinear parameter. The acoustic nonlinear parameter arises in metals due to lattice anharmonicity which is usually very small in comparison to the elastic deformation of the metals. So we can study wave propagation nature for various acceptable values of Υ. Here we are considered the values γ =10000, γ=5000 and γ =2500 respectively. From (8.8) we observe that the material behaves differently in tension and Compression, although the difference is only to the second order. In the literature, such material behavior is sometimes referred to as pseudo elastic. To model materials with identical nonlinear tensile and compressive behavior only the quadratic terms in (8.4) should be used.

Apply (8.8) in (8.6) to get

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(1 - \gamma \frac{\partial u}{\partial x}\right) \frac{\partial^2 u}{\partial x^2} \tag{8.9}$$

Equation (8.9) is the non-linear wave - equation developed by Gol'dberg (1961)

NUMERICAL SOLUTION OF WAVE EQUATION

We apply Finite difference scheme to equation (8.9) with (8.10), (8.11) (given below) so that it becomes a difference equation where $\frac{\partial u}{\partial x}$ is a varying strain value, occurs in the non-linear equation (8.9). Such an equation is reduced to a linear equation by applying quasi- linearization technique. Normally the value of $\frac{\partial u}{\partial x}$ would be assigned its value at the beginning of the time-step; the computations might be repeated at various time levels.

This procedure of reevaluating coefficients is called Quasi-linearization method.

(8.9) can be re-written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} \tag{8.10}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} = 0 \tag{8.11}$$

(8.9) is of the form: $f(u_x, u_{xx}, u_{tt}) = 0$ (8.12)

Apply the quasi-linearization method on the governing equation (8.9) we have

$$f(u_x, u_{xx}, u_{tt})\Big|_{(n)} + (u_x^{n+1} - u_x^{n})\frac{\partial f}{\partial u_x}\Big|_{(n)} + (u_{xx}^{n+1} - u_{xx}^{n})\frac{\partial f}{\partial u_{xx}}\Big|_{(n)}$$

$$+ (u_{tt}^{(n+1)} - u_{tt}^{(n)})\frac{\partial f}{\partial u_{tt}}\Big|_{(n)} = 0 \tag{8.13}$$

(8.13) can be transformed to u(x, t) notation so that

$$(u_{tt} - u_{xx} + \gamma u_x u_{xx})_{(n)} + (u_x^{n+1} - u_x^{n})(\gamma u_{xx})_{(n)} + [(u_{xx}^{(n+1)} - u_{xx}^{(n)})(\gamma u_x)]_{(n)}$$

$$+\left(u_{tt}^{(n+1)} - u_{tt}^{(n)}\right)_{(n)} = 0 \tag{8.14}$$

Finally equation (8.14) after simplification can be written as

$$u_{tt}^{(n+1)} + \gamma u_x^{(n)} u_{xx}^{(n+1)} + \gamma u_{xx}^{(n)} u_x^{(n+1)} - u_{xx}^{(n)} - \gamma u_x^{(n)} u_{xx}^{(n)} = 0 \tag{8.15}$$

(n+1)th stage is iterative so that

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \tag{8.16}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \tag{8.17}$$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = c^2\left(1 - \gamma\frac{\partial u}{\partial x}\right)\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{k^2}{h^2}c^2\left(1 - \gamma\frac{\partial u}{\partial x}\right)\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \beta\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

Where $\beta = \frac{k^2}{h^2}c^2\left(1 - \gamma\frac{\partial u}{\partial x}\right)$

$$u_{i,j+1} = -u_{i,j-1} + 2(1-\beta)\, u_{i,j} + \beta\left(u_{i+1,j} + u_{i-1,j}\right) \tag{8.18}$$

(8.9) satisfy the following conditions

$\alpha = \frac{k^2}{h^2} \leq 1$ for convergence of solution.

the boundary conditions are $u(0,t) = 0 \Rightarrow u(0,jk) = 0$ for $j = 1,2,3\ldots\ldots$

$u(15, t) = -3 \Rightarrow u(15,jk) = -3$ for $j = 1,2,3\ldots..n$ for $t>0$

$\left(\frac{\partial u}{\partial t}\right)_{t=0} = v_0;\ 0 \leq x \leq 7.5$ initial velocity

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0; 7.5 < x \leq 15 \text{ so that}$$

$$\Rightarrow u_{i,1} = u_{i,0} + v_0 k \tag{8.19}$$

$$\Rightarrow u_{i,1} = u_{i,0} \tag{8.20}$$

Initial displacement u (x, 0) = 0.125 sinx

$$\Rightarrow u(ih,0) = 0.125\sin(ih) \tag{8.21}$$

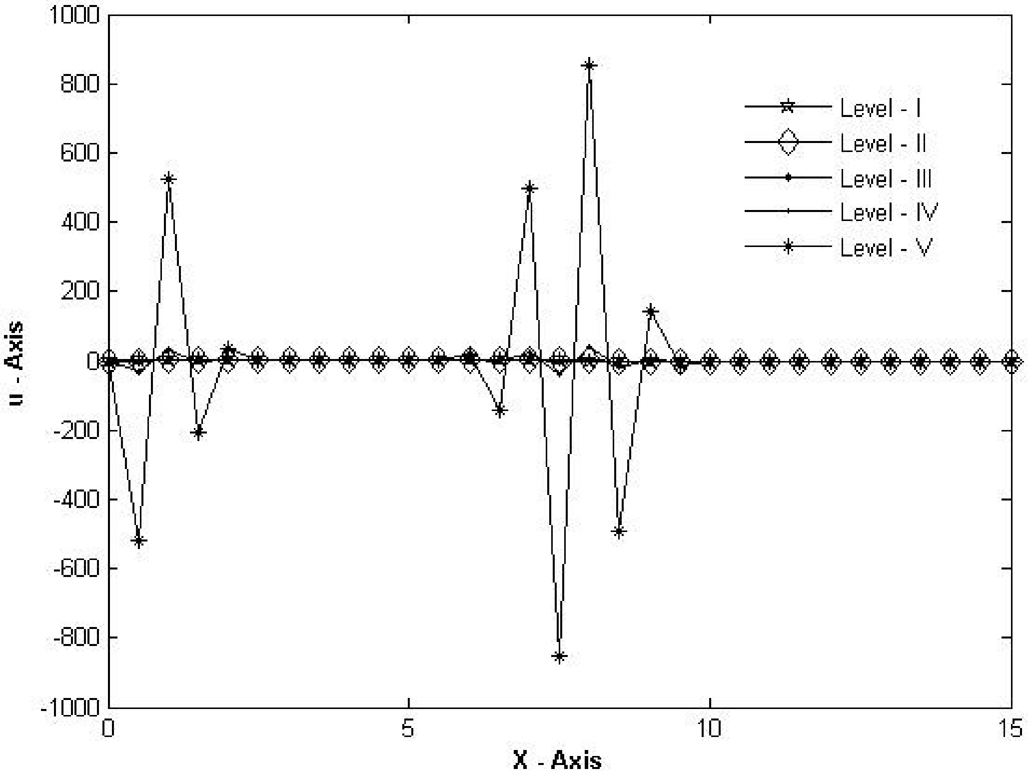
Apply the quasi-linearization technique on (8.18) with (8.19), (8.20) and (8.15) we can get the following results. Also the wave propagation is plotted with various time levels with $v_0 = 5$ m/s .

NUMERICAL RESULTS

CASE-1: at $\gamma = 10000$

X	LEVEL-1	LEVEL-2	LEVEL-3	LEVEL-4	LEVEL-5
0	0	0	0	0	0
0.5	0.559928	-0.17976	-2.03978	-28.8021	-518.078
1	0.5	1.119854	2.814675	27.12049	524.336
1.5	0.5	1	1.402315	-3.58587	-208.491
2	0.5	1	1.5	2.3278	37.22868
2.5	0.5	1	1.5	2	0.710263
3	0.5	1	1.5	2	2.5
3.5	0.5	1	1.5	2	2.5
4	0.5	1	1.5	2	2.5
4.5	0.5	1	1.5	2	2.5
5	0.5	1	1.5	2	2.5
5.5	0.5	1	1.5	2	2.5
6	0.5	1	1.5	2	17.43236
6.5	0.5	1	1.5	-0.7353	-140.268
7	0.5	1	2.257288	21.67987	497.0688
7.5	0.5	0.00019	-2.01452	-39.4301	-851.865
8	0	0.999981	3.514522	41.43012	854.3651
8.5	0	0	-0.75729	-19.6799	-494.568
9	0	0	0	2.735351	142.7683
9.5	0	0	0	0	-14.93
10	0	0	0	0	0
10.5	0	0	0	0	0
11	0	0	0	0	0
11.5	0	0	0	0	0
12	0	0	0	0	0
12.5	0	0	0	0	0
13	0	0	0	0	0
13.5	0	0	0	0	0
14	0	0	0	0	0
14.5	0	0	0	0	0
15	-3	-3	-3	-3	-3

Table. 8.1



1000
800
600
400
200
0
-200
-400
-600
-800
-1000
0
5
10
15
u - Axis
X - Axis
Level - I
Level - II
Level - III
Level - IV
Level - V

Figure. 8.1(a)

CASE-II

at $\gamma = 5000$

X	LEVEL-1	LEVEL-2	LEVEL-3	LEVEL-4	LEVEL-5
0	0	0	0	0	0
0.5	0.559928	0.403166	0.278242	-0.63727	-9.03406
1	0.5	1.063496	1.537378	3.049896	12.76054
1.5	0.5	1	1.507861	1.998268	0.67732
2	0.5	1	1.5	1.993664	2.468401
2.5	0.5	1	1.5	2	2.510962
3	0.5	1	1.5	2	2.5
3.5	0.5	1	1.5	2	2.5
4	0.5	1	1.5	2	2.5
4.5	0.5	1	1.5	2	2.5
5	0.5	1	1.5	2	2.5
5.5	0.5	1	1.5	2	2.5
6	0.5	1	1.5	2	2.408546
6.5	0.5	1	1.5	2.052863	1.822835
7	0.5	1	1.434411	2.558264	9.963272
7.5	0.5	0.47023	0.51342	-0.56705	-12.476
8	0	0.52977	0.98658	2.567053	14.97635
8.5	0	0	0.065589	-0.55826	-7.46327
9	0	0	0	-0.05286	0.677164
9.5	0	0	0	0	0.091954
10	0	0	0	0	0
10.5	0	0	0	0	0
11	0	0	0	0	0
11.5	0	0	0	0	0
12	0	0	0	0	0
12.5	0	0	0	0	0
13	0	0	0	0	0
13.5	0	0	0	0	0
14	0	0	0	0	0
14.5	0	0	0	0	0
15	-3	-3	-3	-3	-3

Table. 8.1(b)

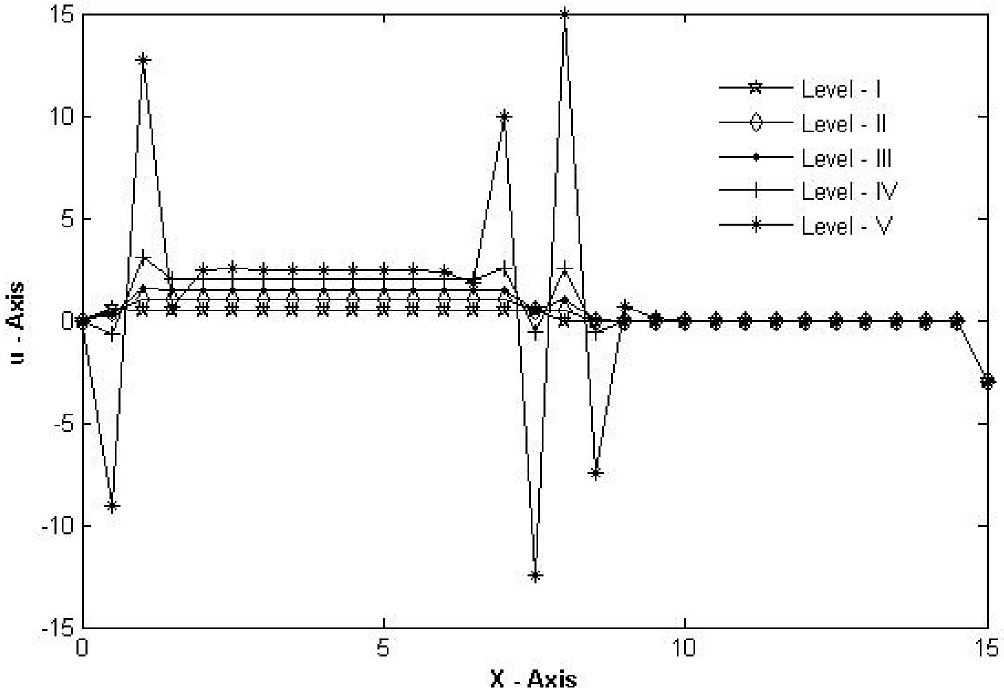
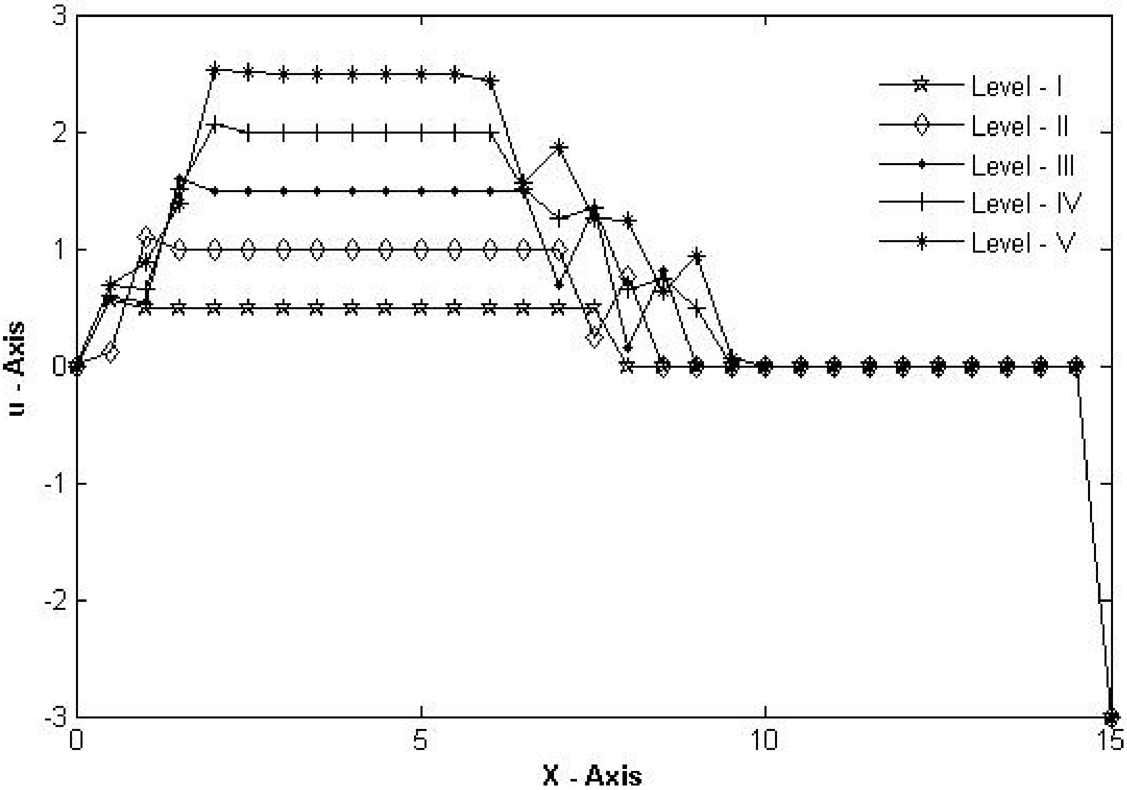

15
10
5
0
-5
-10
-15
0
5
10
15
u - Axis
X - Axis
Level - I
Level - II
Level - III
Level - IV
Level - V

Figure. 8.1(b)

CASE-III: $\gamma = 2500$

X	LEVEL-1(u)	LEVEL-2	LEVEL-3	LEVEL-4	LEVEL-5
0	0	0	0	0	0
0.5	0.559928	0.111702	0.585477	0.685795	0.688854
1	0.5	1.091675	0.545387	0.651041	0.877135
1.5	0.5	1	1.597348	1.508572	1.378224
2	0.5	1	1.5	2.058116	2.53421
2.5	0.5	1	1.5	2	2.507844
3	0.5	1	1.5	2	2.5
3.5	0.5	1	1.5	2	2.5
4	0.5	1	1.5	2	2.5
4.5	0.5	1	1.5	2	2.5
5	0.5	1	1.5	2	2.5
5.5	0.5	1	1.5	2	2.5
6	0.5	1	1.5	2	2.434552
6.5	0.5	1	1.5	1.515122	1.560284
7	0.5	1	0.687793	1.252803	1.866625
7.5	0.5	0.235125	1.344989	1.35211	1.250774
8	0	0.764875	0.155011	0.647888	1.249226
8.5	0	0	0.812207	0.747197	0.633375
9	0	0	0	0.484878	0.939716
9.5	0	0	0	0	0.065448
10	0	0	0	0	0
10.5	0	0	0	0	0
11	0	0	0	0	0
11.5	0	0	0	0	0
12	0	0	0	0	0
12.5	0	0	0	0	0
13	0	0	0	0	0
13.5	0	0	0	0	0
14	0	0	0	0	0
14.5	0	0	0	0	0
15	-3	-3	-3	-3	-3

Table.8.1(c)



u - Axis
X - Axis
Level - I
Level - II
Level - III
Level - IV
Level - V
3
2
1
0
-1
-2
-3
0
5
10
15

Figure.8.1(c)

Non-linear wave propogation:

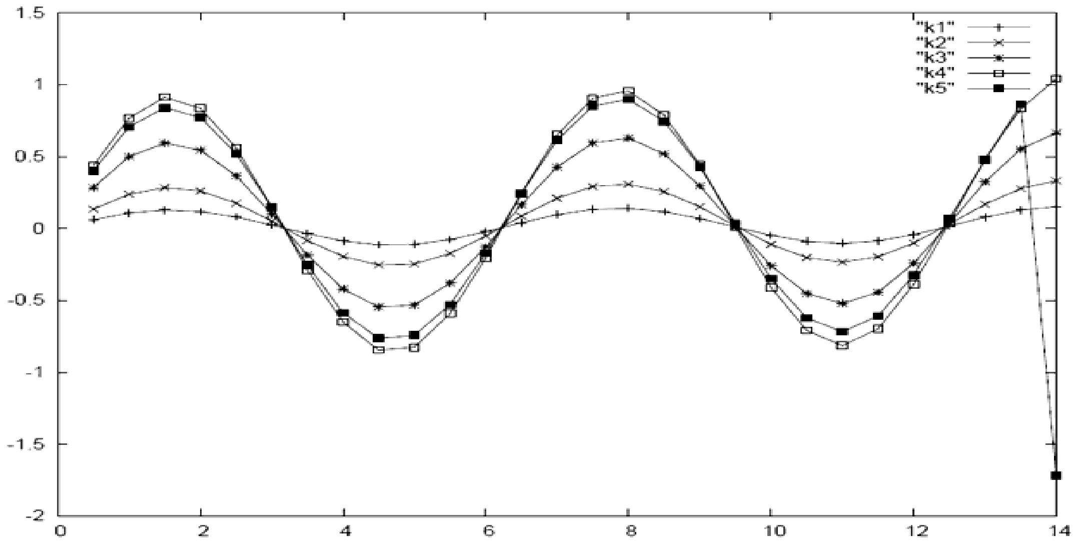


Figure.8.1 (d)

PHYSICAL INTERPRETATION

For all acoustic parameter values (γ)

1. At the lower and the higher positions of the objects the collision might be random; obviously it indicates the inelastic collision. In other words, the loss of kinetic energy may be sustained and converted in to equivalent sound and/or heat dissipated in to the surroundings.

2. At the middle positions of the objects the collision may be uniform; obliviously it indicates the elastic collision. In other wards no loss of kinetic energy is sustained in the collision.

CONCLUSIONS

When ever an impact occurs the velocities of the two objects are changes according to the starting compression force applied at the impact point. An impact occurs a longitudinal sound wave is generated and it propagates in the region up to free end of the second object.

When it reaches to the free end a reflection occurs. That is the only reason the boundary condition at the free end is assumed as negative but small in magnitude.

The displacement in terms of the length of the impact system with respective to time is drawn in Figure 8.1(a) -8.1(d) It gives the following implications.

1) At γ=10000 In the second level the displacement u(x,t) exhibits non-linearity at the middle of the position of the objects and at all other time levels , no non-linearity is observed.

2) At γ=5000 all time levels, displacement sustains with respect to the origin except at time level 5. At end positions at time level 5 Non-linearity is observed.

3) At γ=2500 all the time levels exhibiting the displacements with disturbance at end positions (0-2 cm and 7-10 cm) and the middle position the displacement is found to constant and rises with respective to the time level-1.

4) At lower and higher γ values non-linearity is not observed clearly but it gives the tendency. At middle γ value the non-linearity behavior is clearly observed at higher time level-5.

5) For all acoustic parameter(γ) values displacement u is observed to constant at 2 to 6 units distance with respective to time level.

PART-V

CHAPTER-9

CONCLUSIONS AND DIRECTIONS FOR FUTURE WORK

In this Thesis in Part – I we have studied steady-state convection-diffusion problems in one dimensional space and two dimensional space. Also we have chosen a problem which is related to wave propagation in a non linear medium due to impact of two objects. This problem is also having some commonality with the convection-diffusion problems i.e. convection of molecules within the substance.

Convection-diffusion problems form a class of singular perturbation problems. The numerical treatment of these Singular perturbation problems is far from trivial in view of the boundary layer behavior of the solutions. There is a phenomenal change in the solution at the boundary layer region due to the perturbation parameter which is positive in quantity but very close to zero. The coefficient of the highest order derivative in the convection-diffusion equation.

In Part II we studied the steady state convection- diffusion problems which are solved by applying different numerical methods. It consists of four chapters 2, 3, 4 and 5.

In Chapter 2, we studied a computational method to solve steady state convection – diffusion problem. In this problem an attempt is made to study the asymptotic method to study the solution nature of the same equation. We have observed that, there is a right boundary layer near the argument x=1.

In Chapter 3, we studied a uniformly convergent scheme for convection –diffusion problem namely Allen-Il'in developed scheme and applied to the one-dimensional convection-diffusion problem. We compared the solution with the finite difference methods. In this work a condition is contemplated for convergence It is found that Allen-Il'in scheme converges uniformly through out the specified domain [0,1].

Chapter 4 is devoted to the application of finite element method to solve singularly perturbed two point boundary value problems using cubic B-splines. The basis functions have been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions is employed. A finer mesh has been taken near and around δ where the left boundary layer is located. The proposed Galerkin method has given the computational results which are very much close to the analytical solutions which are available in the literature for a fine mesh size h. The approximate solutions obtained by the developed method are in good agreement with the exact solutions of the selected problems.

In Chapter-5 we studied numerical integration method for solving general steady-state convection-diffusion problems The proposed method is iterative on the deviating argument. The computed results are matching with the exact solution with reasonable accuracy.

In Part-III we discussed a peculiar problem coined by Stynes [66], Artificial-diffusion convection problem and two dimensional convection-diffusion problems. It consists of two chapters, chapter 6 and 7.

Chapter 6 deals with a convection-diffusion problem in one-dimension with variable coefficient wherein an artificial –diffusion term is present. The numerically introduced artificial-diffusion reduces the oscillations in the boundary layer region. As a closed form solution is not available we have solved by using Finite difference methods wherein central difference scheme is employed. The same problem is also solved by classical Frobenious method. These two methods have given reasonably fair results.

In **chapter-7** we presented convection –diffusion problem in two- dimensional space. Convection-diffusion problem in two-dimensional space is solved on a unit square mesh with the prescribed boundary conditions by finite difference method where in central difference scheme is employed. It is observed that there is a boundary layer at the specific values of arguments

Part IV consists of a single Chapter. This chapter aims to study the Numerical study of wave propagation in a non-linear medium due to impact. The problem studied in this part is analogous to those studied in the previous part as the non-linear wave equation possesses convection in nature. Non-linear equation is made linear by quasi-linearization technique.

In all the above problems, Numerical methods are used and the analytical solutions are attained wherever possible. In the Numerical methods for majority of the problems finite difference methods are employed. In chapter-4 we have employed finite element method in order to get a high precision. In a nut-shell the numerical methods presented in this thesis for solving convection-diffusion problems in differential equations have been shown to be accurate and capable over the conventional methods. Above all, these methods are conceptually simple, easy to use and are readily adaptable for computer implementation with a modest amount of problem preparation.

The problems in this thesis are solved in the steady state with Dirichlet's boundary conditions. In future study it is worthwhile to study un-steady convection-diffusion problems with Neumann (derivative) boundary conditions and mixed boundary conditions.

There are likely to be more challenging and numerically involved that what are considered in the present thesis.

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